Polynomial Preserving Recovery for Quadratic Elements on Anisotropic Meshes

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Polynomial preserving gradient recovery technique under anisotropic meshes is further studied for quadratic elements. The analysis is performed for highly anisotropic meshes where the aspect ratios of element sides are unbounded. When the mesh is adapted to the solution that has significant changes in one direction but very little, if any, in another direction, the recovered gradient can be superconvergent. The results further explain why recovery type error estimator is robust even under nonstandard and highly distorted meshes.

Keywords: anisotropic mesh; finite element; recovery; superconvergence

I. INTRODUCTION

Finite element recovery techniques are postprocessing methods that reconstruct numerical approximations from finite element solutions to achieve better results. Recent years witness a revitalization of this field, especially those years when Zienkiewicz-Zhu introduced their estimator based on superconvergence patch recovery (SPR), where they applied least squares fitting over a set of elements surrounding the vertex to smooth the stress (gradient) computed from finite element method [1, 2]. Later, another gradient recovery method, called polynomial preserving recovery (PPR) was proposed [3,4]. Both theoretical analysis and numerical tests reveal that PPR has better or at least the same properties as SPR [5]. In this article, we study PPR for quadratic elements under anisotropic meshes. This is the further extension of a recent study for linear element [6]. Nevertheless, this extension is by no means straightforward. A newly developed tool by Huang–Xu [7] has to be adopted in order to carry on the needed analysis.

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In this article, we consider superconvergence for quadratic elements. We know that the optimal convergence rate of gradient approximation by quadratic element is $O(h^2)$. Our theoretical results shows the convergence rate can reach $O(h^3)$ for mildly structured meshes as well as anisotropic meshes with high aspect ratio in 2-D.

As for references regarding a posteriori error estimates and superconvergence related to this article, the reader is referred to [1, 2, 8–19].

II. PRELIMINARY KNOWLEDGE

A. Model Problem

On a bounded polytopic domain $\Omega \subset R^n$, we consider the boundary value problem: To find $u \in H^1(\Omega)$ that satisfies some well-posed boundary conditions and for linear continuous functional f over H^1

$$B(u, v) = f(v), \quad \forall v \in H^1(\Omega),$$

with bilinear form defined by

$$\begin{aligned} B(u, v) &= \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int_\Omega b_i(x) u \frac{\partial v}{\partial x_i} dx + \int_\Omega cuvdx \\ &= \int_\Omega \nabla u \cdot A(x) \nabla v dx + (u, \boldsymbol{b} \cdot \nabla v) + (cu, v). \end{aligned}$$

We assume the usual strong elliptic condition on $A = A(x)$ and sufficient regularity on all input data, which will be specified later when needed.

Let $\mathcal{T}_h$ be a simplicial partition of $\overline{\Omega}$. The quadratic finite element space is then defined by

$$S_h = \{v : v \in H^1(\Omega), v|_\tau \in P_2, \forall \tau \in \mathcal{T}_h\},$$

where P_k is the set of polynomials of degree no more than k. Then the finite element approximation $u_h \in S_h$ satisfies

$$B(u_h, v) = f(v), \quad \forall v \in S_h.$$

B. Simplification

Let A_0 be a piecewise constant function such that each element $\tau \in \mathcal{T}_h$

$$A_0|_\tau = \frac{1}{|\tau|} \int_\tau A(x) dx.$$

Now we define

$$a(u, v) = \int_\Omega \nabla u \cdot A(x) \nabla v dx, \quad a^\tau(u, v) = \int_\tau \nabla u \cdot A_0 \nabla v dx, \quad e_h = u - h_h,$$

Assume that $a_{ij} \in C^{0,\alpha}(\Omega)$, it is easy to show that

$$|a(e_h, v) - \sum_\tau a^\tau(e_h, v)| \le Ch^\alpha |e_h|_{1,\Omega} |v|_{1,\Omega}, \quad \alpha > 0.$$

Therefore, we shift our analysis to $a^\tau(e_h, v)$. Since $A(x)$ is symmetric positive definite, so is $A_0|\tau$. Then there exists an orthogonal matrix Q_τ such that $A_0|_\tau = Q_\tau^T D_\tau Q_\tau$ with $D_\tau = diag(d_1^\tau, \ldots, d_n^\tau)$. By changing of variable $x = Q_\tau z$, we have

$$a^\tau(e_h, v) = \int_{\tau_z} \nabla_z e_h \cdot D_\tau \nabla_z v \, \det Q_z dz.$$

Note that $\nabla_z = Q_z \nabla_x$, $\det Q_z = \pm 1$, and τ_z is obtained by rotating τ. Therefore, without loss of generality, we may concentrate on the second-order form

$$\int_\tau \nabla e_h \cdot D_\tau \nabla v dx = \int_\tau \sum_{i=1}^n d_i^\tau \frac{\partial e_h}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \; d_i^\tau > 0.$$

and estimate the bilinear form

$$B(e_h, v) = \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla e_h \cdot D_\tau \nabla v dx + \int_\Omega e_h(\boldsymbol{b} \cdot \nabla v + cv) dx.$$

Now, let's focus on domain variation. Following [6], we assume that $\mathcal{T}_h$ can be separated into two parts

$$\mathcal{T}_h = \mathcal{T}_{0,h} \cup \mathcal{T}_{1,h}, \quad \bigcup_{\tau \in \mathcal{T}_{i,h}} \bar{\tau} = \bar{\Omega}_{i,h}, \quad \bar{\Omega} = \bar{\Omega}_{0,h} \cup \bar{\Omega}_{1,h},$$

such that we have the following ϵ-σ condition:

1. Any two triangles that share a common edge in $\mathcal{T}_{0,h}$ form a convex quadrilateral which is an ϵ-perturbation from a parallelogram.
2. $\Omega_{1,h}$ has a small measure: $|\Omega_{1,h}| = O(h^\sigma)(\sigma > 0)$.

Moreover, in the y-coordinate system, the image of $\Omega_{0,h}$ under the triangulation $\mathcal{T}_{0,h}$ is formed by triangles, where each pair of adjacent elements forms a parallelogram. Under this y-coordinate system, we define a "broken norm"

$$\|u\|_{k,\Omega_y}^2 = \sum_{\tau \in \mathcal{T}_{0,h}} \|u\|_{k,\tau_y}^2.$$

By a similar argument as in [6], we have the following theorem.

