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Nodal Superconvergence of SDFEM for Singularly Perturbed Problems

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Abstract In this paper, we analyze the streamline diffusion finite element method for one dimensional singularly perturbed convection-diffusion-reaction problems. Local error estimates on a subdomain where the solution is smooth are established. We prove that for a special group of exact solutions, the nodal error converges at a superconvergence rate of order $(\ln\epsilon^{-1}/N)^{2k}$ (or $(\ln N/N)^{2k}$) on a Shishkin mesh. Here ϵ is the singular perturbation parameter and $2N$ denotes the number of mesh elements. Numerical results illustrating the sharpness of our theoretical findings are displayed.

Keywords Streamline diffusion finite element method · Shishkin mesh · Superconvergence

1 Introduction

We consider the following convection-diffusion-reaction problem

$$\begin{aligned} -\epsilon u'' + au' + bu &= f \quad \text{in } \Omega = (0,1), \\ u &= 0 \quad \text{on } \partial\Omega = \{0,1\} \end{aligned} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is the diffusion parameter, $a = a(x) \geq \alpha > 0$ accounts for the convection, and $b = b(x)$ accounts for the reaction term. The function $f = f(x)$ is a given source term.

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We assume that a, b, and f are sufficiently smooth on $\overline{\Omega}$. It is well-known [11] that the exact solution u of (1.1) exhibits a boundary layer of width $O(\epsilon \ln \epsilon^{-1})$ at the outflow boundary $x = 1$. An efficient way of handling this problem from a numerical method perspective is employing a layer-adapted mesh, such as a Shishkin or a Bakhvalov mesh. A considerable amount of literature has been devoted to theoretical analysis of such methods [11–17, 19, 21, 22].

In [10], Douglas and Dupont proved that when the classical Galerkin method using piecewise continuous polynomials of degree at most k is applied to approximate the solution of (1.1), the error superconverges with order $2k$ at the nodes of the mesh. Although, this result was valid for arbitrary quasi-uniform meshes, it was limited to $\epsilon = O(1)$, and they did not consider the singularly-perturbed regime. Indeed, numerical experiments showed that it is impossible to extend such results to the singularly-perturbed regime on arbitrary meshes. The main reason is that the error in the boundary layer region propagates through the domain resulting in the pollution of the global approximation.

It is well known that the streamline diffusion finite element method (SDFEM) developed by Brooks and Hughes [8] exhibits a good performance in resolving boundary layers [4, 5, 16]. Therefore, they provide more accurate approximations to solutions of singularly perturbed problems. In this paper, we investigate superconvergence properties of SDFEM under Shishkin-type meshes for problem (1.1). We prove that the error at the nodes of the mesh superconverge with the rate of order $(\ln \epsilon^{-1}/N)^{2k}$ or $(\ln N/N)^{2k}$, depending on the choice of the transition point of the mesh. The Shishkin-type meshes which we employ to prove our results consist of $2N$ elements, N of which are located away from the boundary layer, and the remaining N elements are placed inside the boundary layer.

The outline of the rest of the paper is as follows. In Sect. 2, we display the method and state our main results. The proof of these results are provided in Sect. 3. Numerical results verifying the sharpness of our theoretical findings are provided in Sect. 4. We end in Sect. 5 with some concluding remarks.

2 Main Results

In this section, we introduce the SDFEM we will use to approximate the solution of (1.1), and state our main results. The proofs will be provided in Sect. 3. We begin with introducing the layer adapted meshes we will use to approximate the solution of (1.1) using SDFEM. Let

$$\tau = \min\{K_1 \epsilon \ln \epsilon^{-1}, 1/2\} \tag{2.1}$$

for some constant $K_1 \geq k + \frac{1}{2}$, and set

$$H = \frac{1-\tau}{N} \quad \text{and} \quad h = \frac{\tau}{N}.$$

The number τ is called the *transition number*. Let us note that since $\tau \leq 1/2$, we have that $1/(2N) \leq H \leq 1/N$, and hence there exists a constant $C \in (1/2, 1)$ such that $H = C/N$. The *nodes* of the mesh are defined recursively by setting $x_0 = 0$ and

$$x_j = \begin{cases} x_{j-1} + H, & \text{for } j = 1, 2, \ldots, N, \\ x_{j-1} + h, & \text{for } j = N+1, N+2, \ldots, 2N. \end{cases}$$

The set of all nodes, $\{x_0, x_1, \ldots, x_{2N}\}$, will be denoted by $\mathcal{E}_N$. Note that the node $x_N = 1-\tau$ which is called the *transition point*. For $j = 1, 2, \ldots, 2N$, we define $I_j = (x_{j-1}, x_j)$ as the jth element of the finite element mesh $\Omega_N = \bigcup_{j=1}^{2N} I_j$. Since the mesh Ω_N is piecewise uniform we define

$$\Omega_{\mathrm{R}} = \bigcup_{j=1}^{N} I_j \quad \text{and} \quad \Omega_{\mathrm{BL}} = \bigcup_{j=N+1}^{2N} I_j.$$

Clearly, Ω_{R} is a uniform discretization of the interval $(0, 1-\tau)$ of meshsize H, and Ω_{BL} is that of the interval $(1-\tau, 1)$ of meshsize h.

The conforming finite element space is defined as

$$V_N := \{v \in H_0^1(\Omega) : v|_I \in \mathcal{P}^k(I), \forall I \in \Omega_N\}$$

where $\mathcal{P}^k(I)$ is the space of polynomials of degree at most k on I. We will use the standard notation

$$(\varphi, \psi)_{\mathcal{D}} := \sum_{I \in \mathcal{D}} (\varphi, \psi)_I$$

where $\mathcal{D}$ is any subset of Ω_N and $(\varphi, \psi)_I := \int_I \varphi\psi$. We will drop the subscript whenever $\mathcal{D} = \Omega_N$.

We are now ready to define the method. The SDFEM seeks $u_N \in V_N$ such that

$$B_N(u_N, v) = F(v) \quad \text{for all } v \in V_N,$$

where

$$\begin{aligned} B_N(u_N, v) &:= \epsilon(u_N', v') + (au_N', v) + (bu_N, v) \\ &\quad + (-\epsilon u_N'' + au_N' + bu_N, \delta a v') \end{aligned} \tag{2.2}$$

and

$$F(v) := (f, v) + (f, \delta a v').$$

The *artificial diffusion parameter* δ is defined as

$$\delta|_I = \begin{cases} H, & \text{if } I \in \Omega_{\mathrm{R}}, \\ 0, & \text{if } I \in \Omega_{\mathrm{BL}}. \end{cases} \tag{2.3}$$

We assume that N is sufficiently large such that

$$\ln \epsilon^{-1} \le N, \tag{2.4a}$$

$$H \|b\|_{\infty, \Omega_{\mathrm{R}}}^2 \le c_0/2, \tag{2.4b}$$

$$K_2(k+1)\epsilon \ln \epsilon^{-1} \le \frac{H}{3} \le \frac{1}{3N}, \tag{2.4c}$$

where constant $K_2 > 1$ is independent of ϵ and N. The value of K_2 will be determined in (3.15). It is clear that (2.4c) implies that

$$\epsilon \le CN^{-1}, \tag{2.5}$$

Here, $C = [K_2(k+1)]^{-1}$ is independent of ϵ and N.

Note that these assumptions do not constitute a loss of generality since we are interested in singularly perturbed problems and hence $\epsilon \ll 1$ which makes the above assumptions very easy to satisfy even for reasonably small N.

Remark 2.1 This method can be regarded as a coupling of the streamline diffusion method with the classical Galerkin method. Artificial diffusion is added in Ω_{R} to prevent possible oscillations of the approximation outside the boundary layer. Inside the boundary layer we simply apply the classical Galerkin method since the mesh Ω_{BL} is sufficiently fine to resolve the boundary layer.

For any $\mathcal{D} \subseteq \Omega_N$ the Sobolev seminorm on $H^s(\mathcal{D})$ is defined as

$$|v|_{s,\mathcal{D}} := (v^{(s)}, v^{(s)})_{\mathcal{D}}^{1/2}.$$

Accordingly, the Sobolev norm on $H^r(\mathcal{D})$ is defined as

$$\|v\|_{r,\mathcal{D}} := \left(\sum_{s=0}^{r} |v|_{s,D}^2 \right)^{1/2}.$$

We drop the first subscript whenever $r = 0$, and the second one if $\mathcal{D} = \Omega_N$. Similarly, we define the norm

$$|||v|||_{\mathcal{D}} := \left[\epsilon \|v'\|_{\mathcal{D}}^2 + \|\sqrt{\delta} a v'\|_{\mathcal{D}}^2 + \|(b - a'/2)^{1/2} v\|_{\mathcal{D}}^2 \right]^{1/2}, \tag{2.6}$$

for $v \in H^1(\mathcal{D}) := \bigcup_{I \in \mathcal{D}} H^1(I)$.

The following proposition states that the bilinear from B_N is coercive in the $|||\cdot|||$-norm and hence the SDFEM is well defined.

Proposition 2.2 *Suppose that, either $b(x)$ is identically zero and*

$$a'(x) = 0, \quad \textit{for all } x \in \Omega, \tag{2.7a}$$

or there exists a constant $c_0 > 0$ such that

$$b(x) - a'(x)/2 \geq c_0, \quad \textit{for all } x \in \Omega. \tag{2.7b}$$

Then

$$B_N(v, v) \geq \frac{1}{2} |||v|||^2, \quad \textit{for all } v \in V_N. \tag{2.8}$$

We also have the *Galerkin orthogonality property*

$$B_N(u - u_N, v) = 0, \quad \text{for all } v \in V_N, \tag{2.9}$$

which follows from the fact that $B_N(u, v) = 0$ for all v in V_N.

The exact solution of the boundary value problem (1.1) admits a decomposition of the form

$$u = u_{\mathrm{R}} + u_{\mathrm{BL}},$$

where the component functions have the regularity [11, 21]

$$|u_{\mathrm{R}}^{(s)}(x)| \le C \quad \text{and} \quad |u_{\mathrm{BL}}^{(s)}(x)| \le C\epsilon^{-s} e^{-\alpha(1-x)/\epsilon} \tag{2.10}$$

for all x in Ω and $s = 0, 1, 2, \ldots$. Here and elsewhere in the paper, C denotes a generic constant which might not be the same in each appearance. It might depend on the coefficient functions a, b, the right-hand side function f, and the polynomial degree k, but is independent of the singularity parameter ϵ, the mesh number N, and the mesh sizes h and H.

We are now ready to state our main results. The first one is a local error estimate which measures the error away from the boundary layer.

Theorem 2.3 *There exists a constant C independent of ϵ and N such that*

$$|||u - u_N|||_{\Omega_{\mathrm{R}}} \le C N^{-(k+1/2)}. \tag{2.11a}$$

Moreover, if $u_{\mathrm{R}} \in V_N$ and $K_1 \ge 2k + \frac{1}{2}$, then we have

$$|||u - u_N|||_{\Omega_{\mathrm{R}}} \le C \frac{\epsilon^k}{N^{k+\frac{1}{2}}}. \tag{2.11b}$$

Remark 2.4 This is the first local error estimate on a layer-adapted mesh. The estimate (2.11b) can be considered as a superconvergence result in the sense that it contains a positive power of the singularity parameter ϵ. This result is obtained at the expense of assuming that u_{R} is in the finite element space.

Our next result is superconvergence result for the error at the nodes of the mesh.

Theorem 2.5 *If $u_{\mathrm{R}} \in V_N$ and $K_1 \ge 2k + \frac{1}{2}$, there exists a constant C independent of ϵ and N such that*

$$|(u - u_N)(x_j)| \le C \left(\frac{\ln \epsilon^{-1}}{N} \right)^{2k} \tag{2.12}$$

for all $j = 0, 1, \ldots, 2N$.

Remark 2.6 This result can be considered as an extension to the singularly-perturbed regime of the nodal estimate proved by Douglas and Dupont in [10] for the classical Galerkin method. In [3], Celiker and Cockburn proved similar results for the numerical traces of the discontinuous Galerkin approximations to the solution of convection-diffusion problems. However, they did not consider the singularly perturbed regime. They proved that if $\epsilon = O(1)$ then for a wide family of discontinuous Galerkin methods the numerical traces superconverge at a rate of order h^{2k+1} where h is the mesh size of an arbitrary mesh.

