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SUPERCONVERGENCE POINTS OF POLYNOMIAL SPECTRAL INTERPOLATION*

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Abstract. In this work, we study superconvergence properties for some high-order orthogonal polynomial interpolations. At the superconvergent points, the error is smaller by a factor of N^α with $\alpha > 0$, where N is the polynomial degree. The results are two fold: When interpolating function values, we identify those points where the first and second derivatives of the interpolant converge faster; when interpolating the first derivative, we locate those points where the function value of the interpolant superconverges. For the earlier case, we use various Chebyshev polynomials; and for the later case, we also include the counterpart Legendre polynomials.

Key words. superconvergence, interpolation, spectral collocation, analytic function, Chebyshev polynomials, Legendre polynomials

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1. Introduction.

In numerical computation, we often observe that the convergent rate exceeds the best possible global rate at some special points. Those points called superconvergent points, and the phenomenon is called the superconvergence phenomenon, which is well understood for the h-version finite element method; see, e.g., [3, 11, 12, 25, 26, 27, 28, 34, 42, 46] and references therein. As a comparison, a relevant study for the p-version finite element method and the spectral method is lacking. Only very special and simple cases have been discussed in [43, 44, 45].

The study of the superconvergence phenomenon for the h-version method has had a great impact on scientific computing, especially on a posteriori error estimates and adaptive methods, which is well documented in [1, 2, 26, 40] and works cited therein. It is the believe of this author that the scientific community would also benefit from the study of the superconvergence phenomenon of spectral collocation methods as well as related p-version and spectral methods. This work is the first step, where the superconvergence points of some orthogonal polynomial interpolation will be identified.

The most celebrated advantage of spectral methods is the exponential (or geometric) rate of convergence for sufficiently smooth, essentially analytic functions. However, most error bounds in the literature are in the form of $N^{-k}\|u\|_{H^{k+1}}$, where N is the polynomial degree (or trigonometric function degree in the Fourier spectral case). We can see this in almost all books on spectral methods, such as [4, 7, 8, 9, 10, 15, 16, 17, 18, 22, 23, 24, 30, 36, 37]. Ideally, we expect to establish the convergence rate ρ^{-N} for some $\rho > 1$ or $e^{-\sigma N}$ for some $\sigma > 0$. There has been some limited discussion of this type of error bounds in the past, e.g., in [18, 41]; see also [19, 33, 39]. In the framework of p- and hp-finite element methods, results of the exponential convergent rate can be found in [35, 38] and references; see also [20, 21].

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Actually, the exponential rate of convergence for polynomial approximation of analytic functions can be traced back 100 years. The following result is due to Bernstein [5] (see also [29, Theorem 7]): f is analytic on $[-1,1]$ if and only if

$$\sup \lim_{n\to\infty} \sqrt[n]{E_n(f)} = \rho^{-1}, \quad E_n(f) = \inf_{g\in\mathcal{P}_n} \|f-g\|_\infty,$$

where $\mathcal{P}_n$ is the polynomial space of degree no more than n, and $\rho > 1$ is the sum of the half-axes of the maximum elliptic disc B_ρ bounded by the ellipse E_ρ with foci ± 1 that f can be analytically extended to. This fact serves as a starting point of the first part of our analysis in this paper. We shall focus our attention on the approximation properties of polynomial interpolants of analytic functions on $[-1,1]$. We identify those points in $[-1,1]$, where the derivatives are superconvergent, in the sense that the convergent rate gains at least one factor N^{-1}. Three popular Chebyshev polynomial interpolants are discussed here: Chebyshev, Chebyshev–Lobatto, and Chebyshev–Radau.

In the second part of the analysis, we consider polynomial interpolants of the first derivative of a one-order-higher polynomial and identify those points where function values are superconvergent. In this part, both Chebyshev and Legendre polynomials are studied. The knowledge from this part can be used in solving ODEs. However, we would like to indicate that the method presented here is not a new method for the solution of ODEs. The ODE is still solved by the classical method of spectral collocation. It is only when the result is evaluated as a polynomial interpolant that the method here suggests better points (different from the collocation points) where higher accuracy is achieved.

Throughout the paper, we use the following notation for orthogonal polynomials: T_j, the Chebyshev polynomial of the first kind; U_j, the Chebyshev polynomial of the second kind; and L_j, the Legendre polynomial.

2. Interpolation by the Chebyshev polynomials.

In this section, we discuss interpolations by the Chebyshev polynomials of the first kind, Chebyshev–Lobatto polynomials, and Chebyshev–Radau (right and left) polynomials.