Theorem 2.1. *Let $u \in H^4(\Omega) \cap W_\infty^3(\Omega)$ and $u_I \in S_h$ be the solution of the model problem and its quadratic finite element interpolant, respectively. Then*

$$\left| \sum_{\tau \in \mathcal{T}_{0,h}} \int_\tau \nabla e_I \cdot D_\tau \nabla v dx \right| \lesssim (h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon) + h^2 \epsilon (\|u\|_{3,\Omega_{0,h}} + \epsilon)) |v|_{1,\Omega}. \tag{2.1}$$

Proof. Let $e_I = u - u_I$, then, we have

$$\int_\tau \nabla e_I \cdot D_\tau \nabla v dx = I_0(r) + \varepsilon I_1(r) + \varepsilon^2 I_2(r). \tag{2.2}$$

Here,

$$I_0(\tau) = \sum_{i=1}^{n} d_i^\tau \int_{\tau_y} \frac{\partial e_I}{\partial y_i}\frac{\partial v}{\partial y_i} dy = \int_{\tau_y} \nabla_y e_I \cdot D_\tau \nabla_y v dy;$$

$$I_1(\tau) = \sum_{i=1}^{n} \int_{\tau_y} d_i^\tau \left(-\frac{\partial e_I}{\partial y_i}\frac{\partial v}{\partial y_i}\mathrm{div}_x \eta + \frac{\partial e_I}{\partial y_i}\sum_{l=1}^{n}\frac{\partial v}{\partial y_l}\frac{\partial \eta_l}{\partial x_i} + \frac{\partial v}{\partial y_i}\sum_{k=1}^{n}\frac{\partial e_I}{\partial y_k}\frac{\partial \eta_k}{\partial x_i}\right) dy;$$

$$I_2(\tau) = \sum_{i=1}^{n} d_i^\tau \int_{\tau_y} \left(\sum_{k=1}^{n}\frac{\partial e_I}{\partial y_k}\frac{\partial \eta_k}{\partial x_i}\dot{\sum}_{l=1}^{n}\frac{\partial v}{\partial y_l}\frac{\eta_l}{\partial x_i} - \frac{\partial e_I}{\partial y_i}\sum_{l=1}^{n}\frac{\partial v}{\partial y_l}\frac{\partial \eta_l}{\partial x_i} div_x \eta - \frac{\partial v}{\partial y_i}\sum_{k=1}^{n}\frac{\partial e_I}{\partial y_k}\frac{\partial \eta_k}{\partial x_i}\right) dy;$$

ϵ is involved in the conformal transformation of domain variation

$$y = x + \epsilon\eta(x),$$

where η is a piecewise linear function such that $\nabla_x \eta$ is bounded uniformly in ϵ and h. Then, by the standard finite element superconvergence analysis, we are able to derive

$$\left| \sum_{\tau\in\mathcal{T}_{0,h}} I_0(\tau) \right| \lesssim h^{5/2}\|u\|_{4,\Omega_y}|v|_{1,\Omega_y} \lesssim h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon)|v|_{1,\Omega}, \tag{2.3}$$

due to cancellations between parallel sides of adjacent triangles. For the term $I_1(\tau)$, we have

$$\left| \sum_{\tau\in\mathcal{T}_{0,h}} I_1(\tau) \right| \lesssim h^{2}\|u\|_{3,\Omega_y}|v|_{1,\Omega_y} \lesssim h^{2}(\|u\|_{3,\Omega_{0,h}} + \epsilon)|v|_{1,\Omega}. \tag{2.4}$$

Combining (2.3)–(2.4), we obtain

$$\left| \sum_{\tau\in\mathcal{T}_{0,h}} \int_\tau \nabla e_I \cdot D_\tau \nabla v dx \right| \lesssim (h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon) + h^2\epsilon(\|u\|_{3,\Omega_{0,h}} + \epsilon))|v|_{1,\Omega}. \tag{2.5}$$

■

Polynomial Preserving Gradient Recovery. Following [3, 4], we introduce a gradient recovery operator $G_h : S_h \to S_h \times S_h$, which has the two properties below:

1. Polynomial preserving:

$$G_h(p_I) = \nabla p, \quad \forall p \in P_3.$$

   Consequently, it owns the approximation property

$$\|\nabla u - G_h u_I\| \lesssim h^3|u|_{4,\Omega}, \quad \forall u \in H^4(\Omega).$$

2. Boundedness: When there are no two adjacent angles on an element patch adding up to exceed π, we have

$$\|G_h v\| \lesssim |v|_{1,\Omega}, \quad \forall v \in S_h.$$

Theorem 2.2. *Let $u \in H^4(\Omega) \cap W^3_\infty(\Omega)$ and $u_h \in S_h$ be the solution of the model problem and its quadratic finite element approximation respectively. Assume (a) the ϵ-σ mesh condition, (b) the maximum angle condition, and (c) the discrete inf-sup condition. Then the polynomial preserving gradient recovery operator G_h leads to superconvergence in the sense that:*

$$\|\nabla u - G_h u_h\| \lesssim h^3 \|u\|_{4,\Omega} + h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon) + h^2\epsilon(\|u\|_{3,\Omega_{0,h}} + \epsilon) + h^{2+\sigma/2}|u|_{3,\infty,\Omega}.$$

Proof. By the triangle inequality and the polynomial preserving property,

$$\|\nabla u - G_h u_h\| \leq \|\nabla u - G_h u_I\| + \|G_h(u_I - u_h)\| \lesssim h^3|u|_{4,\Omega} + |u_h - u_I|_{1,\Omega}. \tag{2.6}$$

The analysis for the second term on the right-hand side under conditions (a) and (b) is proceeded as the following:

$$\left|\sum_{\tau\in\mathcal{T}_{1,h}} \int_\tau \nabla e_I \cdot D_\tau \nabla v dx\right| \lesssim h^2|u|_{3,\infty,\Omega} \sum_{\tau\in\mathcal{T}_{1,h}} \int_\tau |\nabla v| dx \lesssim h^{2+\sigma/2}|u|_{3,\infty,\Omega}\|\nabla v\|_{0,\Omega}, \tag{2.7}$$

and (by the standard approximation theory)

$$\left|\int_\Omega e_I(\boldsymbol{b}\cdot\nabla v + cv)dx\right| \lesssim h^3|u|_{3,\Omega}|v|_{1,\Omega}. \tag{2.8}$$

Based on (2.5), (2.7), and (2.8), we derive

$$|B(e_I, v)| \lesssim (h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon) + h^2\epsilon(\|u\|_{3,\Omega} + \epsilon) + h^{2+\sigma/2}|u|_{3,\infty,\Omega})|v|_{1,\Omega}. \tag{2.9}$$

Using the discrete inf-sup condition and the strong elliptic assumption, we then have

$$\begin{aligned}\|u_I - u_h\|_{1,\Omega} &\lesssim \sup_{v\in S_h} \frac{B(u_h - u_I, v)}{\|v\|_{1,\Omega}} = \sup_{v\in S_h}\frac{B(e_I, v)}{\|v\|_{1,\Omega}}\\ &\lesssim h^{5/2}(\|u\|_{4,\Omega_{0,h}} + \epsilon) + h^2\epsilon(\|u\|_{3,\Omega} + \epsilon) + h^{2+\sigma/2}|u|_{3,\infty,\Omega}.\end{aligned} \tag{2.10}$$

The conclusion follows by substituting (2.10) into (2.6). ■

III. ERROR ESTIMATES UNDER ANISOTROPIC MESHES

We analyze errors under triangulation of the regular and Union-Jack patterns. We consider the worst case for the regular pattern where the maximum angle condition is violated at the extremal case $\theta \to 0$, where a patch is compressed at a fixed ratio, i.e., $\theta_2 = k_2\theta$, $\theta_3 = k_3\theta$, and $k_2, k_3 > 0$.