Remark 2.7 Motivated by the findings in [3], Xie and Zhang [18] provided numerical evidence that the numerical traces of the local discontinuous Galerkin method maintain the $2k+1$-superconvergence rate in the singularly-perturbed regime provided that a Shishkin-type mesh is employed. A proof of this result has not appeared in the literature yet. However, Xie and Zhang [19] were able to prove an analogous result for a slightly simplified problem, namely, in the absence of the diffusion term. In [20], Xie et al. provided numerical evidence that a similar superconvergence phenomenon takes place for two-dimensional singularly perturbed convection-diffusion problems.

Remark 2.8 As we will see in Sect. 3, our proof of Theorem 2.5 relies on the estimate (2.11b) of Theorem 2.3. Hence, we need the assumption $u_{\mathrm{R}} \in V_N$. On the other hand, our numerical results (see Sect. 4) show that this assumption is *not* necessary.

Remark 2.9 The estimate (2.12) depends weakly on ϵ since it contains the factor $\ln \epsilon^{-1}$. It is therefore not uniform, but can still be regarded as a superconvergence rate since it is of order N^{-2k} up to a logarithmic factor. Numerical results which we display in Sect. 4 show that this factor does exist. In this sense, the estimate (2.12) is sharp.

Remark 2.10 It has been shown in [2] and [3] that the nodal superconvergence information can be exploited to postprocess the approximate solution in an element-by-element fashion to obtain a new approximation which converges much faster than the original approximation. The local nature of the postprocessing ensures that the computational cost involved is negligible compared to the cost of computing the original solution. We will consider similar postprocessing techniques of the SDFEM approximation in a forthcoming paper.

Remark 2.11 Let us note that the choice (2.1) of the transition parameter which was borrowed from [18] is a slight variation of what has appeared in the literature. If we use the classical choice [12–14]

$$\tau = \min\{K_1 \epsilon \ln N, 1/2\} \tag{2.13}$$

then we obtain the following variations of Theorem 2.5 and Theorem 2.3, respectively.

Theorem 2.12 *Suppose $K_1 \geq 2k + \frac{1}{2}$ is sufficiently large such that*

$$N^{-(K_1 - 2k)} \leq \epsilon^{\frac{1}{2}}. \tag{2.14}$$

Then there exists a constant C independent of ϵ and N such that

$$|||u - u_N|||_{\Omega_{\mathrm{R}}} \leq C N^{-(k+1/2)}. \tag{2.15a}$$

Moreover, if $u_{\mathrm{R}} \in V_N$ and $K_1 \geq 2k + \frac{1}{2}$, then we have

$$|||u - u_N|||_{\Omega_{\mathrm{R}}} \leq C N^{-(2k+\frac{1}{2})}. \tag{2.15b}$$

Theorem 2.13 *If $u_{\mathrm{R}} \in V_N$ and the assumption (2.14) holds true, there exists a constant C independent of ϵ and N such that*

$$|(u - u_N)(x_j)| \leq C \left(\frac{\ln N}{N}\right)^{2k} \tag{2.16}$$

for all $j = 0, 1, \ldots, 2N$.

Remark 2.14 The nodal error estimates (2.12) and (2.16) holds under the assumption (2.7a)–(2.7b). The proofs can be extended to traditional finite element method (FEM) by removing the stabilization term in scheme (2.2). However, repeating the proofs in next section for FEM we see that $\epsilon^{-\frac{1}{2}}$ will appear in the error bounds and is not removable. It is because the ϵ-weighted energy norm introduced by FEM scheme, compared with the SD-norm (2.6), lacks an extra term $\|\sqrt{\delta} a v'\|_{\mathcal{D}}$.

3 Proofs

In this section we provide proofs of the results stated in Sect. 2. The outline of this section is as follows. We begin with proving the coercivity of the bilinear form, Proposition 2.2. Section 3.2 is devoted to the proof of Theorem 2.3. In Sect. 3.3 we prove the nodal superconvergence result, Theorem 2.5. Finally, in Sect. 3.4 we provide some of the main ingredients of the proof of Theorem 2.13.

3.1 Proof of Proposition 2.2

We will use the following inverse inequality. Let $K = (a, b)$ and $h_K = b - a$. There exists a constant C which depends solely on r such that

$$\|v'\|_{0,K} \le \frac{C}{h_K}\|v\|_{0,K} \quad \text{for all } v \in \mathcal{P}^r(K). \tag{3.1}$$

Proof (Proposition 2.2). For any $v \in V_N$ we have

$$\begin{aligned} B_N(v, v) &= \epsilon\|v'\|^2 + \|(b - a'/2)^{1/2}v\|^2 + \|\sqrt{\delta}av'\|^2 + (-\epsilon v'' + bv, \delta av') \\ &= |||v|||^2 + (-\epsilon v'' + bv, \delta av') \end{aligned} \tag{3.2}$$

where we have used the fact that $v = 0$ on $\partial\Omega$. If $b(x)$ is identically zero, we have

$$\begin{aligned} |(-\epsilon v'' + bv, \delta av')| &= |(-\epsilon v'', \delta av')| \\ &\le \frac{1}{2}\left(\|\sqrt{\delta}\epsilon v''\|^2 + \|\sqrt{\delta}av'\|^2\right) && \text{by Young's inequality} \\ &= \frac{1}{2}\left(H\epsilon^2\|v''\|^2_{\Omega_R} + \|\sqrt{\delta}av'\|^2\right) && \text{by (2.3)} \\ &\le \frac{1}{2}\left(\frac{\epsilon^2}{H}\|v'\|^2_{\Omega_R} + \|\sqrt{\delta}av'\|^2\right) && \text{by (3.1)} \\ &\le \frac{1}{2}\left(\epsilon\|v'\|^2_{\Omega_R} + \|\sqrt{\delta}av'\|^2\right) && \text{by (2.5)} \\ &\le \frac{1}{2}|||v|||^2 && \text{by (2.6).} \end{aligned}$$

If $b(x)$ is not identically zero then

$$\begin{aligned} &|(-\epsilon v'' + bv, \delta av')| \\ &\le \|\sqrt{\delta}\epsilon v''\|^2 + \|\sqrt{\delta}bv\|^2 + \frac{1}{2}\|\sqrt{\delta}av'\|^2 && \text{by Young's inequality} \\ &\le \frac{\epsilon^2}{H}\|v'\|^2_{\Omega_R} + \frac{H}{c_0}\|b\|^2_{\infty,\Omega_R}\|\sqrt{c_0}v\|^2_{\Omega_R} + \frac{1}{2}\|\sqrt{\delta}av'\|^2 && \text{by (2.3) and (3.1)} \\ &\le \frac{\epsilon}{2}\|v'\|^2_{\Omega_R} + \frac{1}{2}\|\sqrt{c_0}v\|^2_{\Omega_R} + \frac{1}{2}\|\sqrt{\delta}av'\|^2 && \text{by (2.4b) and (2.5)} \\ &\le \frac{1}{2}|||v|||^2 && \text{by (2.6).} \end{aligned}$$

Therefore, (2.8) holds in either case by (3.2). This completes the proof. □

3.2 Proof of Theorem 2.3

We proceed in several steps. First, we introduce a cut-off function and state its main properties. Step 2 contains two preliminary lemmas. In Step 3, we introduce a weighted seminorm, $|||\cdot|||_\omega$. In Step 4, we define the Lagrange interpolation operator that will be used for our error analysis, and state its approximation properties. Step 5 is the main step in which we obtain an expression for the $|||\cdot|||_\omega$-seminorm of the projection of the error and estimate each term of the resulting expression. Finally, in Step 6, we combine all the estimates to prove Theorem 2.3.

Step 1: *The cut-off function.* In this step we introduce a *cut-off function* which will be instrumental throughout the proof of Theorem 2.3. Let $\varphi(t)$ be function satisfying the following conditions. We suppose that there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 \le \varphi(t) \le C_2, &\quad \text{for } t \le 1,\\ \varphi(t) = C_3 e^{-t}, &\quad \text{for } t \ge 0, \qquad (3.3)\\ \varphi(t) = 3 - \frac{1}{\ln|t|+1}, &\quad \text{for } t \le -1;\end{aligned}$$

$$\varphi'(t) < 0, \quad \text{for } t \in (-\infty,\infty); \tag{3.4}$$

$$\begin{aligned} |\varphi^{\ell}(t)| \le C_2|\varphi(t)|, &\quad (1 \le \ell \le k+1) \text{ for } t \in (-\infty,\infty),\\ |\varphi^{\ell}(t)| \le C_2|\varphi'(t)|, &\quad (2 \le \ell \le k+1) \text{ for } t \in (-\infty,\infty), \qquad (3.5)\\ |\varphi(t)| \le C_2(|t|+1)(\ln(|t|+1)+1)^2|\varphi'(t)|, &\quad \text{for } t \in (-\infty,\infty),\end{aligned}$$

where $0 < C_3 < 1$. Finally, setting

$$RO(\mathcal{D},\nu) = \frac{\max_{x\in\mathcal{D}}|\nu(x)|}{\min_{x\in\mathcal{D}}|\nu(x)|},$$

we assume that

$$RO(\widehat{I},\varphi) + RO(\widehat{I},\varphi') \le C_2 \tag{3.6}$$

for any interval $\widehat{I}$ of length 1. The cut-off function ω is defined as

$$\omega(x) = \begin{cases} \varphi(\frac{x-A}{\rho_{\mathrm{R}}}), & \text{if } x \in \Omega_{\mathrm{R}},\\ 0, & \text{if } x \in \Omega_{\mathrm{BL}}.\end{cases}$$

Here

$$A = 1 - \tau - \rho_{\mathrm{BL}}(k+1)\ln\epsilon^{-1}, \tag{3.7}$$

and

$$\rho_{\mathrm{R}} = K_2 N^{-1}, \qquad \rho_{\mathrm{BL}} = K_2\epsilon, \tag{3.8}$$

where K_2 is the constant in (2.4c).

Remark 3.1 We note that $\omega(x)$ is a one-dimensional version of the function introduced by Guzmán in [7] which was a variation of an analogous function employed by Johnson et al. in [9]. Guzmán's modification was necessary in order to handle the case where $b(x)$ is not bounded away from zero from below, i.e., in the absence of the reaction term.

Some of the basic properties of $\omega(x)$ which follow directly from the properties (3.3)–(3.6) are gathered in the following lemma.

Lemma 3.2 *The cut-off function $\omega(x)$ has the following regularity properties*

$$|\omega^{(\alpha)}(x)| \le C\rho_{\mathrm{R}}^{-\alpha}|\omega(x)|, \quad 1 \le \alpha \le k+1, \; x \in \Omega_N, \tag{3.9a}$$

$$|\omega^{(\alpha)}(x)| \le C\rho_{\mathrm{R}}^{-\alpha+1}|\omega'(x)|, \quad 1 \le \alpha \le k+1, \; x \in \Omega_N, \tag{3.9b}$$

$$|\omega(x)| \le C(\ln N)^2|\omega'(x)|, \quad x \in (0, 1-A), \tag{3.9c}$$

$$|\omega(x)| = \rho_{\mathrm{R}}|\omega'(x)|, \quad x \in (1-A, 1), \tag{3.9d}$$

$$RO(I,\omega) \text{ and } RO(I,\omega') \quad \text{are bounded independently of } N \text{ and } \epsilon, I \in \Omega_N. \tag{3.9e}$$

Step 2: *A preliminary lemma.*

Lemma 3.3 *Let $v \in \mathcal{P}^k(I_N)$. If C_3 is sufficiently small, then*

$$\omega(x_N^-)\|v\|_{I_N} \le \sqrt{C_3}C(k)\|\omega v\|_{I_N}, \tag{3.10}$$

for some constant $C(k)$ which only depends solely on the polynomial degree k.

Proof By assumption (2.4c), the point A defined by (3.7) lies in I_N. In fact, $A \in (x_{N-1/3}, x_N)$, where the points $\{x_{N-2/3}, x_{N-1/3}\}$ equipartition I_N.