2.1. Statement of the results.

We interpolate (or collocate) a smooth function u at a set of $N+1$ special points on $[-1,1]$ in the following sense: Find $u_N \in \mathcal{P}_N[-1,1]$ such that

$$u_N(x_k) = u(x_k), \qquad -1 \le x_0 < x_1 < \cdots < x_N \le 1. \tag{2.1}$$

Here the interpolation points x_k are zeros of the aforementioned four sets of special polynomials. Our goal is to identify those points y_j where u_N', the derivative of the interpolant, superconverges to u' in the sense that

$$N^\alpha |(u-u_N)'(y_j)| \le C \max_{x\in[-1,1]} |(u-u_N)'(x)|, \quad \alpha > 0.$$

If y_j's are the same for a whole class of functions, or in other words, y_j's are *independent* of the particular choice of u, we say that they are derivative superconvergent points for the interpolation to this class of functions. Here we consider analytic functions. The reason is that the spectral method is most effective when the solution is analytic or piecewise analytic. Indeed, for many problems in real life, the solutions are piecewise analytic.

We state the superconvergence results in this subsection. The proof is postponed to the following subsection.

2.1.1. Chebyshev interpolant. Interpolating at the zeros of T_{N+1},

$$T_{N+1}(x_k) = \cos(N+1)\theta_k = 0, \quad x_k = \cos\theta_k, \quad \theta_k = \frac{2k+1}{2N+2}\pi, \quad k = 0, 1, \ldots, N. \tag{2.2}$$

PROPOSITION 2.1. *For the Chebyshev interpolant, the first derivative superconverges at zeros of* U_N*, which are*

$$y_k = \cos\frac{k\pi}{N+1}, \quad k = 1, 2, \ldots, N; \tag{2.3}$$

the second derivative superconverges at $\cos\theta_k$ *with* θ_k *satisfying the following equation:*

$$(N+1)\cos(N+1)\theta\sin\theta = \sin(N+1)\theta\cos\theta. \tag{2.4}$$

This set of $\cos\theta_k$ *are close to interior zeros (not including* $k = 0, N$ *in* (2.2)*) of* T_{N+1} *for large* N*.*

2.1.2. Chebyshev–Lobatto interpolant. Interpolating at the zeros of $T_{N+1} - T_{N-1}$,

$$(T_{N+1} - T_{N-1})(x_k) = 2\sin N\theta_k \sin\theta_k = 0, \quad x_k = \cos\frac{k\pi}{N}, \quad k = 0, 1, \ldots, N. \tag{2.5}$$

We also call this set of points the second type of Chebyshev points.

PROPOSITION 2.2. *For the Chebyshev–Lobatto interpolant, the first derivative superconverges at* $\cos\theta_k$ *with* θ_k *satisfying the following equation:*

$$N\cos N\theta\sin\theta + \sin N\theta\cos\theta = 0. \tag{2.6}$$

This set of θ_k*'s are close to zeros of* $\cos N\theta$ *for large* N*, i.e.,* $\theta_k \approx \frac{2k-1}{2N}\pi$*, when* θ_k *is away from* 0 *and* π*.*

As for superconvergent points of the second derivative, θ_k*'s satisfy the following equation:*

$$(N^2 - 1)\sin N\theta\sin\theta = 2N\cos N\theta\cos\theta. \tag{2.7}$$

This set of θ_k*'s are close to interior zeros (not including* 0 *and* π*) of* $\sin N\theta$ *for large* N*, i.e.,* $\theta_k \approx \frac{k\pi}{N}$*, when* θ_k *is away from* 0 *and* π*.*

Remark 2.1. We see that when interpolating at the zeros of $T_{N+1}(x)$, the derivative of the interpolant superconverges at the zeros of $U_N(x)$, and the second derivative superconvergent points go back almost to the zeros of $T_{N+1}(x)$ except the two ends; when interpolating at the zeros of $(T_{N+1} - T_{N-1})(x) = \gamma_N(1 - x^2)U_{N-1}(x)$, the derivative of the interpolant superconverges almost at the zeros of $T_N(x)$, and the second derivative superconvergent points go back almost to the zeros of $U_{N-1}(x)$.

2.1.3. Chebyshev–Radau (right) interpolant. Interpolating at the zeros of $T_{N+1} - T_N$,

$$\begin{aligned} (T_{N+1} - T_N)(x_k) &= -2\sin\left(N + \frac{1}{2}\right)\theta_k \sin\frac{\theta_k}{2} = 0, \\ x_k &= \cos\frac{2k\pi}{2N+1}, \quad k = 0, 1, \ldots, N. \end{aligned} \tag{2.8}$$

PROPOSITION 2.3. *For the right Chebyshev–Radau interpolant, the first derivative superconverges at* $\cos\theta_k$ *with* θ_k *satisfying the following equation:*

$$(2N+1)\cos\left(N+\frac{1}{2}\right)\theta\sin\frac{\theta}{2}+\sin\left(N+\frac{1}{2}\right)\theta\cos\frac{\theta}{2}=0. \tag{2.9}$$

This set of $\theta_k s$ *are close to zeros of* $\cos(N+\frac{1}{2})\theta$ *for large* N*, i.e.,* $\theta_k \approx \frac{2k-1}{2N+1}\pi$*, when* θ_k *is away from* 0*.*

As for superconvergent points of the second derivative, θ_k *satisfies the following equation:*

$$(2N^2+2N+1)\sin\left(N+\frac{1}{2}\right)\theta\sin\frac{\theta}{2}=(2N+1)\cos\left(N+\frac{1}{2}\right)\theta\cos\frac{\theta}{2}. \tag{2.10}$$