To simplify the matter, we analyze the case $\boldsymbol{b} = 0$ and $c = 0$. We need the following integral identity [7] for $v_Q \in P_2(\tau)$,

$$\begin{aligned}\int_\tau \nabla(u - u_I)\cdot\nabla v_Q = \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{l_k}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2}\\ + O(h^{4-j})|u|_{4,p,\tau}|v_Q|'_{2-j,q,\tau}, \quad j = 0, 1.\end{aligned}$$

where k is modulo 3, angle θ_k is opposite of side l_k (with length ℓ_k), $\boldsymbol{t}_k$ ($\boldsymbol{n}_k$) is the counter-clockwise unit tangential (outer-normal) vector on side l_k. Denote

$$M_k = \frac{\sin 2\theta_k}{\sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3}.$$

We can express

$$\begin{aligned}
a_k^0(\tau) &= M_k \left(\frac{l_{k-1}^2 l_k}{180} \cos^2\theta_{k+1} + \frac{l_1 l_2 l_3}{120} \cos\theta_{k-1} \cos\theta_{k+1} \right. \\
&\quad \left. - \frac{l_k^2 l_{k+1}}{180} \cos\theta_{k-1} - \frac{l_{k+1}^2 l_{k-1}}{180} \cos^2\theta_{k-1} \cos\theta_{k+1} \right), \\
a_k^1(\tau) &= M_k \left(-\frac{l_{k-1}^2 l_k}{90} \sin\theta_{k+1} \cos\theta_{k+1} - \frac{l_1 l_2 l_3}{120} \cos\theta_{k-1} \sin\theta_{k+1} \right. \\
&\quad - \frac{l_{k+1}^2 l_{k-1}}{90} \cos\theta_{k-1} \sin\theta_{k-1} \cos\theta_{k+1} + \frac{l_1 l_2 l_3}{120} \cos\theta_{k+1} \sin\theta_{k-1} \\
&\quad \left. - \frac{l_k^2 l_{k+1}}{180} \sin\theta_{k-1} + \frac{l_{k+1}^2 l_{k-1}}{180} \cos^2\theta_{k-1} \sin\theta_{k+1} \right), \\
a_k^2(\tau) &= M_k \left(\frac{l_{k-1}^2 l_k}{180} \sin^2\theta_{k+1} - \frac{l_1 l_2 l_3}{120} \sin\theta_{k-1} \sin\theta_{k+1} \right. \\
&\quad \left. + \frac{l_{k+1}^2 l_{k-1}}{90} \cos\theta_{k-1} \sin\theta_{k-1} \sin\theta_{k+1} - \frac{l_{k+1}^2 l_{k-1}}{180} \sin^2\theta_{k-1} \cos\theta_{k+1} \right), \\
a_k^3(\tau) &= M_k \frac{l_{k+1}^2 l_{k-1}}{180} \sin^2\theta_{k-1} \sin\theta_{k+1},
\end{aligned}$$

and

$$\begin{aligned}
b_k^0(\tau) &= \frac{1}{1440} \left(\frac{l_k^4 \left(l_{k+1}^2 - l_{k-1}^2\right)}{2|\tau|} + \frac{l_{k-1}^4 \cos^3\theta_{k+1}}{\sin\theta_{k+1}} - \frac{l_{k+1}^4 \cos^3\theta_{k-1}}{\sin\theta_{k-1}} \right), \\
b_k^1(\tau) &= \frac{1}{1440} \left(-3 l_{k-1}^4 \cos^2\theta_{k+1} - 3 l_{k+1}^4 \cos^2\theta_{k-1} \right), \\
b_k^2(\tau) &= \frac{1}{1440} \left(3 l_{k-1}^4 \cos\theta_{k+1} \sin\theta_{k+1} - 3 l_{k+1}^4 \cos\theta_{k-1} \sin\theta_{k-1} \right), \\
b_k^3(\tau) &= \frac{1}{1440} \left(-l_{k+1}^4 \sin^2\theta_{k-1} - l_{k-1}^4 \sin^2\theta_{k+1} \right).
\end{aligned}$$

In Fig. 1, we let $\theta_2 = k_2\theta$, $\theta_3 = k_3\theta$. It is straightforward to verify

$$\begin{aligned}
\frac{\partial^3 u}{\partial \boldsymbol{t_1}^3} &= \frac{\partial^3 u}{\partial x^3}, \\
\frac{\partial^3 u}{\partial \boldsymbol{t}_2^3} &= -\cos^3 k_3\theta \frac{\partial^3 u}{\partial x^3} + (3\cos^2 k_3\theta \sin k_3\theta) \frac{\partial^3 u}{\partial x^2 \partial y} \\
&\quad - (3\cos k_3\theta \sin^2 k_3\theta) \frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 k_3\theta \frac{\partial^3 u}{\partial y^3},
\end{aligned}$$

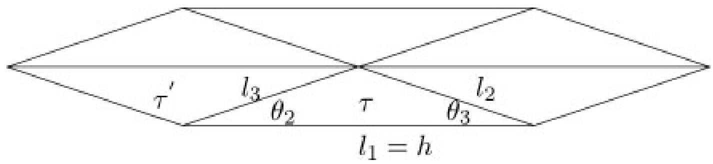

FIG. 1. Anisotropic mesh.

$$\frac{\partial^3 u}{\partial \boldsymbol{t_3}^3} = -(\cos^3 k_2\theta)\frac{\partial^3 u}{\partial x^3} - 3(\sin^2 k_2\theta)(\cos k_2\theta)\frac{\partial^3 u}{\partial x \partial y^2} - 3(\sin k_2\theta)(\cos^2 k_2\theta)\frac{\partial^3 u}{\partial x^2 \partial y} - (\sin^3 k_2\theta)\frac{\partial^3 u}{\partial y^3},$$

$$\frac{\partial^3 u}{\partial \boldsymbol{n_1} \partial \boldsymbol{t_1}^2} = -\frac{\partial^3 u}{\partial x^2 \partial y},$$

$$\frac{\partial^3 u}{\partial \boldsymbol{n_2} \partial \boldsymbol{t_2}^2} = (\cos^2 k_3\theta)(\sin k_3\theta)\frac{\partial^3 u}{\partial x^3} + [(\cos^3 k_3\theta) - (2\cos k_3\theta)(\sin^2 k_3\theta)]\frac{\partial^3 u}{\partial x^2 \partial y} - [2(\cos^2 k_3\theta)(\sin k_3\theta) - (\sin^3 k_3\theta)]\frac{\partial^3 u}{\partial x \partial y^2} + (\sin^2 k_3\theta)(\cos k_3\theta)\frac{\partial^3 u}{\partial y^3},$$

$$\frac{\partial^3 u}{\partial \boldsymbol{n_3} \partial \boldsymbol{t_3}^2} = -(\cos^2 k_2\theta)(\sin k_2\theta)\frac{\partial^3 u}{\partial x^3} + [(\cos^3 k_2\theta) - 2(\sin^2 k_2\theta)(\cos k_2\theta)]\frac{\partial^3 u}{\partial x^2 \partial y} + [2(\cos^2 k_2\theta)(\sin k_2\theta) - (\sin^3 k_2\theta)]\frac{\partial^3 u}{\partial x \partial y^2} + (\sin^2 k_2\theta)(\cos k_2\theta)\frac{\partial^3 u}{\partial y^3},$$

Note that $\sin\theta_1 = \sin(\pi - \theta_2 - \theta_3) = \sin(k_2 + k_3)\theta$. Let $\ell_1 = h$, $\ell_2 \sim h$, $\ell_3 \sim h$. Therefore, as $\theta \to 0$, for $k = 1, 2, 3$, we have

$$\begin{array}{ll} \mathbf{1}: a_k^0 \sim h^3\theta^{-2}, & \mathbf{2}: a_k^1 \sim h^3\theta^{-1}, \\ \mathbf{3}: a_k^2 \sim h^3, & \mathbf{4}: a_k^3 \sim h^3\theta, \\ \mathbf{5}: b_k^0 \sim h^4\theta^{-1}, & \mathbf{6}: b_k^1 \sim h^4 \\ \mathbf{7}: b_k^2 \sim h^4\theta, & \mathbf{8}: b_k^3 \sim h^4\theta^2. \end{array} \tag{3.1}$$

To balance a_k^0, a_k^1 and b_k^0, we need following conditions:

$$\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\tau} = C(u,\tau)\theta^2, \left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\partial\tau} = C(u,\partial\tau)\theta, \left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_{0,\tau} = C(u,\tau)\theta. \tag{3.2}$$

Here and hereafter, $C(u,\tau)(C(u,\partial\tau))$ means a generic constant that depends on u and $\tau(\partial\tau)$, but independent of h and θ.