To prove (3.10), without loss of generality, we assume that v is an element of a basis for $\mathcal{P}^k(I_N)$. By a straightforward calculation one gets

$$\|v\|_{I_N} \le C(k)\|v\|_{(x_{N-1},x_{N-2/3})},$$

for some constant $C(k)$ which depends solely on k. It is not difficult to compute the precise value of $C(k)$.

Taking sufficiently small C_3, we can adjust the value of $\varphi(t)$ on $(-1, 0)$ such that the definition of $\omega(x)$ satisfies

$$\omega(x_N^-) \le \sqrt{C_3} \min_{x\in(x_{N-1},x_{N-2/3})} \omega(x),$$

and hence

$$\omega(x_N^-)\|v\|_{I_N} \le \sqrt{C_3}C(k)\|\omega v\|_{(x_{N-1},x_{N-2/3})} \le \sqrt{C_3}C(k)\|\omega v\|_{I_N}.$$

This completes the proof. □

Step 3: *A weighted seminorm.* Let v be a function in $L^2(\Omega_N)$. On the set of interior nodes $\{x_1, x_2, \ldots, x_{2N-1}\}$ we define the *jump* of v as

$$[\![v]\!](x) = v(x^-) - v(x^+), \quad \text{where } v(x^\pm) = \lim_{t\downarrow 0} v(x \pm t).$$

For $v \in H^1(\Omega_{\mathrm{R}})$ we define the weighted seminorm

$$|\!|\!|v|\!|\!|_\omega^2 := \epsilon\|\omega v'\|^2 + H\|a\omega v'\|^2 + \|(b - a'/2)^{\frac{1}{2}}\omega v\|^2 + \|(a\omega|\omega'|)^{\frac{1}{2}} v\|^2 + \frac{1}{2}(av^2[\![\omega^2]\!])(x_N). \tag{3.11}$$

Step 4: *The Lagrange interpolation operator.* Let $K=(a,b)$ be an interval and set $h_K = b-a$. Let $\varphi \in H^1(K)$ be an arbitrary function. The Lagrange interpolation $\mathbb{I}\varphi$ of φ is defined as the element of $\mathcal{P}^k(K)$ such that

$$(\mathbb{I}\varphi)(\zeta_i) = \varphi(\zeta_i) \quad \text{for } i = 0, 1, \dots, k \text{ where } \zeta_i = a + i\frac{h_K}{k}.$$

We will use the following notation throughout the paper

$$e_u := u - u_N, \qquad \xi_u := u - \mathbb{I}u, \qquad \psi_u := \mathbb{I}u - u_N.$$

The following lemma contains basic the approximation property of the interpolation operator $\mathbb{I}$.

Lemma 3.4 *Let $\varphi \in H^{k+1}(K)$ then there exists a constant C that depends solely on k such that*

$$|\varphi - \mathbb{I}\varphi|_{s,K} \le C h_K^{r+1-s} |\varphi|_{r+1,K} \tag{3.12}$$

for any s and r such that $0 \le s \le r \le k$.

A proof of this lemma can be found in many classical references such as [1, 6].

An application of Lemma 3.4 and using the decomposition $u = u_{\mathrm{R}} + u_{\mathrm{BL}}$ of the exact solution, we obtain the following result. We omit the proof since it uses techniques similar to those employed in [21].

Lemma 3.5 *There exists a constant C such that*

$$\|\xi_u\|_{\Omega_{\mathrm{R}}} \le \frac{C}{N^{k+1}}, \qquad \sqrt{\epsilon}\|\xi_u'\|_{\Omega_{\mathrm{R}}} + \sqrt{H}\|a\xi_u'\|_{\Omega_{\mathrm{R}}} \le \frac{C}{N^{k+1/2}}, \qquad \|\xi_u''\|_{\Omega_{\mathrm{R}}} \le \frac{C}{N^{k-1}}, \tag{3.13a}$$

$$\|\xi_u\|_{\Omega_{\mathrm{BL}}} \le C\sqrt{\epsilon}\left(\frac{\ln \epsilon^{-1}}{N}\right)^{k+1}, \qquad \sqrt{\epsilon}\|\xi_u'\|_{\Omega_{\mathrm{BL}}} \le C\left(\frac{\ln \epsilon^{-1}}{N}\right)^{k}. \tag{3.13b}$$

Furthermore, if $u_{\mathrm{R}} \in V_N$ and $K_1 \ge 2k + \frac{1}{2}$, then

$$\begin{aligned} \|\xi_u\|_{\Omega_{\mathrm{R}}} &\le C\frac{\epsilon^k}{N^{k+1}}, \qquad \sqrt{\epsilon}\|\xi_u'\|_{\Omega_{\mathrm{R}}} + \sqrt{H}\|a\xi_u'\|_{\Omega_{\mathrm{R}}} \le C\frac{\epsilon^k}{N^{k+1/2}}, \\ \|\xi_u''\|_{\Omega_{\mathrm{R}}} &\le C\frac{\epsilon^k}{N^{k-1}}. \end{aligned} \tag{3.14}$$

Step 5: *An estimate of $|||\psi_u|||_\omega$.*

Lemma 3.6 *If K_2 and N are sufficiently large such that*

$$L := \max\left(K_2^{-\frac{1}{2}},\ N^{-1}(\ln N)^2,\ \epsilon N\right) \tag{3.15}$$

is sufficiently small (*see* (2.5)) *then*

$$|||\psi_u|||_\omega^2 \le C\left[\epsilon^2 H\|\xi_u''\|_{\Omega_{\mathrm{R}}}^2 + H\|a\xi_u'\|_{\Omega_{\mathrm{R}}}^2 + \epsilon\|\xi_u'\|_{\Omega_{\mathrm{R}}}^2 + H^{-1}\|\xi_u\|_{\Omega_{\mathrm{R}}}^2\right]. \tag{3.16}$$

Proof Since the proof is quite involved we proceed in several steps.

Step 5a: *An expression for* $\|\psi_u\|_\omega^2$.

Lemma 3.7 *Let* $E_u := \omega^2\psi_u - \mathbb{I}(\omega^2\psi_u)$ *and define* $\lambda_u \in \mathcal{P}^1(\Omega_N)$ *as*

$$\lambda_u(x) = \begin{cases} (\omega^2\psi_u)(x_N^-)\frac{x-x_{N-1}}{x_N-x_{N-1}}, & x \in (x_{N-1}, x_N), \\ 0, & \textit{otherwise}. \end{cases}$$

Then

$$\|\psi_u\|_\omega^2 = R_1 + \cdots + R_7 \tag{3.17}$$

where

$$\begin{aligned} R_1 &= B_N(\psi_u, E_u), & R_2 &= -B_N(\xi_u, \mathbb{I}(\omega^2\psi_u)), \\ R_3 &= B_N(\xi_u, \lambda_u), & R_4 &= B_N(\psi_u, \lambda_u), \\ R_5 &= -2\epsilon(\psi_u', \omega\omega'\psi_u), & R_6 &= -H(-\epsilon\psi_u'' + b\psi_u, a(\omega^2\psi_u)')_{\Omega_\mathrm{R}}, \\ R_7 &= -2H(a\psi_u', a\omega\omega'\psi_u)_{\Omega_\mathrm{R}}. \end{aligned}$$

Proof By the definition of $\|\cdot\|_\omega$-norm

$$\begin{aligned} \|\psi_u\|_\omega^2 &= \epsilon(\omega\psi_u', \omega\psi_u') + H(a\omega\psi_u', a\omega\psi_u')_{\Omega_\mathrm{R}} + ((b - a'/2)\omega\psi_u, \omega\psi_u) \\ &\quad - (a\omega\psi_u, \omega'\psi_u) + \frac{1}{2}(a\psi_u^2[\![\omega^2]\!])(x_N). \end{aligned} \tag{3.18}$$

By simple algebra, we have

$$(\omega\psi_u', \omega\psi_u') = (\psi_u', (\omega^2\psi_u)') - 2(\psi_u', \omega\omega'\psi_u). \tag{3.19}$$

Integrating by parts, and using the continuity of a, ψ_u, and that of ω except at x_N, and the fact that $\psi_u = 0$ on $\partial\Omega$, we get

$$-\frac{1}{2}(a'\omega\psi_u, \omega\psi_u) = -\frac{1}{2}(a\psi_u^2[\![\omega^2]\!])(x_N) + (a\psi_u, \omega\omega'\psi_u) + (a\psi_u', \omega^2\psi_u). \tag{3.20}$$

Using (3.19) and (3.20) in (3.18) we get

$$\begin{aligned} \|\psi_u\|_\omega^2 = \epsilon(\psi_u', (\omega^2\psi_u)') &- 2\epsilon(\psi_u', \omega\omega'\psi_u) \\ &+ H(a\psi_u', a(\omega^2\psi_u)')_{\Omega_\mathrm{R}} - 2H(a\psi_u', a\omega\omega'\psi_u)_{\Omega_\mathrm{R}} \\ &+ (b\psi_u, \omega^2\psi_u) + (a\psi_u', \omega^2\psi_u). \end{aligned}$$

Adding and subtracting the term $H(-\epsilon\psi_u'' + b\psi_u, a(\omega^2\psi_u)')_{\Omega_\mathrm{R}}$ we see that

$$\|\psi_u\|_\omega^2 = B_N(\psi_u, \omega^2\psi_u) + R_5 + R_6 + R_7.$$

Finally, by the Galerkin orthogonality property (2.9),

$$\begin{aligned} B_N(\psi_u, \omega^2\psi_u) &= B_N(\psi_u, E_u) + B_N(\psi_u, \mathbb{I}(\omega^2\psi_u) - \lambda_u) + B_N(\psi_u, \lambda_u) \\ &= B_N(\psi_u, E_u) + B_N(e_u - \xi_u, \mathbb{I}(\omega^2\psi_u) - \lambda_u) + B_N(\psi_u, \lambda_u) \end{aligned}$$

$$
\begin{aligned}
&= B_N(\psi_u, E_u) - B_N(\xi_u, \mathbb{I}(\omega^2\psi_u) - \lambda_u) + B_N(\psi_u, \lambda_u) \\
&= R_1 + R_2 + R_3 + R_4.
\end{aligned}
$$

This completes the proof. □

Step 5b: *An estimate for* R_1. We begin with proving the error estimates

$$
\begin{aligned}
\|\omega^{-1}E_u'\|_{\Omega_{\mathrm{R}}} &\le CK_2^{-\frac{1}{2}}N^{\frac{1}{2}}\|(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{\Omega_{\mathrm{R}}}, \\
\|\omega^{-1}E_u\|_{\Omega_{\mathrm{R}}} &\le CK_2^{-\frac{1}{2}}N^{-\frac{1}{2}}\|(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{\Omega_{\mathrm{R}}},
\end{aligned}
\tag{3.21}
$$

which will be frequently used in the sequel. Using standard approximation theory, (3.1), and (3.9), one obtains

$$
\begin{aligned}
\left\|E_u'\right\|_{I_j}^2 &= \|(\omega^2\psi_u - \mathbb{I}(\omega^2\psi_u))'\|_{I_j}^2 \\
&\le C\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} H^{2k}\left\|D^\alpha\omega D^\beta\omega D^\gamma\psi_u\right\|_{I_j}^2 \\
&\le C\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} H^{2k}\rho_{\mathrm{R}}^{-2(\alpha+\beta)+1}H^{-2\gamma}\|\omega(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{I_j}^2 \\
&\le C\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} K_2^{-2(\alpha+\beta)+1}N^{-2k+2(\gamma+\alpha+\beta)-1}\|\omega(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{I_j}^2 \\
&\le CK_2^{-1}N\|\omega(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{I_j}^2
\end{aligned}
$$

for all $I_j \in \Omega_{\mathrm{R}}$. For the second inequality of the above estimate, we used (3.9e), the inverse inequality (3.1), and (3.9e) once again. Therefore, it follows from (3.9e) that

$$
\begin{aligned}
\left\|\omega^{-1}E_u'\right\|_{\Omega_{\mathrm{R}}}^2 &= \sum_{j=1}^N \|\omega^{-1}(\omega^2\psi_u - \mathbb{I}(\omega^2\psi_u))'\|_{I_j}^2 \\
&\le \sum_{j=1}^N \frac{1}{\min\limits_{x\in I_j}\omega(x)}\|(\omega^2\psi_u - \mathbb{I}(\omega^2\psi_u))'\|_{I_j}^2 \\
&\le C\sum_{j=1}^N \frac{1}{\min\limits_{x\in I_j}\omega(x)}K_2^{-1}N\|\omega(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{I_j}^2 \\
&\le C\sum_{j=1}^N RO(I_j,\omega)K_2^{-1}N\|(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{I_j}^2 \\
&\le CK_2^{-1}N\|(\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{\Omega_{\mathrm{R}}}^2.
\end{aligned}
$$

This completes the proof of the first part of (3.21). The second estimate can be proved by a similar argument. We omit the details.