Lemma 3.1. *If a_k^0, a_k^1, b_k^0, $k = 1, 2, 3$, are balanced, then the coefficients $a_k^s(\tau)$ and $b_k^s(\tau)$ given in above have the following properties:*

(1)

$$a_k^s(\tau) = O(h^3), \quad b_k^s(\tau) = O(h^4);$$

(2) *Assume that τ and τ' are two adjacent equivalent triangles. Then*

$$a_k^s(\tau) = a_k^s(\tau'), \quad b_k^s(\tau) = b_k^s(\tau');$$

(3) *If τ and τ' form a pair of $O(h^2)$ approximate equivalent triangles (namely the lengths of any two opposite edges differ only by $O(h^2)$), then*

$$a_k^s(\tau) - a_k^s(\tau') = O(h^4), \quad b_k^s(\tau) - b_k^s(\tau') = O(h^5).$$

Also, by direct computation, one easily get

$$\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_1^2} = \frac{\partial^2 v_Q}{\partial x^2}, \quad \frac{\partial^2 v_Q}{\partial \boldsymbol{t}_2^2} = \frac{\partial^2 v_Q}{\partial x^2}\cos^2\theta_3 - 2\frac{\partial^2 v_Q}{\partial x \partial y}\cos\theta_3 \sin\theta_3 + \frac{\partial^2 v_Q}{\partial y^2}\sin^2\theta_3,$$
$$\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_3^2} = \frac{\partial^2 v_Q}{\partial x^2}\cos^2\theta_2 + 2\frac{\partial^2 v_Q}{\partial x \partial y}\cos\theta_2 \sin\theta_2 + \frac{\partial^2 v_Q}{\partial y^2}\sin^2\theta_2.$$

Noting that the scale in the x-direction is h while in the y-direction, the scale is $h\theta$, one gets

$$\frac{\partial v_Q}{\partial y}\sin\theta_i = O\left(\frac{\partial v_Q}{\partial x}\right), \frac{\partial^2 v_Q}{\partial y^2}\sin^2\theta_i = O\left(\frac{\partial^2 v_Q}{\partial x^2}\right).$$

Therefore, under the regular partition of anisotropic mesh, see Fig. 1,

$$\frac{\partial^2 v_Q}{\partial t_k^2} \sim \frac{1}{h^2}, \quad k = 1, 2, 3. \tag{3.3}$$

Similarly,

$$\frac{\partial v_Q}{\partial t_k} \sim \frac{1}{h}, \quad k = 1, 2, 3. \tag{3.4}$$

Thus,

$$\left|\frac{\partial^2 v_Q}{\partial t_k^2}\right|_{L^2(\tau)} \lesssim h^{-1}\left|\frac{\partial v_Q}{\partial t_k}\right|_{L^2(\tau)}, \quad k = 1, 2, 3. \tag{3.5}$$

Hence, by Lemma 3.1, Hölder's inequality and (3.3)–(3.5), we have

$$\left(a_1^0(\tau)\int_\tau - a_1^0(\tau')\int_{\tau'}\right)\frac{\partial^3 u}{\partial \boldsymbol{t}_1^3}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_1^2} \lesssim h^3\theta^{-2}C(u,\tau)\theta^2|v_Q|_{1,\tau} \le h^3 C(u,\tau)\|v_Q\|_{1,\tau}.$$

Similarly, we obtain the estimate for other cases.

Hence, for $k = 1, 2, 3$ and $s = 0, 1, 2$, we have

$$\left(a_k^s(\tau)\int_\tau - a_k^s(\tau')\int_{\tau'}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2} \lesssim h^3 C(u,\tau)\|v_Q\|_{1,\tau}.$$

However, we can not obtain a similar estimate for boundary terms under such conditions as above because

$$\left|\frac{\partial v_Q}{\partial t_k}\right|_{L^2(\gamma)} \lesssim h^{-1/2}\theta^{-1/2}\left|\frac{\partial v_Q}{\partial t_k}\right|_{L^2(\tau)}. \tag{3.6}$$

To get rid of $\theta^{-1/2}$, condition (3.2) is refined to

$$\begin{aligned}\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\tau} &= C(u,\tau)\theta^2, \quad \left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,\tau} = C(u,\tau)\theta,\\ \left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\partial\tau} &= C(u,\partial\tau)\theta^{3/2}, \quad \left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,\partial\tau} = C(u,\partial\tau)\theta^{1/2}.\end{aligned} \tag{3.7}$$

Thus, assume condition (3.7), and (3.3)–(3.5), we obtain:

$$\begin{aligned}\left(b_1^0(\tau) - b_1^0(\tau')\int_{l_1}\right)\frac{\partial^3 u}{\partial \boldsymbol{t}_1^3}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_1^2} &\lesssim h^5\theta^{-1}C(u,l_1)\theta^{3/2}\left|\frac{\partial^2 v_Q}{\partial t_1^2}\right|_{L^2(l_1)} && \text{(By (3.1))}\\ &\lesssim h^{9/2}C(u,l_1)\left|\frac{\partial^2 v_Q}{\partial t_1^2}\right|_{L^2(\tau)} && \text{(By (3.6))}\\ &\lesssim h^{7/2}C(u,l_1)|v_Q|_{1,\tau} && \text{(By (3.5))}\\ &\lesssim h^{7/2}C(u,l_1)\|v_Q\|_{1,\tau}.\end{aligned}$$

By following the same method, we can obtain the estimate for the rest of cases. For the sake of demonstration, we provide the proof of one more case here.

$$\begin{aligned}\left(b_2^1(\tau) - b_2^1(\tau')\int_{l_2}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_2\boldsymbol{t}_2^2}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_2^2} &\lesssim h^5 C(u,l_2)\theta^{1/2}\left|\frac{\partial^2 v_Q}{\partial t_2^2}\right|_{L^2(l_1)} && \text{(By (3.1))}\\ &\lesssim h^{9/2}C(u,l_2)\left|\frac{\partial^2 v_Q}{\partial t_2^2}\right|_{L^2(\tau)} && \text{(By (3.6))}\\ &\lesssim h^{7/2}C(u,l_2)|v_Q|_{1,\tau} && \text{(By(3.5))}\\ &\lesssim h^{7/2}C(u,l_2)\|v_Q\|_{1,\tau}.\end{aligned}$$

Hence, for any $k = 1, 2, 3$ and $s = 0, 1, 2$, we have

$$\left(b_k^s(\tau) - b_k^s(\tau')\int_{l_k}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2} \lesssim h^{7/2}C(u,\gamma)\|v_Q\|_{1,\tau},$$

where γ is the common edge of τ and τ'. In addition, due to cancellations between parallel sides of adjacent triangles, we have the following identity [7]:

$$\begin{aligned}(\nabla(u-u_I),\nabla v_h) = \sum_{e=\tau\bigcap\tau'}\sum_{s=0}^{3}&\left(a_e^s(\tau)\int_\tau - a_e^s(\tau')\int_{\tau'}\right.\\ &\left.+\left[b_e^s(\tau)-b_e^s(\tau')\right]\int_e\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2} + O(h^{4-s})|u|_{4,p,\Omega}|v_h|'_{2-s,q,\Omega}.\end{aligned}$$

Following the above identity, we derive the following theorem.

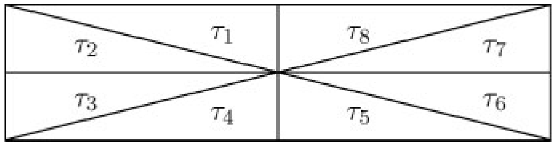

FIG. 2. Union-Jack Pattern.

Theorem 3.1. *Let $u \in H^4(\Omega_a)$ and let $\Omega_a \subset \Omega$ contain anisotropic uniform triangles of type in Fig. 1 with $\theta \in (0, \pi/4]$ ($\theta \in (\pi/4, \pi/2)$ can be treated similarly by shifting the focus directions). Assume that*

$$\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\tau} = C(u,\tau)\theta^2, \quad \left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_{0,\tau} = C(u,\tau)\theta,$$

$$\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\partial\tau} = C(u,\partial\tau)\theta^{3/2}, \quad \left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_{0,\partial\tau} = C(u,\partial\tau)\theta^{1/2}.$$

Then we have

$$\left|\sum_{\tau\subset\Omega_a}\int_\tau \nabla e_I \cdot \mathcal{D}_\tau \nabla v_h\right| \lesssim h^3 |v_h|_{1,\Omega_a}, \quad \forall v_h \in S_h^0(\Omega_a).$$

Remark 1. From Theorem 3.1, we see that if u has very little activity in the x-direction, the degenerating limit $\theta = 0$ in a sample triangle τ can be allowed, which is equivalent to allow the maximum angle in τ be as close as possible to π. The rate of convergence is maintained at $O(h^3)$.

Remark 2. In this case, two adjacent triangles form a parallelogram or a quasi-parallelogram, so cancellation instead of doubling between parallel sides occurs.

Remark 3. In the result, the constant depends on u and Ω_a, but it is independent of h and θ.

Next, let us consider the model problem under Union-Jack Patch, see Fig. 2.

In this case, we can not analyze the problem in a similar fashion as we did in the previous case since some two adjacent triangles, say τ_1 and τ_8 does not form a quasi-parallelogram. The integral on the side shared by these two triangles doubles instead of canceling. However, if we group triangles τ_1 and τ_5, τ_2 and τ_6, τ_3 and τ_7, τ_4, and τ_8, cancelations also happens under certain conditions. Here we only focus on τ_1 and τ_5 and we allow them to form a pair of $O(h^2)$ equivalent triangles, but we restrict $\theta_2 = \frac{\pi}{2}$. For other pairs, we can analyze them in exactly the same way (Fig. 3).

From the graph above, we know that θ_1, θ_2, and θ_3 of these two triangles are almost the same. Moreover, both of these pairs are in count-clockwise order. Hence,

$$a_k^s(\tau_1) - a_k^s(\tau_5) = O(h^4), \quad b_k^s(\tau_1) - b_k^s(\tau_5) = O(h^5), \qquad k = 1,2,3; s = 0,1,2,3.$$

Also, both the unit tangent and out-normal vectors on the corresponding sides are opposite.

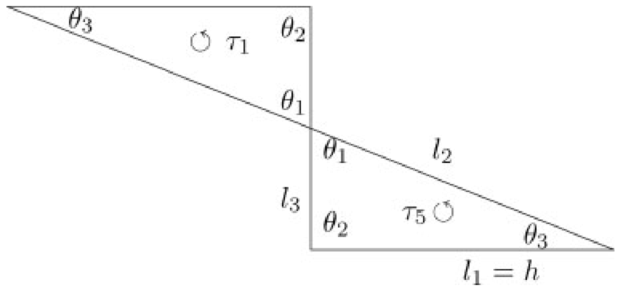

FIG. 3. A group: τ_1 and τ_5.

We also assume that the patch is compressed at a fixed ratio as in the case of regular mesh, i.e., $\theta_3 = k\theta$. Hence, $l_1 = h, l_2 = O(h), l_3 = O(h)\theta$, as $\theta \to 0$. By simple computation, we obtain

$$
\begin{aligned}
&\mathbf{1}: a_1^0 \sim h^3, && a_2^0 = 0, && a_3^0 \sim h^3\theta^3,\\
&\mathbf{2}: a_1^1 \sim h^3\theta, && a_2^1 = 0, && a_3^1 \sim h^3\theta^2,\\
&\mathbf{3}: a_1^2 \sim h^3\theta^2, && a_2^2 = 0, && a_3^2 \sim h^3\theta,\\
&\mathbf{4}: a_1^3 \sim h^3\theta^3, && a_2^3 = 0, && a_3^3 \sim h^3,\\
&\mathbf{5}: b_1^0 \sim h^4\theta^{-1}, && b_2^0 \sim h^4\theta^{-1}, && b_3^0 \sim h^4\theta^3,\\
&\mathbf{6}: b_1^1 \sim h^4, && b_2^1 \sim h^4, && b_3^1 \sim h^4\theta^2,\\
&\mathbf{7}: b_1^2 \sim h^4\theta, && b_2^2 \sim h^4\theta, && b_3^2 \sim h^4\theta,\\
&\mathbf{8}: b_1^3 \sim h^4\theta^2, && b_2^3 \sim h^4\theta^2, && b_3^3 \sim h^4.
\end{aligned}
\tag{3.8}
$$

On the other hand,

$$
\begin{aligned}
\frac{\partial^3 u}{\partial \boldsymbol{t_1}^3} &= \frac{\partial^3 u}{\partial x^3},\\
\frac{\partial^3 u}{\partial \boldsymbol{n_3}^3} &= -\frac{\partial^3 u}{\partial x^3},\\
\frac{\partial^3 u}{\partial \boldsymbol{t}_2^3} &= -\cos^3 k\theta \frac{\partial^3 u}{\partial x^3} + (3\cos^2 k\theta \sin k\theta)\frac{\partial^3 u}{\partial x^2 \partial y} - (3\cos k\theta \sin^2 k\theta)\frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 k\theta \frac{\partial^3 u}{\partial y^3},\\
\frac{\partial^3 u}{\partial \boldsymbol{n_2 t_2}^2} &= \cos^2 k\theta \sin k\theta \frac{\partial^3 u}{\partial x^3} + (\cos^3 k\theta - 2\sin^2 k\theta \cos k\theta)\frac{\partial^3 u}{\partial x^2 \partial y}\\
&\quad + (\sin^3 k\theta - 2\cos^2 k\theta \sin k\theta)\frac{\partial^3 u}{\partial x \partial y^2} + \sin^3 k\theta \cos\theta \frac{\partial^3 u}{\partial y^3},
\end{aligned}
$$