By the definition of B_N and Cauchy-Schwarz inequality

$$
R_1 \le \sum_{I\in\Omega_{\mathrm{R}}} \|\!|\psi_u|\!\|_I\left(\epsilon^{\frac{1}{2}}\|E_u'\|_I + H^{-\frac{1}{2}}\|E_u\|_I + H^{\frac{1}{2}}\|aE_u'\|_I\right).
$$

Multiplying and dividing by ω and using (3.9e) we get

$$R_1 \leq C |||\omega\psi_u||| \Big(\epsilon^{\frac{1}{2}} \|\omega^{-1} E_u'\|_{\Omega_R} + H^{-\frac{1}{2}} \|\omega^{-1} E_u\|_{\Omega_R} + H^{\frac{1}{2}} \|a\omega^{-1} E_u'\|_{\Omega_R} \Big). \tag{3.22}$$

Applying the estimates (3.21) to the right side of the above inequality we get

$$\begin{aligned} &\epsilon^{\frac{1}{2}} \|\omega^{-1} E_u'\|_{\Omega_R} + H^{-\frac{1}{2}} \|\omega^{-1} E_u\|_{\Omega_R} + H^{\frac{1}{2}} \|a\omega^{-1} E_u'\|_{\Omega_R} \\ &\quad \leq CL \|(a\omega|\omega'|)^{\frac{1}{2}} \psi_u\|_{\Omega_R} \leq CL |||\psi_u|||_{\omega}. \end{aligned} \tag{3.23}$$

It is easy to see that

$$|||\omega\psi_u||| \leq |||\psi_u|||_{\omega}. \tag{3.24}$$

Inserting, (3.23) and (3.24) into (3.22) we get

$$R_1 \leq CL |||\psi_u|||_{\omega}^2. \tag{3.25}$$

Step 5c: *An estimate for* R_2. By definition of the bilinear form B_N we have

$$R_2 = R_{2,a} + R_{2,b} + R_{2,c} + R_{2,d} \tag{3.26}$$

where

$$\begin{aligned} R_{2,a} &= -\epsilon(\xi_u', (\mathbb{I}(\omega^2\psi_u))'), \\ R_{2,b} &= -(a\xi_u', \mathbb{I}(\omega^2\psi_u)), \\ R_{2,c} &= -(b\xi_u, \mathbb{I}(\omega^2\psi_u)), \\ R_{2,d} &= H(\epsilon\xi_u'' - a\xi_u' - b\xi_u, a(\mathbb{I}(\omega^2\psi_u))')_{\Omega_R}. \end{aligned}$$

Since $\mathbb{I}(\omega^2\psi_u) = \omega^2\psi_u - E_u$, using Cauchy-Schwarz inequality and Young's inequality

$$xy \leq \frac{\mu}{2} x^2 + \frac{1}{2\mu} y^2 \tag{3.27}$$

for any $x, y \in \mathbb{R}$, and any $\mu > 0$, we get

$$\begin{aligned} R_{2,a} &\leq -\epsilon(\xi_u', (\omega^2\psi_u)') + \epsilon(\xi_u', E_u') \\ &\leq C\epsilon \|\omega\xi_u'\|_{\Omega_R}^2 + \epsilon \|a^{\frac{1}{2}} \omega' \psi_u\|_{\Omega_R}^2 + \frac{1}{16}\epsilon \|\omega\psi_u'\|_{\Omega_R}^2 + \epsilon \|\omega^{-1} E_u'\|_{\Omega_R}^2. \end{aligned}$$

Then we use (3.9) and (3.21) to simplify the above estimate as

$$\begin{aligned} R_{2,a} &\leq C\epsilon \|\omega\xi_u'\|_{\Omega_R}^2 + \epsilon \|\rho_R^{-\frac{1}{2}} (a\omega|\omega'|)^{\frac{1}{2}} \psi_u\|_{\Omega_R}^2 + \frac{1}{16}\epsilon \|\omega\psi_u'\|_{\Omega_R}^2 + \epsilon \|\omega^{-1} E_u'\|_{\Omega_R}^2 \\ &\leq C\epsilon \|\omega\xi_u'\|_{\Omega_R}^2 + CL \|(a\omega|\omega'|)^{\frac{1}{2}} \psi_u\|_{\Omega_R}^2 + \frac{1}{16}\epsilon \|\omega\psi_u'\|_{\Omega_R}^2. \end{aligned} \tag{3.28}$$

$R_{2,b}$ can be estimated in a similar way. Indeed, using integration by parts and some algebraic manipulations we get

$$\begin{aligned} R_{2,b} &= (\xi_u, a(\omega^2\psi_u - E_u)') + (\xi_u, a'(\omega^2\psi_u - E_u)) \\ &\le C\frac{1}{H}\|\omega\xi_u\|^2_{\Omega_\mathrm{R}} + \frac{H}{\rho_\mathrm{R}}\|(a\omega|\omega'|)^{\frac{1}{2}}\psi_u\|^2_{\Omega_\mathrm{R}} + \frac{1}{16}H\|a\omega\psi_u'\|^2_{\Omega_\mathrm{R}} + H\|a\omega^{-1}E_u'\|^2_{\Omega_\mathrm{R}} \\ &\quad + C\frac{1}{H}\|\omega\xi_u\|^2_{\Omega_\mathrm{R}} + H\|a'\omega\psi_u\|^2_{\Omega_\mathrm{R}} + H\|a'\omega^{-1}E_u\|^2_{\Omega_\mathrm{R}}. \end{aligned}$$

Using (3.9) and (3.21), we get

$$R_{2,b} \le C\frac{1}{H}\|\omega\xi_u\|^2_{\Omega_\mathrm{R}} + CL\|(a\omega|\omega'|)^{\frac{1}{2}}\psi_u\|^2_{\Omega_\mathrm{R}} + \frac{1}{16}H\|a\omega\psi_u'\|^2_{\Omega_\mathrm{R}}. \tag{3.29}$$

Similarly,

$$\begin{aligned} R_{2,c} &\le C\frac{1}{H}\|b\omega\xi_u\|^2_{\Omega_\mathrm{R}} + H\|\omega\psi_u\|^2_{\Omega_\mathrm{R}} + \|\omega^{-1}E_u\|^2_{\Omega_\mathrm{R}} \\ &\le C\frac{1}{H}\|\omega\xi_u\|^2_{\Omega_\mathrm{R}} + CL\|(a\omega|\omega'|)^{\frac{1}{2}}\psi_u\|_{\Omega_\mathrm{R}}, \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} R_{2,d} &\le C(H\epsilon^2\|\omega\xi_u''\|^2_{\Omega_\mathrm{R}} + H\|a\omega\xi_u'\|^2_{\Omega_\mathrm{R}} + H\|b\omega\xi_u\|^2_{\Omega_\mathrm{R}}) \\ &\quad + H\|a\omega^{-1}E_u'\|^2_{\Omega_\mathrm{R}} + \frac{H}{\rho_\mathrm{R}}\|(a\omega|\omega'|)^{\frac{1}{2}}\psi_u\|^2_{\Omega_\mathrm{R}} + \frac{1}{16}H\|a\omega\psi_u'\|^2_{\Omega_\mathrm{R}} \\ &\le C(H\epsilon^2\|\omega\xi_u''\|^2_{\Omega_\mathrm{R}} + H\|a\omega\xi_u'\|^2_{\Omega_\mathrm{R}} + H\|b\omega\xi_u\|^2_{\Omega_\mathrm{R}}) \\ &\quad + CL\|(a\omega|\omega'|)^{\frac{1}{2}}\psi_u\|^2_{\Omega_\mathrm{R}} + \frac{1}{16}H\|a\omega\psi_u'\|^2_{\Omega_\mathrm{R}}. \end{aligned} \tag{3.31}$$

Combining the estimates (3.28)–(3.31) with (3.26) we obtain

$$\begin{aligned} R_2 &\le C\left(H\epsilon^2\|\omega\xi_u''\|^2_{\Omega_\mathrm{R}} + H\|a\omega\xi_u'\|^2_{\Omega_\mathrm{R}} + \epsilon\|\omega\xi_u'\|^2_{\Omega_\mathrm{R}} + H^{-1}\|\omega\xi_u\|^2_{\Omega_\mathrm{R}}\right) \\ &\quad + CL|||\psi_u|||^2_\omega + \frac{1}{8}\left(\epsilon\|\omega\psi_u'\|^2_{\Omega_\mathrm{R}} + H\|a\omega\psi_u'\|^2_{\Omega_\mathrm{R}}\right). \end{aligned} \tag{3.32}$$

Step 5d: *An estimate for* R_3. A simple calculation shows that

$$\|\lambda_u\|_{I_N} \le \sqrt{\frac{H}{3}}|(\omega^2\psi_u)(x_N^-)|, \qquad \|\lambda_u'\|_{I_N} \le \sqrt{\frac{1}{H}}|(\omega^2\psi_u)(x_N^-)|. \tag{3.33}$$

Using the definition of B_N we get

$$\begin{aligned} R_3 &\le \epsilon\|\xi_u'\|_{I_N}\|\lambda_u'\|_{I_N} + \left(\|a\xi_u'\|_{I_N} + \|b\xi_u\|_{I_N}\right)\|\lambda_u\|_{I_N} \\ &\quad + H\| - \epsilon\xi_u'' + a\xi_u' + b\xi_u\|_{I_N}\|\lambda_u'\|_{I_N}. \end{aligned}$$

By (3.9), (3.33), and the assumptions $b \in L^\infty(\mathcal{I})$, we obtain

$$\begin{aligned} R_3 \le C\Big[&\epsilon\|\xi_u'\|_{I_N}H^{-\frac{1}{2}}|(\omega^2\psi_u)(x_N^-)| + \left(\|a\xi_u'\|_{I_N} + \|\xi_u\|_{I_N}\right)H^{\frac{1}{2}}|(\omega^2\psi_u)(x_N^-)| \\ &+ H\| - \epsilon\xi_u'' + a\xi_u' + b\xi_u\|_{I_N}H^{-\frac{1}{2}}|(\omega^2\psi_u)(x_N^-)|\Big]. \end{aligned}$$

Hence,

$$R_3 \le C\left[\epsilon^{\frac{1}{2}}\|\xi_u'\|_{I_N} + H^{\frac{1}{2}}\|a\xi_u'\|_{I_N} + H^{\frac{1}{2}}\epsilon\|\xi_u''\|_{I_N} + H^{\frac{1}{2}}\|\xi_u\|_{I_N}\right]|(\omega^2\psi_u)(x_N^-)|. \tag{3.34}$$

By (3.3), (3.4), and (3.8), we have

$$|(\omega^2\psi_u)(x_N^-)| \le C_w\omega(x_N^-)\left[\frac{1}{2}(a\psi_u^2[\![\omega^2]\!])(x_N)\right]^{\frac{1}{2}}, \tag{3.35}$$

where $C_w = [\frac{2\omega(x_N^-)}{\alpha[\![\omega^2]\!](x_N)}]^{\frac{1}{2}}$ is independent of ϵ and N. Using (3.35) and (3.3) in (3.34), we obtain

$$\begin{aligned} R_3 &\le C\omega(x_N^-)\left[\epsilon^{\frac{1}{2}}\|\xi_u'\|_{I_N} + H^{\frac{1}{2}}\|a\xi_u'\|_{I_N} + H^{\frac{1}{2}}\epsilon\|\xi_u''\|_{I_N} + H^{\frac{1}{2}}\|\xi_u\|_{I_N}\right] \\ &\quad\times\left[\frac{1}{2}(a\psi_u^2[\![\omega^2]\!])(x_N)\right]^{\frac{1}{2}} \\ &\le C\left[\epsilon\|\omega\xi_u'\|_{I_N}^2 + H\|a\omega\xi_u'\|_{I_N}^2 + H\epsilon^2\|\omega\xi_u''\|_{I_N}^2 + H\|\omega\xi_u\|_{I_N}^2\right] \\ &\quad+\frac{1}{8}(a\psi_u^2[\![\omega^2]\!])(x_N), \end{aligned} \tag{3.36}$$

where C depends on a, b, and C_w.

Step 5e: *An estimate for* R_4. By (3.9b), (3.9e), (2.4), and (3.33), we have

$$\begin{aligned} R_4 &\le \epsilon\|\psi_u'\|_{I_N}\|\lambda_u'\|_{I_N} + \left(\|a\psi_u'\|_{I_N} + \|b\psi_u\|_{I_N}\right)\|\lambda_u\|_{I_N} \\ &\quad + H\|-\epsilon\psi_u'' + a\psi_u' + b\psi_u\|_{I_N}\|a\lambda_u'\|_{I_N} \\ &\le \epsilon\|\psi_u'\|_{I_N}H^{-\frac{1}{2}}\left|(\omega^2\psi_u)(x_N^-)\right| + \left(\|a\psi_u'\|_{I_N} + \|b\psi_u\|_{I_N}\right)\sqrt{\frac{H}{3}}\left|(\omega^2\psi_u)(x_N^-)\right| \\ &\quad + H\left[\epsilon\|\psi_u''\|_{I_N} + \|a\psi_u'\|_{I_N} + \|b\psi_u\|_{I_N}\right]\|a\|_{I_N}H^{-\frac{1}{2}}|(\omega^2\psi_u)(x_N^-)| \\ &\le \left(\epsilon H^{-\frac{1}{2}}\|\psi_u'\|_{I_N} + \sqrt{\frac{H}{3}}\|b\psi_u\|_{I_N} + \epsilon H^{\frac{1}{2}}\|\psi_u''\|_{I_N}\right)\left|(\omega^2\psi_u)(x_N^-)\right| \\ &\quad + H^{\frac{1}{2}}\|a\psi_u'\|_{I_N}\left(\sqrt{\frac{1}{3}} + \|a\|_{I_N}\right)|(\omega^2\psi_u)(x_N^-)|. \end{aligned}$$

Using (3.35) and inverse inequality we obtain

$$\begin{aligned} &\left(\epsilon H^{-\frac{1}{2}}\|\psi_u'\|_{I_N} + \sqrt{\frac{H}{3}}\|b\psi_u\|_{I_N} + \epsilon H^{\frac{1}{2}}\|\psi_u''\|_{I_N}\right)\left|(\omega^2\psi_u)(x_N^-)\right| \\ &\quad\le C_w\left(\sqrt{\frac{\epsilon}{H}} + \sqrt{\frac{H}{3}}\frac{C\ln N}{\alpha}\right)|\!|\!|\omega\psi_u|\!|\!|\left[\frac{1}{2}(a\psi_u^2[\![\omega^2]\!])(x_N)\right]^{\frac{1}{2}} \\ &\quad\le CL|\!|\!|\psi_u|\!|\!|_\omega^2 + \frac{1}{8}(a\psi_u^2[\![\omega^2]\!])(x_N). \end{aligned}$$

Meanwhile, it follows from (3.10) and assumption $\|a\|_{W^1_\infty} \le C$ that

$$
\begin{aligned}
&H^{\frac{1}{2}} \|a\psi_u'\|_{I_N} \left(\sqrt{\frac{1}{3}} + \|a\|_{I_N}\right) |(\omega^2\psi_u)(x_N^-)| \\
&\le C_w \left(\sqrt{\frac{1}{3}} + \|a\|_{I_N}\right) \sqrt{C_3} C(k) H^{\frac{1}{2}} \|a\omega\psi_u'\|_{I_N} \left[\frac{1}{2}(a\psi_u^2 [\![\omega^2]\!])(x_N)\right]^{\frac{1}{2}} \\
&\le \frac{1}{4} H \|a\omega\psi_u'\|_{I_N}^2 + \frac{1}{8}(a\psi_u^2 [\![\omega^2]\!])(x_N),
\end{aligned}
$$

where we take sufficiently small C_3 such that $[C_w(\sqrt{\frac{1}{3}} + \|a\|_{I_N})\sqrt{C_3}C(k)]^2 \le \frac{1}{4}$. Combining the above two inequalities we have

$$
R_4 \le CL \|\!|\psi_u|\!\|_\omega^2 + \frac{1}{4} H \|a\omega\psi_u'\|_{I_N}^2 + \frac{1}{4}(a\psi_u^2 [\![\omega^2]\!])(x_N). \tag{3.37}
$$

Step 5f: *An estimate for* R_5. By (3.9) and (3.27)

$$
R_5 \le \frac{\epsilon}{16} \|\omega\psi_u'\|_{\Omega_R}^2 + CL \|(a\omega|\omega'|)^{\frac{1}{2}} \psi_u\|_{\Omega_R}^2. \tag{3.38}
$$

Step 5g: *An estimate for* R_6. Using (2.4), (3.9), (3.1), and (3.27)

$$
\begin{aligned}
R_6 &\le CH^{-1}\epsilon^2 \|\omega\psi_u'\|_{\Omega_R}^2 + CH \|b\omega\psi_u\|_{\Omega_R}^2 + \frac{H}{\rho_R} \|(a\omega|\omega'|)^{\frac{1}{2}} \psi_u\|_{\Omega_R}^2 + \frac{1}{16} H \|a\omega\psi_u'\|_{\Omega_R}^2 \\
&\le CL \|\!|\psi_u|\!\|_\omega^2 + \frac{1}{16} H \|a\omega\psi_u'\|_{\Omega_R}^2.
\end{aligned} \tag{3.39}
$$

Step 5h: *An estimate for* R_7. By (3.9) and (3.27)

$$
R_7 \le \frac{1}{16} H \|a\omega\psi_u'\|_{\Omega_R}^2 + CL \|(a\omega|\omega'|)^{\frac{1}{2}} \psi_u\|_{\Omega_R}^2. \tag{3.40}
$$

Step 5i: *Combining the estimates for* R_1*–*R_7. Inserting the estimates (3.25), (3.32), (3.36), and (3.37)–(3.40) into (3.17), recalling the definition, (3.11), of $\|\!|\psi_u|\!\|_\omega^2$, and carrying out some simple algebra we obtain

$$
\begin{aligned}
\|\!|\psi_u|\!\|_\omega^2 \le\, & C\left(\epsilon^2 H \|\omega\xi_u''\|_{\Omega_R}^2 + H \|a\omega\xi_u'\|_{\Omega_R}^2 + \epsilon \|\omega\xi_u'\|_{\Omega_R}^2 + H^{-1} \|\omega\xi_u\|_{\Omega_R}^2\right) \\
& + \left(CL + \frac{1}{2}\right) \|\!|\psi_u|\!\|_\omega^2.
\end{aligned}
$$

Let us note that $w(x) \le C$ for any $x \in \Omega_R$. Assuming that L is small enough such that $CL \le \frac{1}{3}$ completes the proof of (3.16). □

Step 6: *Proof of Theorem 2.3*. It follows from the properties of ω that

$$
\|(b - a'/2)^{\frac{1}{2}} \psi_u\|_{\Omega_R} \le \omega_R \|(b - a'/2)^{\frac{1}{2}} \omega\psi_u\|_{\Omega_R} \le \omega_R \|\!|\psi_u|\!\|_\omega, \tag{3.41}
$$

and that

$$
\sqrt{\epsilon} \|\psi_u'\|_{\Omega_R} + \sqrt{H} \|a\psi_u'\|_{\Omega_R} \le \omega_R(\sqrt{\epsilon} \|\omega\psi_u'\|_{\Omega_R} + \sqrt{H} \|a\omega\psi_u'\|_{\Omega_R}) \le C\omega_R \|\!|\psi_u|\!\|_\omega, \tag{3.42}
$$

where

$$\omega_{\mathrm{R}} := \max_{x\in\Omega_{\mathrm{R}}} \frac{1}{\omega(x)} = \frac{1}{\omega(x_N^-)}$$

by (3.4). The definition of $\omega(x)$, (3.3), and (2.4c) yield

$$\omega_{\mathrm{R}} = C_3^{-1} e^{\frac{1-\tau-A}{\rho_{\mathrm{R}}}} = C_3^{-1} e^{(k+1)\epsilon N \ln \epsilon^{-1}} \leq C_3^{-1} e^{\frac{1}{3K_2}},$$

which is a constant independent of ϵ and N. Combining (3.41) and (3.42) we get

$$|||\psi_u|||_{\Omega_{\mathrm{R}}} \leq C|||\psi_u|||_{\omega}. \tag{3.43}$$

By Lemma 3.5, Lemma 3.3, and Lemma 3.6, we have

$$\begin{aligned} |||\psi_u|||_{\omega} &\leq C(\epsilon^2 H\|\xi_u''\|_{\Omega_{\mathrm{R}}}^2 + H\|a\xi_u'\|_{\Omega_{\mathrm{R}}}^2 + \epsilon\|\xi_u'\|_{\Omega_{\mathrm{R}}}^2 + H^{-1}\|\xi_u\|_{\Omega_{\mathrm{R}}}^2)^{\frac{1}{2}} \\ &\leq CN^{-(k+\frac{1}{2})}. \end{aligned} \tag{3.44}$$

Inserting (3.44) into (3.43), we obtain (2.11a).

If $u_{\mathrm{R}} \in V_N$ and $K_1 \geq 2k + \frac{1}{2}$ then the error bound in (3.44) will be changed by using (3.14) as

$$|||\psi_u|||_{\omega} \leq C\frac{\epsilon^k}{N^{k+\frac{1}{2}}}.$$

This completes the proof of Theorem 2.3.

3.3 Proof of Theorem 2.5

We proceed in several steps. In Step 1, we introduce the Green's function associated with the boundary value problem (1.1) and state its regularity and approximation properties. In Step 2, we prove an estimate for $|||e_u|||$. Step 3 provides an explicit expression for the nodal error in terms of the Green's function. Each term of this expression is then estimated in Steps 4 and 5. Finally, these estimates are put together in Step 6 to complete the proof of Theorem 2.5.

Step 1: *The Green's function.* Let $y \in \Omega$ be an arbitrary point. The Green's function, $\phi_y(x)$, associated with the boundary value problem (1.1) and the point y is defined as the solution of

$$-\epsilon\phi_y'' - (a\phi_y)' + b\phi_y = 0 \quad \text{in } \Omega\setminus\{y\}, \tag{3.45a}$$

$$[\![\phi_y]\!](y) = 0, \tag{3.45b}$$

$$\epsilon[\![\phi_y']\!](y) = 1, \tag{3.45c}$$

$$\phi_y = 0 \quad \text{on } \partial\Omega. \tag{3.45d}$$

The following lemma contains the regularity estimates for ϕ_y and its derivatives.

Lemma 3.8 *There exists a constant C such that*

$$|\phi_y^{(s)}(x)| \leq C(1+\epsilon^{-s}e^{-\alpha x/\epsilon}), \qquad \textit{for all } x \in (0, y), \tag{3.46a}$$

$$|\phi_y^{(s)}(x)| \leq C(1+\epsilon^{-s}e^{-\alpha(x-y)/\epsilon}), \quad \textit{for all } x \in (y, 1), \tag{3.46b}$$

for any $s = 0, 1, 2, \ldots$.

The proof of this lemma will be omitted since it can be proved by standard techniques. We refer to Theorem 2 in Chap. 8 of [11] for details.

A simple computation using Lemma 3.8 shows that there exists a constant C such that

$$\|\phi_y\|_{r+1,\Omega_N} \le C\epsilon^{-r-1/2} \quad \text{for all } r \ge 0. \tag{3.47}$$

This estimate together with the approximation properties, (3.12), of the interpolation operator $\mathbb{I}$ yields the following lemma.