It follows that if

$$
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{\partial\tau} = C(u, \partial\tau)\theta, \tag{3.9}
$$

then b_1^0 and b_2^0 are balanced. From Appendix, we see that under the Union-Jack mesh,

$$\left|\frac{\partial^2 v}{\partial t_k^2}\right|_{L^2(\tau)} \lesssim h^{-1}\theta^{-1}|v|_{1,\tau} \lesssim h^{-1}\theta^{-1}\|v\|_{1,\tau}, \quad k=1,2,3;$$
$$\left|\frac{\partial^2 v}{\partial t_k^2}\right|_{L^2(\gamma)} \lesssim h^{-3/2}\theta^{-1}|v|_{1,\tau} \lesssim h^{-3/2}\theta^{-1}\|v\|_{1,\tau}, \quad k=1,2,3. \tag{3.10}$$

Thus, we require condition (3.9) to be refined to

$$\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\partial\tau} = C(u,\partial\tau)\theta^2, \quad \left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\tau} = C(u,\tau)\theta, \quad \left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,\partial\tau} = C(u,\partial\tau)\theta. \tag{3.11}$$

Hence, if $u \in W^{4,p}(\Omega)$ and condition (3.11) holds, then we obtain

$$\begin{aligned}\left(a_k^s(\tau_1)\int_{\tau_1} + a_k^s(\tau_5)\int_{\tau_5}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2} &= a_k^s(\tau_1)\left\{\int_{\tau_1}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2} + \int_{\tau_5}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2}\right\}\\ &\quad + \left(a_k^s(\tau_5) - a_k^s(\tau_1)\right)\int_{\tau_5}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2}\\ &= I_1 + I_2.\end{aligned}$$

Since $\boldsymbol{n_k}$ and $\boldsymbol{t_k}$ of corresponding sides of τ_1 and τ_5 are opposite, we conclude that the sign of

$$\int_{\tau_1}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2} \quad \text{and} \quad \int_{\tau_5}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2}$$

are opposite.

Therefore, for $k=1$ and $s=0$, we obtain

$$\begin{aligned} I_1 &\lesssim h\left|a_1^0(\tau)\right| \cdot C(u,\tau_1\cup\tau_5)\theta \cdot \left|\frac{\partial^2 v_Q}{\partial t_k^2}\right|_{0,\tau_1\cup\tau_5} && \text{(By (3.8) and (3.11))}\\ &\lesssim h^4 C(u,\tau_1\cup\tau_5)\theta\left|\frac{\partial^2 v_Q}{\partial t_k^2}\right|_{0,\tau_1\cup\tau_5}\\ &\lesssim h^3 C(u,\tau_1\cup\tau_5)\|v\|_{1,\tau_1\cup\tau_5}. && \text{(By (3.10))}\\ I_2 &\lesssim h^4 \cdot C(u,\tau_1\cup\tau_5)\theta \cdot \left|\frac{\partial^2 v_Q}{\partial t_k^2}\right|_{0,\tau_1\cup\tau_5} && \text{(By (3.8) and (3.11))}\\ &\lesssim h^4 C(u,\tau_1\cup\tau_5)\theta\left|\frac{\partial^2 v_Q}{\partial t_k^2}\right|_{0,\tau_1\cup\tau_5}\\ &\lesssim h^3 C(u,\tau_1\cup\tau_5)\|v\|_{1,\tau_1\cup\tau_5}. && \text{(By (3.10))}\end{aligned}$$

By following the same argument, we can obtain the same estimate for all other k's and s's. Hence,

$$I_i \lesssim h^3 C(u,\tau_1\cup\tau_5)\|v\|_{1,\tau_1\cup\tau_5}, \quad k=1,2,3,\ s=0,1,2,3,\ i=1,2.$$

Summing up I_1 and I_2, we obtain,

$$\left(a_k^s(\tau_1)\int_{\tau_1} + a_k^s(\tau_5)\int_{\tau_5}\right)\frac{\partial^3 u}{\partial n_k^s \partial t_k^{3-s}}\frac{\partial^2 v_Q}{\partial t_k^2} \lesssim h^3 C(u,\tau_1\cup\tau_5)\|v\|_{1,\tau_1\cup\tau_5}.$$

Now, let's consider terms involving integration on the boundary.

$$\begin{aligned}\left(b_k^s(\tau_1)\int_{l_1} + b_k^s(\tau_5)\int_{l_5}\right)\frac{\partial^3 u}{\partial \boldsymbol{n_k}^s \partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2} &= b_k^s(\tau_1)\left\{\int_{l_1}\frac{\partial^3 u}{\partial \boldsymbol{n_k}^s \partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2} + \int_{l_5}\frac{\partial^3 u}{\partial \boldsymbol{n_k}^s \partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2}\right\}\\ &\quad + \left(b_k^s(\tau_5) - b_k^s(\tau_1)\right)\int_{l_5}\frac{\partial^3 u}{\partial \boldsymbol{n_k}^s \partial \boldsymbol{t_k}^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t_k}^2}\\ &= L_1 + L_2.\end{aligned}$$

By the same analysis as I_1, we know that the sign of

$$\int_{l_1}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2} \quad \text{and} \int_{l_5}\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2}$$

are opposite. Hence, for $k = 1$ and $s = 0$,

$$\begin{aligned}L_1 &\lesssim h\left|b_1^0(\tau)\right| \cdot C(u,l_1\cup l_5)\theta^2 \cdot \left|\frac{\partial^2 v_Q}{\partial t_1^2}\right|_{0,\, l_1\cup l_5}, \quad \text{(By (3.8) and (3.11))}\\ &\lesssim h^{7/2} C(u,l_1\cup l_5)\left|\frac{\partial^2 v_Q}{\partial t_1^2}\right|_{0,\tau}, \qquad \text{(By (3.10))}\\ &\lesssim h^{7/2} C(u,l_1\cup l_5)\|v_Q\|_{1,\tau}.\\ L_2 &\lesssim h^5\theta^{-1} \cdot C(u,l_1\cup l_5)\theta^2 \cdot \left|\frac{\partial^2 v_Q}{\partial t_1^2}\right|_{0,\, l_1\cup l_5}, \quad \text{(By (3.8) and (3.11))}\\ &\lesssim h^{7/2} C(u,l_1\cup l_5)\left|\frac{\partial v_Q^2}{\partial t_1^2}\right|_{0,\tau}, \qquad \text{(By (3.10))}\\ &\lesssim h^{7/2} C(u,l_1\cup l_5)\|v_Q\|_{1,\tau}.\end{aligned}$$

Thus,

$$\left(b_1^0(\tau_1)\int_{l_1} + b_1^0(\tau_5)\int_{l_5}\right)\frac{\partial^3 u}{\partial \boldsymbol{t}_k^3}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_1^2} \lesssim h^{7/2} C(u,l_1\cup l_5)\|v_Q\|_{1,\tau}.$$

By the same argument as above, we can obtain

$$\left(b_k^s(\tau_1)\int_{l_1} + b_k^s(\tau_5)\int_{l_5}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-k}}\frac{\partial^2 v_Q}{\partial \boldsymbol{t}_k^2} \lesssim h^{7/2} C(u,l_1\cup l_5)\|v_Q\|_{1,\tau_1\cup\tau_5} \quad k = 1,2,3,\ s = 0,1,2,3.$$

Again, we provide the proof of one more case for demonstration. For $k = 2, s = 1$,

$$
\begin{aligned}
L_1 &\lesssim h\left|b_2^1(\tau)\right| \cdot C(u, l_1 \cup l_5)\theta \cdot \left|\frac{\partial^2 v_Q}{\partial t_2^2}\right|_{0,\, l_1\cup l_5}, \text{ (By (3.8) and (3.11))}\\
&\lesssim h^{7/2} C(u, l_1 \cup l_5)\left|\frac{\partial^2 v_Q}{\partial t_2^2}\right|_{0,\tau}, \qquad \text{(By (3.10))}\\
&\lesssim h^{7/2} C(u, l_1 \cup l_5)\|v_Q\|_{1,\tau}.\\
L_2 &\lesssim h^5 \cdot C(u, l_1 \cup l_5)\theta \cdot \left|\frac{\partial^2 v_Q}{\partial t_2^2}\right|_{0,\, l_1\cup l_5}, \text{ (By (3.8) and (3.11))}\\
&\lesssim h^{7/2} C(u, l_1 \cup l_5)\left|\frac{\partial^2 v_Q}{\partial t_2^2}\right|_{0,\tau}, \qquad \text{(By (3.10))}\\
&\lesssim h^{7/2} C(u, l_1 \cup l_5)\|v_Q\|_{1,\tau}.
\end{aligned}
$$

Our result can be concluded as the following theorem:

Theorem 3.2. *Let $u \in H^5(\Omega_a)$ and let Ω contain anisotropic uniform triangles of type in Figure 2 with $\theta \in (0, \pi/4]$ ($\theta \in (\pi/4, \pi/2)$ can be treated similarly by shifting the focus directions). Assume that*

$$
\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\partial\tau} = C(u, \partial\tau)\theta, \quad \left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\tau} = C(u, \tau)\theta, \quad \left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_{0,\partial\tau} = C(u, \partial\tau)\theta.
$$

Then we have

$$
\left|\sum_{\tau\subset\Omega_a}\int_\tau \nabla e_I \cdot \mathcal{D}_\tau \nabla v_h\right| \lesssim h^3 |v_h|_{1,\Omega_a}, \quad \forall v_h \in S_h^0(\Omega_a).
$$

By the similar argument as in Theorem 2.2, we have following theorems.

Theorem 3.3. *Let $u \in H^4(\Omega) \cap W_\infty^3(\Omega)$ and $u_h \in S_h$ be the solution of the model problem and its quadratic finite element approximation, respectively. Assume (a) ϵ-σ mesh condition, (b) $\Omega_a \subset \Omega_{0,h}$ contains anisotropic uniform triangles of type in Fig. 1 with $\theta \in (0, \pi/4]$ and elements in $\Omega \backslash \Omega_a$ satisfy the maximum angle condition, and (c) the discrete inf-sup condition. Furthermore, assume that the condition in Theorem 3.1 is satisfied. Then the polynomial preserving gradient recovery operator G_h has the following error bound:*

$$
\|\nabla u - G_h u_h\| \le h^3 C_1(u, \Omega) + h^{5/2}(C_2(u, \Omega_{0,h}) + \epsilon) + h^2\epsilon(C_3(u, \epsilon) + \epsilon) + h^{2+\sigma/2} C_4(u, \Omega).
$$

Theorem 3.4. *Let $u \in H^5(\Omega) \cap W_\infty^3(\Omega)$ and $u_h \in S_h$ be the solution of the model problem and its quadratic finite element approximation, respectively. Assume (a) ϵ-σ mesh condition, (b) $\Omega_a \subset \Omega_{0,h}$ contains anisotropic uniform triangles of type in Fig. 2 with $\theta \in (0, \pi/4]$ and elements in $\Omega \backslash \Omega_a$ satisfy the maximum angle condition, and (c) the discrete inf-sup condition. Furthermore, assume that the condition in Theorem 3.2 is satisfied. Then the polynomial preserving gradient recovery operator G_h has the following error bound:*

$$
\|\nabla u - G_h u_h\| \le h^3 C_1(u, \Omega) + h^{5/2}(C_2(u, \Omega_{0,h}) + \epsilon) + h^2\epsilon(C_3(u, \epsilon) + \epsilon) + h^{2+\sigma/2} C_4(u, \Omega).
$$

TABLE I. L^2 errors for both meshes on [0,1].

Partition	Regular	Union-Jack
2×20	2.2602e + 000	3.4208e + 000
4×40	3.8200e − 001	4.4553e − 001
8×80	5.0715e − 002	4.8220e − 002
16×160	5.6078e − 003	7.2927e − 003
32×320	6.3853e − 004	1.4361e − 003

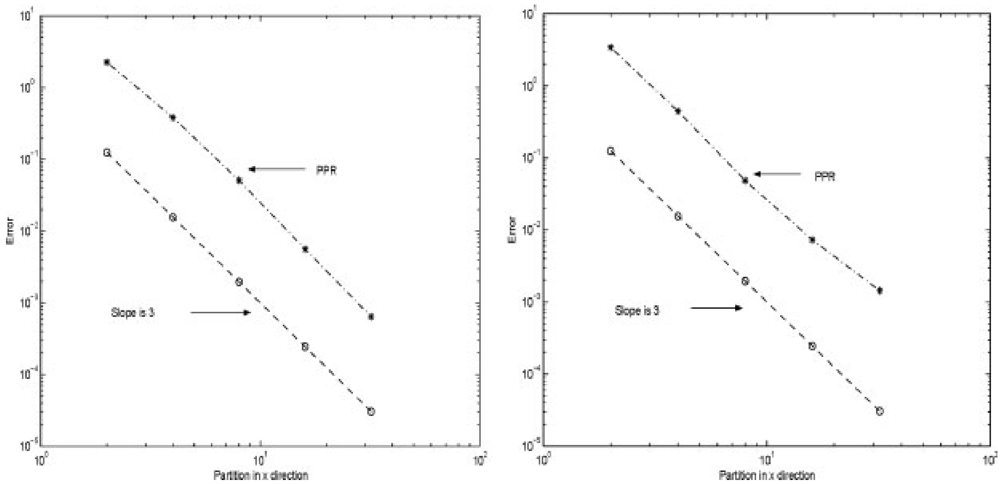

FIG. 4. L_2 error for regular meshes and L_2 error for Union-Jack meshes.

IV. NUMERICAL EXPERIMENT

We compute the Neumann boundary equation below on the unit square domain:

$$\begin{cases} -\Delta u + u = f, \\ \dfrac{\partial u}{\partial \mu} = g. \end{cases} \tag{4.1}$$

Here, we choose proper functions f and g such that $u = \sin(\pi x)\sin(10\pi y)$. The L_2 Error on the whole domain under regular and Union-Jack patterns are shown in Table I:

Corresponding graph is in Fig. 4.

APPENDIX: INVERSE INEQUALITY FOR UNION-JACK ANISOTROPIC MESHES

Let us focus on one τ in Fig. A1 since we can obtain the results for regular mesh in a similar way. We observe that $h = \theta H$ as $\theta \to 0$.