Lemma 3.9 *There exists a constant C such that*

$$\|\phi_{x_j}\|_{\Omega_R} + \|\mathbb{I}\phi_{x_j}\|_{\Omega_R} \le C, \tag{3.48a}$$

$$\|\phi_{x_j}'\|_{\Omega_R} + \|(\mathbb{I}\phi_{x_j})'\|_{\Omega_R} \le C(\epsilon^{-\frac{1}{2}} + H^{-1}), \tag{3.48b}$$

$$\|(\phi_{x_j} - \mathbb{I}\phi_{x_j})^{(s)}\|_{\Omega_{BL}} \le Ch^{k+1-s}\epsilon^{-k-1/2}, \tag{3.48c}$$

for $s = 0, 1$ and $j = 1, 2, \dots, 2N-1$.

Proof Using (3.46) we can bound $\|\phi_{x_j}'\|_{\Omega_R}$ by $C\epsilon^{-\frac{1}{2}}$. $\|(\mathbb{I}\phi_{x_j})'\|_{\Omega_R}$ can be bounded by CH^{-1} when inverse inequality (3.1) and (3.48a) are used. □

Step 2: An estimate for $|||e_u|||$.

Theorem 3.10 *There exists a constant C such that*

$$|||e_u||| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k. \tag{3.49}$$

Proof Since $e_u = \xi_u + \psi_u$, by triangle inequality

$$|||e_u||| \le |||\xi_u||| + |||\psi_u|||. \tag{3.50}$$

We begin with bounding $|||\xi_u|||$. By the definition, (2.6), of the $|||\cdot|||$-norm and δ, (2.3), we have

$$|||\xi_u|||^2 = \epsilon\|\xi_u'\|^2 + H\|a\xi_u'\|_{\Omega_R}^2 + \|(b - a'/2)^{\frac{1}{2}}\xi_u\|^2 \tag{3.51}$$

By the approximation results (3.13) of Lemma 3.5 we have

$$\epsilon\|\xi_u'\|^2 + H\|a\xi_u'\|_{\Omega_R}^2 + \|(b - a'/2)^{\frac{1}{2}}\xi_u\|^2 \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{2k},$$

where we have used the boundedness assumptions on b and a'. Inserting these estimates into (3.51) we get

$$|||\xi_u||| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k \tag{3.52}$$

Next, we obtain an estimate for $|||\psi_u|||$. By the coercivity, (2.8), and the Galerkin orthogonality, (2.9), and the definition, (2.2), of the bilinear from B_N, we have

$$\begin{aligned}\frac{1}{2}|||\psi_u|||^2 &\le B_N(\psi_u, \psi_u)\\ &= B_N(-\xi_u, \psi_u)\end{aligned}$$

$$
\begin{aligned}
&= -(\epsilon\xi_u', \psi_u') - (a\xi_u', \psi_u) - (b\xi_u, \psi_u) - (-\epsilon\xi_u'' + a\xi_u' + b\xi_u, \delta a\psi_u') \\
&= -\epsilon(\xi_u', \psi_u') + (\xi_u, a\psi_u') - ((b-a')\xi_u, \psi_u) \\
&\quad - (-\epsilon\xi_u'' + a\xi_u' + b\xi_u, \delta a\psi_u') \qquad (3.53)
\end{aligned}
$$

where we have used integration by parts on the term $(a\xi_u', \psi_u)$ in the last step. Next, we estimate each one of the terms appearing above. Using the approximation results (3.13) of Lemma 3.5, definition of the $|||\cdot|||$-norm, and the assumptions (2.5) we get

$$
\begin{aligned}
|\epsilon(\xi_u', \psi_u')| &\le \sqrt{\epsilon}\|\xi_u'\|\sqrt{\epsilon}\|\psi_u'\| \\
&\le \sqrt{\epsilon}(\|\xi_u'\|_{\Omega_{\mathrm{R}}} + \|\xi_u'\|_{\Omega_{\mathrm{BL}}})|||\psi_u||| \\
&\le C\left(\frac{1}{N^{k+1/2}} + \left(\frac{\ln\epsilon^{-1}}{N}\right)^k\right)|||\psi_u||| \\
&\le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k |||\psi_u|||.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|(\xi_u, a\psi_u')| &\le |(\xi_u, a\psi_u')_{\Omega_{\mathrm{R}}}| + |(\xi_u, a\psi_u')_{\Omega_{\mathrm{BL}}}| \\
&\le H^{-1/2}\|\xi_u\|_{\Omega_{\mathrm{R}}} H^{1/2}\|a\psi_u'\|_{\Omega_{\mathrm{R}}} + C\epsilon^{-1/2}\|\xi_u\|_{\Omega_{\mathrm{BL}}}\epsilon^{1/2}\|\psi_u'\|_{\Omega_{\mathrm{BL}}} \\
&\le CH^{-1/2}\frac{1}{N^{k+1}}|||\psi_u||| + C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{k+1}|||\psi_u||| \\
&\le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k |||\psi_u|||.
\end{aligned}
$$

Following similar steps, by the boundedness of b and a' we get

$$
\begin{aligned}
|((b-a')\xi_u, \psi_u)| &\le \|b-a'\|_\infty |((b-a'/2)^{1/2}/c_0^{1/2}\xi_u, \psi_u)| \\
&\le C\|\xi_u\|\|(b-a'/2)^{1/2}\psi_u\| \\
&\le C\|\xi_u\||||\psi_u||| \\
&\le C\sqrt{\epsilon}\left(\frac{\ln\epsilon^{-1}}{N}\right)^{k+1}|||\psi_u||| \\
&\le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k |||\psi_u|||.
\end{aligned}
$$

The above computation is for the case b is not identically zero so that the assumption (2.7b) holds. On the other hand, if b is identically zero this term vanishes by the assumption (2.7a).

The last term is estimated as follows

$$
\begin{aligned}
|(-\epsilon\xi_u'' + a\xi_u' + b\xi_u, \delta a\psi_u')| &\le |\sqrt{H}(-\epsilon\xi_u'' + a\xi_u' + b\xi_u, \sqrt{H}a\psi_u')_{\Omega_{\mathrm{R}}}| \\
&\le \sqrt{H}(\epsilon\|\xi_u''\|_{\Omega_{\mathrm{R}}} + \|a\xi_u'\|_{\Omega_{\mathrm{R}}} + \|b\xi_u\|_{\Omega_{\mathrm{R}}})\sqrt{H}\|a\psi_u'\|_{\Omega_{\mathrm{R}}}
\end{aligned}
$$

$$\leq C\left(\frac{\epsilon\sqrt{H}}{N^{k-1}}+\frac{1}{N^{k-1/2}}+\sqrt{\epsilon}\left(\frac{\ln\epsilon^{-1}}{N}\right)^{k+1}\right)|||\psi_u|||$$
$$\leq C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{k}|||\psi_u|||.$$

Collecting these estimates in (3.53) and canceling $|||\psi_u|||$ on both sides we get

$$|||\psi_u||| \leq C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{k}. \tag{3.54}$$

The proof is completed once we combine (3.52), (3.54), and (3.50). □

Step 3: *An expression for the nodal error.*

Lemma 3.11 *Let $x_j \in \mathcal{E}_N$ be an arbitrary node. Then*

$$e_u(x_j) = T_1 + T_2 \tag{3.55}$$

where

$$T_1 = B_N(e_u, \phi_{x_j} - \mathbb{I}\phi_{x_j}),$$
$$T_2 = H(\epsilon e_u'' - a e_u' - b e_u, a\phi_{x_j}')_{\Omega_R}.$$

Proof Multiplying both sides of (3.45a) by e_u and integrating by parts we get

$$e_u(x_j) = (\epsilon e_u', \phi_{x_j}') + (a e_u', \phi_{x_j}) + (b e_u, \phi_{x_j})$$

where we have used the jump properties, (3.45b)–(3.45c), of ϕ_{x_j}. Recalling the definition, (2.2), of the bilinear form B_N and the particular choice, (2.3), of the artificial diffusion parameter δ we get that

$$e_u(x_j) = B_N(e_u, \phi_{x_j}) + T_2.$$

The result follows from the Galerkin orthogonality property, (2.9). □

Step 4: *An estimate for T_1*. If we set

$$T_{1,a} = (\epsilon e_u', (\phi_{x_j} - \mathbb{I}\phi_{x_j})'),$$
$$T_{1,b} = (a e_u', \phi_{x_j} - \mathbb{I}\phi_{x_j}),$$
$$T_{1,c} = (b e_u, \phi_{x_j} - \mathbb{I}\phi_{x_j}),$$
$$T_{1,d} = H(-\epsilon e_u'' + a e_u' + b e_u, a(\phi_{x_j} - \mathbb{I}\phi_{x_j})')_{\Omega_R},$$

then $T_1 = T_{1,a} + T_{1,b} + T_{1,c} + T_{1,d}$. By triangle and Cauchy-Schwarz inequalities

$$|T_{1,a}| \leq \sqrt{\epsilon}\|e_u'\|_{\Omega_R}\sqrt{\epsilon}\|(\phi_{x_j} - \mathbb{I}\phi_{x_j})'\|_{\Omega_R} + \sqrt{\epsilon}\|e_u'\|_{\Omega_{BL}}\sqrt{\epsilon}\|(\phi_{x_j} - \mathbb{I}\phi_{x_j})'\|_{\Omega_{BL}}. \tag{3.56}$$

By the local estimate (2.11b) and the estimate (3.48b) we get

$$\begin{aligned} \sqrt{\epsilon}\|e_u'\|_{\Omega_{\mathrm{R}}} &\le |||e_u|||_{\Omega_{\mathrm{R}}} \le C\frac{\epsilon^k}{N^{k+\frac{1}{2}}}, \\ \sqrt{\epsilon}\|(\phi_{x_j} - \mathbb{I}\phi_{x_j})'\|_{\Omega_{\mathrm{R}}} &\le C(1+\epsilon^{\frac{1}{2}}N). \end{aligned} \tag{3.57}$$

By Theorem 3.10, the estimate (3.48c), and the definitions of τ and h we have

$$\begin{aligned} \sqrt{\epsilon}\|e_u'\|_{\Omega_{\mathrm{BL}}} &\le |||e_u||| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k, \\ \sqrt{\epsilon}\|(\phi_{x_j} - \mathbb{I}\phi_{x_j})'\|_{\Omega_{\mathrm{BL}}} &\le C\sqrt{\epsilon}h^k\epsilon^{-k-1/2} \le C\left(\frac{K_1\epsilon\ln\epsilon^{-1}}{N}\right)^k\epsilon^{-k}. \end{aligned} \tag{3.58}$$

Inserting the estimates (3.57) and (3.58) into (3.56) we get

$$|T_{1,a}| \le C\left[\frac{1}{N^{2k}} + \left(\frac{\ln\epsilon^{-1}}{N}\right)^{2k}\right] \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{2k} \tag{3.59}$$

since $N < \epsilon^{-1}$ by assumption (2.5).

To estimate $T_{1,b}$ we begin with

$$|T_{1,b}| \le H^{\frac{1}{2}}\|ae_u'\|_{\Omega_{\mathrm{R}}}H^{-\frac{1}{2}}\|\phi_{x_j} - \mathbb{I}\phi_{x_j}\|_{\Omega_{\mathrm{R}}} + \epsilon^{\frac{1}{2}}\|ae_u'\|_{\Omega_{\mathrm{BL}}}\epsilon^{-\frac{1}{2}}\|\phi_{x_j} - \mathbb{I}\phi_{x_j}\|_{\Omega_{\mathrm{BL}}}. \tag{3.60}$$

The local estimate (2.11b) and (3.48a) yield

$$\begin{aligned} H^{\frac{1}{2}}\|ae_u'\|_{\Omega_{\mathrm{R}}} &\le |||e_u|||_{\Omega_{\mathrm{R}}} \le C\frac{\epsilon^k}{N^{k+\frac{1}{2}}}, \\ H^{-\frac{1}{2}}\|\phi_{x_j} - \mathbb{I}\phi_{x_j}\|_{\Omega_{\mathrm{R}}} &\le H^{-\frac{1}{2}}. \end{aligned} \tag{3.61}$$

By Theorem 3.10 and (3.48c) we get

$$\begin{aligned} \epsilon^{\frac{1}{2}}\|ae_u'\|_{\Omega_{\mathrm{BL}}} &\le |||e_u||| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^k, \\ \epsilon^{-\frac{1}{2}}\|\phi_{x_j} - \mathbb{I}\phi_{x_j}\|_{\Omega_{\mathrm{BL}}} &\le Ch^{k+1}\epsilon^{-k-1} \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{k+1}. \end{aligned} \tag{3.62}$$

Inserting the estimates (3.61) and (3.62) into (3.60) we get

$$|T_{1,b}| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{2k}. \tag{3.63}$$

By a similar argument we obtain

$$|T_{1,c}| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{2k}. \tag{3.64}$$

Consider $T_{1,d}$. Using (3.14), (2.11b), and (3.1) we estimate e_u'' on $\Omega_{\rm R}$ as

$$\begin{aligned}\|e_u''\|_{\Omega_{\rm R}} &\le \|\xi_u''\|_{\Omega_{\rm R}} + \|\psi_u''\|_{\Omega_{\rm R}}\\ &\le \frac{\epsilon^k}{N^{k-1}} + \frac{1}{H}\frac{1}{\sqrt{H}}\sqrt{H}\|\psi_u'\|_{\Omega_{\rm R}}\\ &\le C\left(\frac{\epsilon^k}{N^{k-1}} + N^{\frac{3}{2}}\frac{\epsilon^k}{N^{k+\frac{1}{2}}}\right)\\ &\le C\frac{\epsilon^k}{N^{k-1}}. \end{aligned} \tag{3.65}$$

By (2.11b)

$$H(\|ae_u'\|_{\Omega_{\rm R}} + \|be_u\|_{\Omega_{\rm R}}) \le C|||e_u|||_{\Omega_{\rm R}} \le C\frac{\epsilon^k}{N^{k+1}}. \tag{3.66}$$

By (3.65)–(3.66) and (3.57) we get

$$\begin{aligned}|T_{1,d}| &\le H(\epsilon\|e_u''\|_{\Omega_{\rm R}} + \|ae_u'\|_{\Omega_{\rm R}} + \|be_u\|_{\Omega_{\rm R}})\|(\phi_{x_j} - \mathbb{I}\phi_{x_j})'\|_{\Omega_{\rm R}}\\ &\le \left[\frac{\epsilon^{k+1}}{N^k} + \frac{\epsilon^{k+\frac{3}{2}}}{N^k}\left(\ln\epsilon^{-1}\right)^k + \frac{\epsilon^k}{N^{k+1}}\right](\epsilon^{-\frac{1}{2}} + N)\\ &\le C\frac{(\ln\epsilon^{-1})^k}{N^{2k}}. \end{aligned} \tag{3.67}$$

Collecting the estimates (3.59), (3.63), (3.64), and (3.67) we get

$$|T_1| \le C\left(\frac{\ln\epsilon^{-1}}{N}\right)^{2k}. \tag{3.68}$$

Step 5: *An estimate for* T_2. Proceeding as in the estimate of $T_{1,d}$ above and invoking (3.47) with $r = 0$ we get

$$\begin{aligned}|T_2| &\le CH(\epsilon\|e_u''\|_{\Omega_{\rm R}} + \|e_u'\|_{\Omega_{\rm R}} + \|be_u\|_{\Omega_{\rm R}})\|\phi_{x_j}'\|_{\Omega_{\rm R}}\\ &\le C\frac{\epsilon^k}{N^{k+1}}\epsilon^{-\frac{1}{2}} \le C\frac{1}{N^{2k+\frac{1}{2}}}. \end{aligned} \tag{3.69}$$

Step 6: *Proof of the estimate* (*2.12*). This follows trivially from (3.55) and the estimates (3.68) and (3.69).

This completes the proof of Theorem 2.5.

3.4 Proof of Theorem 2.13

Since the proof is very similar to that of Theorem 2.5, we only point out the main ingredients that need to be modified in Sect. 3.2.

The first main ingredient is a new interpolation operator $\widetilde{\mathbb{I}} : H^1(\Omega) \to V_N$ which gives a zero interpolant on $\Omega_{\rm R}$ and coincides with $\mathbb{I}$ on $\Omega_{\rm BL} \setminus I_{N+1}$. The interpolation of $\varphi \in H^1(\Omega)$ is defined on the element I_{N+1} as

$$\widetilde{\mathbb{I}}\varphi = \ell_\varphi + \mathbb{I}\varphi$$

where

$$\ell_\varphi(x) = -\varphi(x_N)\frac{x_{N+1}-x}{h} \quad \text{for } x \in I_{N+1}.$$

This definition is similar to that of the interpolation operator in Sect. 3 of [21]. Using assumption (2.14) and (2.10) we can prove the following bounds for $\ell_{u_{\rm BL}}$:

$$\begin{aligned}\|\ell_{u_{\rm BL}}\|^2_{I_{N+1}} &= u^2_{\rm BL}(x_N)\frac{1}{h^2}\int_{x_N}^{x_{N+1}}(x-x_{N+1})^2dx \\ &= \frac{h}{3}u^2_{\rm BL}(x_N) \le Che^{-2\alpha\tau/\epsilon} \le ChN^{-2K_1} \le C\frac{\epsilon^2 \ln N}{N^{4k+1}},\end{aligned}$$

and

$$\|\ell'_{u_{\rm BL}}\|^2_{I_{N+1}} = u^2_{\rm BL}(x_N)\frac{1}{h^2}\int_{x_N}^{x_{N+1}}dx \le Ch^{-1}e^{-2\alpha\tau/\epsilon} \le C\frac{1}{N^{4k-1}\ln N},$$

Taking square root yields

$$\|\ell_{u_{\rm BL}}\|_{I_{N+1}} \le C\frac{\epsilon\sqrt{\ln N}}{N^{2k+\frac{1}{2}}}, \quad \text{and} \quad \|\ell'_{u_{\rm BL}}\|_{I_{N+1}} \le C\frac{1}{\sqrt{\ln N}N^{2k-\frac{1}{2}}}. \tag{3.70}$$

Using these estimates, the decomposition $u = u_{\rm R} + u_{\rm BL}$, and denoting

$$\widetilde{\xi}_u = (u_{\rm R} - \mathbb{I}u_{\rm R}) + (u_{\rm BL} - \widetilde{\mathbb{I}}u_{\rm BL}), \tag{3.71}$$

we can prove the following variation of Lemma 3.5.

Lemma 3.12 *Under the assumption* (2.14), *there exists a constant* C *such that*

$$\begin{aligned}\|\widetilde{\xi}_u\|_{\Omega_{\rm R}} &\le \frac{C}{N^{k+1}}, \qquad \sqrt{\epsilon}\|\widetilde{\xi}'_u\|_{\Omega_{\rm R}} + \sqrt{H}\|a\widetilde{\xi}'_u\|_{\Omega_{\rm R}} \le \frac{C}{N^{k+1/2}}, \\ \|\widetilde{\xi}''_u\|_{\Omega_{\rm R}} &\le C\left(\frac{1}{N^{k-1}} + \frac{1}{\epsilon N^{2k}}\right), \\ \|\widetilde{\xi}_u\|_{\Omega_{\rm BL}} &\le C\sqrt{\epsilon}\left(\frac{\ln N}{N}\right)^{k+1}, \qquad \sqrt{\epsilon}\|\widetilde{\xi}'_u\|_{\Omega_{\rm BL}} \le C\left(\frac{\ln N}{N}\right)^{k}.\end{aligned} \tag{3.72}$$

Furthermore, if $u_{\rm R} \in V_N$, *we have* $\widetilde{\xi}_u = u_{\rm BL} - \widetilde{\mathbb{I}}u_{\rm BL} = u_{\rm BL}$ *on* $\Omega_{\rm R}$ *and*

$$\begin{aligned}\|\widetilde{\xi}_u\|_{\Omega_{\rm R}} &\le C\frac{\epsilon}{N^{2k}}, \qquad \sqrt{\epsilon}\|\widetilde{\xi}'_u\|_{\Omega_{\rm R}} \le C\frac{\sqrt{\epsilon}}{N^{2k}}, \\ \sqrt{H}\|a\widetilde{\xi}'_u\|_{\Omega_{\rm R}} &\le C\frac{1}{N^{2k+\frac{1}{2}}}, \qquad \|\widetilde{\xi}''_u\|_{\Omega_{\rm R}} \le C\frac{1}{\epsilon N^{2k}}.\end{aligned} \tag{3.73}$$

Proof (1) We first prove the interpolation error bounds when $u_{\rm R} \in V_N$. Notice that $\widetilde{\xi}_u = u_{\rm BL} - \widetilde{\mathbb{I}}u_{\rm BL} = u_{\rm BL}$ on $\Omega_{\rm R}$. Inequalities (3.73) can be carried out by direct calculations based on (2.10) and (2.14). For instance,

$$\|u_{\rm BL}\|_{\Omega_{\rm R}} \le C\left(\int_0^{1-\tau} e^{-2\alpha(1-x)/\epsilon}dx\right)^{\frac{1}{2}} \le C\sqrt{\epsilon}e^{-\alpha\tau/\epsilon} \le C\sqrt{\epsilon}N^{-K_1} \le C\epsilon N^{-2k}.$$

(2) Recall the interpolation operator (3.71). The error estimates of $\xi_{u_\mathrm{R}} = u_\mathrm{R} - \mathbb{I}u_\mathrm{R}$ follows the standard approximation theory and (2.10)

$$\|\xi_{u_\mathrm{R}}\| \leq \frac{C}{N^{k+1}}, \qquad \sqrt{\epsilon}\|\xi'_{u_\mathrm{R}}\| + \sqrt{H}\|a\xi'_{u_\mathrm{R}}\| \leq \frac{C}{N^{k+1/2}}, \qquad \|\xi''_{u_\mathrm{R}}\| \leq \frac{C}{N^{k-1}}. \tag{3.74}$$

Since the estimates of $u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL}$ on Ω_R is already shown in (3.73), to prove (3.72) only requires error estimates of $u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL}$ on Ω_BL. Because of the definition of operator $\widetilde{\mathbb{I}}$ on Ω_BL, we have

$$\|u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL}\|_{\Omega_\mathrm{BL}} \leq \|u_\mathrm{BL} - \mathbb{I}u_\mathrm{BL}\|_{\Omega_\mathrm{BL}} + \|\ell_{u_\mathrm{BL}}\|_{I_{N+1}},$$

and

$$\sqrt{\epsilon}\|(u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL})'\|_{\Omega_\mathrm{BL}} \leq \sqrt{\epsilon}\|(u_\mathrm{BL} - \mathbb{I}u_\mathrm{BL})'\|_{\Omega_\mathrm{BL}} + \sqrt{\epsilon}\|\ell'_{u_\mathrm{BL}}\|_{I_{N+1}}.$$

Then, it follows standard approximation theory, (2.10), (2.14) and (3.70) that

$$\|u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL}\|_{\Omega_\mathrm{BL}} \leq C\sqrt{\epsilon}\left(\frac{\ln N}{N}\right)^{k+1}, \qquad \sqrt{\epsilon}\|(u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL})'\|_{\Omega_\mathrm{BL}} \leq C\left(\frac{\ln N}{N}\right)^{k}. \tag{3.75}$$

Combining (3.73)–(3.75) establishes the error estimates in (3.72). □

Consequently, Theorem 3.10 is modified as follows.

Theorem 3.13 *Under the assumption* (2.14), *there exists a constant C such that*

$$|||e_u||| \leq C\left(\frac{\ln N}{N}\right)^{k}.$$

Finally, the proof of Theorem 2.13 follows the same procedure as we did for the proof of Theorem 2.5.