Theorem. *Let γ be the boundary of τ, and $v \in S_h$, then*

$$\left\| \frac{\partial^2 v}{\partial t_k^2} \right\|_{L^2(\tau)} \lesssim \frac{1}{H\theta} \|v\|_{1,\tau}$$

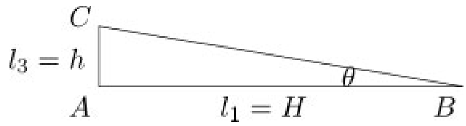

FIG. A1. One element of Union-Jack pattern.

and

$$\left\|\frac{\partial^2 v}{\partial t_k^2}\right\|_{L^2(\gamma)} \lesssim \frac{1}{H^{3/2}\theta}\|v\|_{1,\tau}$$

Proof. Without loss of generality, we assume that vertexes of τ are $A(0,0)$, $B(H,0)$, and $C(0,h)$. Then v is of the form

$$v = a\left(\frac{x}{H}\right)^2 + b\left(\frac{x}{H}\right)\left(\frac{y}{h}\right) + c\left(\frac{y}{h}\right)^2 + d\frac{x}{H} + e\frac{y}{h} + f,$$

where a, b, c, d, e, f are constants.

The results are obvious if v is a constant or a linear function, then the results follow from the classic inverse inequality. From now on, we assume that the coefficient of y^2 term is not 0, i.e., $c \neq 0$.

$$\begin{aligned}
v_x &= \frac{2ax}{H^2} + \frac{by}{Hh} + \frac{d}{H}, \quad v_y = \frac{2cy}{h^2} + \frac{bx}{Hh} + \frac{e}{h},\\
v_{xx} &= \frac{2a}{H^2}, \quad v_{xy} = \frac{b}{hH}, \quad v_{yy} = \frac{2c}{h^2},\\
\frac{\partial^2 v}{\partial t_1^2} &= \frac{\partial^2 v}{\partial x^2}, \; \frac{\partial^2 v}{\partial t_3^2} = \frac{\partial^2 v}{\partial y^2}, \; \frac{\partial^2 v}{\partial t_2^2} = \cos^2\theta\frac{\partial^2 v}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 v}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 v}{\partial y^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\frac{\partial^2 v}{\partial t_1^2}\right\|_{L^2(\tau)} &\sim H^{-2}\cdot\sqrt{hH} = \frac{h^{1/2}}{H^{3/2}},\\
\left\|\frac{\partial^2 v}{\partial t_1^2}\right\|_{L^2(l_1)} &\sim H^{-2}\cdot H^{1/2} = H^{-3/2},\\
\left\|\frac{\partial^2 v}{\partial t_2^2}\right\|_{L^2(\tau)} &\leq \left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L^2(\tau)} + \left\|2\theta\frac{\partial^2 v}{\partial x\partial y}\right\|_{L^2(\tau)} + \left\|\frac{\partial^2 v}{\partial y^2}\theta^2\right\|_{L^2(\tau)}\\
&\sim H^{-2}\sqrt{hH} + 2\theta\cdot\frac{1}{hH}\cdot\sqrt{hH} + \frac{1}{h^2}\theta^2\sqrt{hH}\\
&\sim \frac{h^{1/2}}{H^{3/2}},
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{\partial^2 v}{\partial t_2^2}\right\|_{L^2(l_2)} &\leq \left\|\frac{\partial^2 v}{\partial x^2}\right\|_{L^2(l_2)} + \left\|2\theta\frac{\partial^2 v}{\partial x\partial y}\right\|_{L^2(l_2)} + \left\|\frac{\partial^2 v}{\partial y^2}\theta^2\right\|_{L^2(l_2)} \\
&\sim H^{-2}\sqrt{H} + 2\theta\cdot\frac{1}{hH}\cdot\sqrt{H} + \frac{1}{h^2}\theta^2\sqrt{H} \\
&\sim \frac{1}{H^{3/2}}, \\
\left\|\frac{\partial^2 v}{\partial t_3^2}\right\|_{L^2(\tau)} &= \left\|\frac{\partial^2 v}{\partial y^2}\right\|_{L^2(\tau)} \sim h^{-2}\sqrt{hH} \sim \frac{H^{1/2}}{h^{3/2}}, \\
\left\|\frac{\partial^2 v}{\partial t_3^2}\right\|_{L^2(l_3)} &= \left\|\frac{\partial^2 v}{\partial y^2}\right\|_{L^2(l_3)} \sim h^{-2}\sqrt{h} \sim \frac{1}{h^{3/2}}.
\end{aligned}$$

Now, let's compute $|v|_{1,\tau}$.

$$\begin{aligned}
|v|_{1,\tau}^2 &= \int_\tau \left[\left(\frac{2ax}{H^2}+\frac{by}{hH}+\frac{d}{H}\right)^2 + \left(\frac{2cy}{h^2}+\frac{bx}{hH}+\frac{e}{h}\right)^2\right]dxdy \\
&= \int_0^H \int_0^{h-\frac{h}{H}x} \left[\left(\frac{2ax}{H^2}+\frac{by}{hH}+\frac{d}{H}\right)^2 + \left(\frac{2cy}{h^2}+\frac{bx}{hH}+\frac{e}{h}\right)^2\right]dxdy \\
&\sim \frac{1}{2}\int_0^H \int_0^{h} \left[\left(\frac{2ax}{H^2}+\frac{by}{hH}+\frac{d}{H}\right)^2 + \left(\frac{2cy}{h^2}+\frac{bx}{hH}+\frac{e}{h}\right)^2\right]dxdy \\
&= \frac{4a^2}{H^4}\frac{H^3h}{3} + \frac{b^2}{h^2H^2}\frac{h^3H}{3} + \frac{d^2}{H^2}hH + \frac{4ab}{hH^3}\frac{H^2h^2}{4} + \frac{4ad}{H^3}\frac{H^2h}{2} + \frac{2bd}{hH^2}\frac{h^2H}{2} \\
&\quad + \frac{4c^2}{h^4}\frac{h^3H}{3} + \frac{b^2}{h^2H^2}\frac{H^3h}{3} + \frac{e^2}{H^2}hH + \frac{4bc}{h^3H}\frac{h^2H^2}{4} + \frac{4ec}{h^3}\frac{h^2H}{2} + \frac{2be}{h^2H}\frac{H^2h}{2} \\
&\sim \frac{h}{H} + \frac{H}{h} \qquad \left(\frac{h}{H}\to 0 \text{ as } \theta\to 0\right) \\
&= \frac{H}{h}
\end{aligned}$$

Hence, $|v|_{L^2(\tau)} \lesssim \frac{\sqrt{H}}{\sqrt{h}}$. we obtain

$$\left\|\frac{\partial^2 v}{\partial t_k^2}\right\|_{L^2(\tau)} \lesssim \frac{1}{h}|v|_{1,\tau} \lesssim \frac{1}{H\theta}\|v\|_{1,\tau}, \quad k = 1,2,3.$$

$$\left\|\frac{\partial^2 v}{\partial t_k^2}\right\|_{L^2\gamma} \lesssim \frac{1}{h\sqrt{H}}|v|_{1,\tau} \lesssim \frac{1}{H^{3/2}\theta}\|v\|_{1,\tau}, \quad k = 1,2,3.$$

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