Remark 3.14 The above analysis motivates the definition of the interpolation operator $\widetilde{\mathbb{I}}$. Indeed, it is straightforward to check that under the assumption (2.14) error estimates of $\|u_\mathrm{BL} - \mathbb{I}u_\mathrm{BL}\|_{\Omega_\mathrm{R}}$ and $\|(u_\mathrm{BL} - \mathbb{I}u_\mathrm{BL})'\|_{\Omega_\mathrm{R}}$ will contain a factor of ϵ^{-1} preventing us from proving error estimates which are uniform in ϵ. On the other hand, $\widetilde{\mathbb{I}}u_\mathrm{BL}$ is a continuous interpolant which vanishes on Ω_R. Therefore, we only used the regularity of u_BL on Ω_R in the interpolation error estimates of $\|u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL}\|_{\Omega_\mathrm{R}}$ and $\|(u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL})'\|_{\Omega_\mathrm{R}}$ and avoided the appearance of ϵ^{-1}. Meanwhile, the factor ϵ^{-1} in the error bound of $\|(u_\mathrm{BL} - \widetilde{\mathbb{I}}u_\mathrm{BL})''\|_{\Omega_\mathrm{R}}$ is always offset, because the bilinear form (2.2) shows that there is always an ϵ factor in the coefficient of the second order derivative term.

4 Numerical Results

In this section, we display numerical results verifying our theoretical findings and demonstrate that they are sharp. We consider (1.1) with $a = 1$, $b = 1$, and $f(x) = x$. The exact solution is then

$$u(x) = -1 + x + \frac{-e^{-\sqrt{1+4\epsilon}/\epsilon + (1+\sqrt{1+4\epsilon})x/2\epsilon} + e^{(1-\sqrt{1+4\epsilon})x/2\epsilon}}{1 - e^{-\sqrt{1+4\epsilon}/\epsilon}}.$$

Note that the exact solution violates the assumption $u_{\mathrm{R}} \in V_N$. In order to verify the results of Theorem 2.5 and Theorem 2.13 we define the discrete ℓ^∞-norm

$$\|e_u\|_{\ell^\infty} := \max_{j=0,1,\ldots,2N} |e_u(x_j)|.$$

To test the result of Theorem 2.3 we also compute $|||e_u|||_{\Omega_{\mathrm{R}}}$.

We implement the SDFEM with the choices (2.1) and (2.13) of the transition number τ. The constant K_1 appearing in the definition of τ is taken to be $K_1 = 2k + \frac{1}{2}$.

The column "n" indicates the number of elements in the mesh that was used to compute the SDFEM approximation. More explicitly, we use a mesh with $2N$ elements where $N = 2^n$. In order to observe the order of convergence of the error, at each refinement of the mesh, we compute the approximate order of convergence as follows. Let $e_u^{(i)}$ denote the error of approximation using a mesh with $n = i$, then

$$\log_2 \frac{\|e_u^{(i)}\|_{\ell^\infty}}{\|e_u^{(i+1)}\|_{\ell^\infty}}$$

is the approximate order of convergence. We display this quantity in the columns labeled "order". A similar remark is valid for the $|||\cdot|||$-norm. Whenever the errors are too small, and the round-off errors kick in, we do not display the error and the resulting approximate order of convergence since the numbers are no longer reliable.

In Table 3 the approximate order of convergence is computed according to the formula

$$\log_p \frac{\|e_u^{(i)}\|_{\ell^\infty}}{\|e_u^{(i+1)}\|_{\ell^\infty}} \quad \text{with } p = 2\frac{\ln 2^{i+1}}{\ln 2^{i+2}} = 2\frac{i+1}{i+2}.$$

This is a reflection of the fact that, due to Theorem 2.13, we are expecting a convergence with respect to a power of $\ln N/N$.

We see from Table 1 that the local error $|||e_u|||_{\Omega_{\mathrm{R}}}$ indeed converges with order $N^{k+1/2}$ as predicted by Theorem 2.3. Let us note that numerical results which we do not display here verified the same orders of convergence when we use (2.13) as the transition number.

Table 1 History of convergence for $|||e_u|||_{\Omega_{\mathrm{R}}}$. $\tau = K_1 \epsilon \ln \epsilon^{-1}$

n	k	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
		Error	Order	Error	Order	Error	Order
4	1	2.97e–03	–	2.98e–03	–	2.98e–03	–
5		1.05e–03	1.50	1.05e–03	1.50	1.05e–03	1.50
6		3.71e–04	1.50	3.71e–04	1.50	3.71e–04	1.50
7		1.32e–04	1.50	1.31e–04	1.50	1.31e–04	1.50
8		4.68e–05	1.49	4.64e–05	1.50	4.64e–05	1.50
4	2	2.44e–05	–	2.47e–05	–	2.47e–05	–
5		4.30e–06	2.50	4.36e–06	2.50	4.36e–06	2.50
6		7.58e–07	2.50	7.72e–07	2.50	7.72e–07	2.50
7		1.33e–07	2.51	1.36e–07	2.50	1.37e–07	2.50
8		2.35e–08	2.51	2.41e–08	2.50	2.42e–08	2.50
4	3	1.25e–07	–	1.28e–07	–	1.28e–07	–
5		1.11e–08	3.50	1.13e–08	3.50	1.14e–08	3.49
6		9.75e–10	3.50	1.00e–09	3.50	1.52e–09	2.91
7		8.62e–11	3.50	9.02e–11	3.47	1.13e–09	4.24

Table 2 History of convergence for $\|e_u\|_{\ell^\infty}$. $\tau = K_1 \epsilon \ln \epsilon^{-1}$

n	k	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
		Error	Order	Error	Order	Error	Order
4	1	2.73e–02	–	5.66e–02	–	8.70e–02	–
5		5.95e–03	2.20	1.50e–02	1.91	2.73e–02	1.67
6		1.48e–03	2.01	3.38e–03	2.15	5.94e–03	2.20
7		3.66e–04	2.02	8.26e–04	2.03	1.48e–03	2.01
8		9.15e–05	2.00	2.06e–04	2.01	3.65e–04	2.01
4	2	6.44e–03	–	2.01e–02	–	3.87e–02	–
5		5.62e–04	3.52	2.50e–03	3.00	6.44e–03	2.59
6		3.27e–05	4.10	1.77e–04	3.82	5.62e–04	3.52
7		2.08e–06	3.97	1.06e–05	4.06	3.27e–05	4.10
8		1.29e–07	4.01	6.56e–07	4.02	2.08e–06	3.97
4	3	1.48e–03	–	7.00e–03	–	1.71e–02	–
5		5.16e–05	4.85	4.06e–04	4.11	1.48e–03	3.52
6	3	9.29e–07	5.79	1.04e–05	5.29	5.15e–05	4.85
7		1.42e–08	6.05	1.54e–07	6.07	9.32e–07	5.79
8		2.20e–10	6.01	2.51e–09	5.94	1.58e–08	5.89

Table 3 History of convergence for $\|e_u\|_{\ell^\infty}$. $\tau = K_1 \epsilon \ln N$

n	k	$\epsilon = 10^{-4}$		$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$	
		Error	Order	Error	Order	Error	Order
4	1	2.21e–03	–	2.21e–03	–	2.21e–03	–
5		8.49e–04	2.03	8.50e–04	2.03	8.50e–04	2.03
6		3.03e–04	2.02	3.03e–04	2.02	3.03e–04	2.02
7		1.02e–04	2.01	1.02e–04	2.01	1.03e–04	2.01
8		3.33e–05	2.01	3.34e–05	2.01	3.34e–05	2.00
4	2	6.99e–05	–	6.99e–05	–	6.99e–05	–
5		1.07e–05	3.99	1.07e–05	3.99	1.07e–05	3.99
6		1.37e–06	4.03	1.37e–06	4.03	1.36e–06	4.04
7		1.59e–07	3.99	1.59e–07	3.99	1.56e–07	4.02
8		1.70e–08	4.00	1.69e–08	4.00	1.78e–08	3.88
4	3	2.86e–06	–	2.85e–06	–	2.86e–06	–
5		1.58e–07	6.16	1.58e–07	6.16	1.60e–07	6.14
6		7.58e–09	5.94	7.57e–09	5.94	9.58e–09	5.51
7		3.01e–10	5.98	2.91e–10	6.05	2.38e–09	–
8		1.03e–11	6.02	6.63e–11	–	–	–

In Table 2 we verify the $2k$-th order convergence of Theorem 2.5. Similarly, Table 3 confirms the result of Theorem 2.13.

Next, we investigate the presence of the factor $(\ln \epsilon^{-1})^{2k}$ in the estimate (2.12) of Theorem 2.5. In order to do that, we fix the polynomial degree, k, and the number of elements, $2N$, and vary ϵ. Observe that, if the error estimate (2.12) is sharp then the ratio of two errors

Table 4

k	4^k	$(9/4)^k$	$(16/9)^k$
1	4.00	2.25	1.78
2	16.00	5.06	3.16
3	64.00	11.39	5.62

Table 5 The examination of the convergence rate factor $(\ln \epsilon^{-1})^{2k}$ of the estimate (2.11) of Theorem 2.5

n	k	$\frac{\|e_u\|_{\ell^\infty}(\epsilon_1)}{\|e_u\|_{\ell^\infty}(\epsilon_3)}$	$\frac{\|e_u\|_{\ell^\infty}(\epsilon_2)}{\|e_u\|_{\ell^\infty}(\epsilon_3)}$	$\frac{\|e_u\|_{\ell^\infty}(\epsilon_1)}{\|e_u\|_{\ell^\infty}(\epsilon_2)}$
5	1	4.57	2.53	1.81
6		4.01	2.28	1.75
7		4.03	2.25	1.79
8		3.99	2.25	1.78
5	2	11.45	4.46	2.57
6		17.19	5.41	3.18
7		15.72	5.10	3.08
8		16.05	5.07	3.17
5	3	28.78	7.88	3.66
6		55.47	11.16	4.97
7		65.87	10.90	6.04
8		71.76	11.40	6.30

computed with two different values of ϵ, say ϵ_1 and ϵ_2, is

$$\frac{\|e_u\|_{\ell^\infty}(\epsilon_1)}{\|e_u\|_{\ell^\infty}(\epsilon_2)} \approx \left(\frac{\ln \epsilon_1^{-1}}{\ln \epsilon_2^{-1}}\right)^{2k}.$$

We thus set

$$\epsilon_1 = 10^{-8}, \qquad \epsilon_2 = 10^{-6}, \qquad \epsilon_3 = 10^{-4}$$

and compute the ratios of the errors

$$\frac{\|e_u\|_{\ell^\infty}(\epsilon_i)}{\|e_u\|_{\ell^\infty}(\epsilon_j)} \quad \text{for } (i,j) = (1,3),\ (2,3),\ (1,2). \tag{4.1}$$

We then refine the mesh and repeat the process. Note that

$$\left(\frac{\ln \epsilon_1^{-1}}{\ln \epsilon_3^{-1}}\right)^{2k} = 4^k, \qquad \left(\frac{\ln \epsilon_2^{-1}}{\ln \epsilon_3^{-1}}\right)^{2k} = \left(\frac{9}{4}\right)^k, \qquad \left(\frac{\ln \epsilon_1^{-1}}{\ln \epsilon_2^{-1}}\right)^{2k} = \left(\frac{16}{9}\right)^k, \tag{4.2}$$

and we expect the error ratios to be close to these numbers. For comparison, the exact values of these numbers are displayed in Table 4.

We see from Table 5 that the error ratios (4.1) are close to what we expect from (4.2). This verifies that the factor $(\ln \epsilon^{-1})^{2k}$ is indeed present in the estimate (2.12) which confirms the sharpness of our estimate.

5 Concluding Remarks

We considered SDFEM for one dimensional singularly-perturbed convection-diffusion-reaction problems. We proved that on Shishkin-type meshes the nodal error superconverges with a rate of order $(\ln \epsilon^{-1}/N)^{2k}$ or $(\ln N/N)^{2k}$, depending on the choice of the transition point of the mesh. Our result can be considered as an extension to the singularly-perturbed regime of the nodal estimate proved by Douglas and Dupont in [10]. In [3], Celiker and Cockburn proved a superconvergence result similar to that of [10] for the discontinuous Galerkin method. However, our result is the first such result for singularly-perturbed problems. In a forthcoming paper, we will consider an element-by-element postprocessing resulting in a new approximation that converges with the same rate as that of the nodal error throughout the computational domain.

The other part of our main result is a local error estimate. We prove that, in a suitably defined norm, the error of the SDFEM converges uniformly in ϵ in the part of the mesh where the exact solution is regular. In other words, we prove uniform-in-ϵ convergence away from the boundary layer.

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