This set of θ_k*'s are close to interior zeros (not including* 0*) of* $\sin(N+\frac{1}{2})\theta$ *for large* N*, i.e.,* $\theta_k \approx \frac{2k\pi}{2N+1}$*, when* θ_k *is away from* 0*.*

2.1.4. Chebyshev–Radau (left) interpolant. Interpolating at the zeros of $T_{N+1}+T_N$,

$$\begin{gathered}(T_{N+1}+T_N)(x_k)=2\cos\left(N+\frac{1}{2}\right)\theta_k\cos\frac{\theta_k}{2}=0,\\ x_k=\cos\frac{2k+1}{2N+1}\pi,\quad k=0,1,\dots,N.\end{gathered} \tag{2.11}$$

PROPOSITION 2.4. *For the left Chebyshev–Radau interpolant, the first derivative superconverges at* $\cos\theta_k$ *with* θ_k *satisfying the following equation:*

$$(2N+1)\sin\left(N+\frac{1}{2}\right)\theta\cos\frac{\theta}{2}+\cos\left(N+\frac{1}{2}\right)\theta\sin\frac{\theta}{2}=0. \tag{2.12}$$

This set of θ_k*'s are close to zeros of* $\sin(N+\frac{1}{2})\theta$ *for large* N*, i.e.,* $\theta_k \approx \frac{2k}{2N+1}\pi$*, when* θ_k *is away from* π*.*

As for superconvergent points of the second derivative, θ_k *satisfies the following equation:*

$$(2N^2+2N+1)\cos\left(N+\frac{1}{2}\right)\theta\cos\frac{\theta}{2}=(2N+1)\sin\left(N+\frac{1}{2}\right)\theta\sin\frac{\theta}{2}. \tag{2.13}$$

This set of θ_k*'s are close to interior zeros (not including* π*) of* $\cos(N+\frac{1}{2})\theta$ *for large* N*, i.e.,* $\theta_k \approx \frac{2k+1}{2N+1}\pi$*, when* θ_k *is away from* π*.*

Remark 2.2. Propositions 2.3 and 2.4 say that when interpolating at the right (left) Radau points, the derivative of the interpolant superconverges almost at the left (right) Radau points except $x=-1$ ($x=1$), and the second derivative superconvergent points almost go back to the interior right (left) Radau points.

Remark 2.3. In this section, we provide derivative superconvergence points as roots of some polynomial equations, as well as the approximated values of those points. Numerical data in the last section demonstrate that these approximations are usually good enough for practical purposes. Therefore, they could be used as the initial guesses, e.g., the Newton iteration, if more accurate values are desired. On the other hand, there are quick and simple ways to find roots of sums of orthogonal polynomials by expressing the problem as an eigenvalue problem with the matrix being upper Hessenberg; see [14].

2.2. Analysis. Let u be analytic on $I = [-1, 1]$. According to Bernstein [5], u can be analytically extended to B_ρ, which is enclosed by an ellipse E_ρ with ± 1 as foci and $\rho > 1$ as the sum of its semimajor and semiminor:

$$E_\rho : \qquad z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1} e^{-i\theta}), \qquad 0 \le \theta \le 2\pi.$$

We consider polynomial $u_N \in \mathcal{P}_N$ who interpolates u at $N+1$ points $-1 \le x_0 < x_1 < \cdots < x_N \le 1$. The error equation is, according to [13, p. 68], expressed as

$$u(x) - u_N(x) = \frac{1}{2\pi i} \int_{E_\rho} \frac{\omega_{N+1}(x)}{\omega_{N+1}(z)} \frac{u(z)}{z - x} dz, \quad \omega_{N+1}(x) = c\Pi_{j=0}^{N}(x - x_j). \tag{2.14}$$

By direct differentiation of (2.14) (cf. [33]), one obtains the error equation for the derivative

$$u'(x) - u_N'(x) = \frac{1}{2\pi i} \int_{E_\rho} \left(\frac{\omega_{N+1}'(x)}{z - x} + \frac{\omega_{N+1}(x)}{(z - x)^2} \right) \frac{u(z)}{\omega_{N+1}(z)} dz, \tag{2.15}$$

and the second derivative

$$u''(x) - u_N''(x) = \frac{1}{2\pi i} \int_{E_\rho} \left(\frac{\omega_{N+1}''(x)}{z - x} + \frac{2\omega_{N+1}'(x)}{(z - x)^2} + \frac{2\omega_{N+1}(x)}{(z - x)^3} \right) \frac{u(z)}{\omega_{N+1}(z)} dz. \tag{2.16}$$

We need to estimate $\omega_{N+1}(x)$, $\omega_{N+1}'(x)$, and $\omega_{N+1}''(x)$ on $[-1, 1]$ to establish the error bounds. However, by each differentiation, we lose at least one power of N.

Key observation. Let us examine the error equation (2.15). At the N special points $\omega_{N+1}'(x) = 0$, we have only the second term, which is usually smaller than the first term in magnitude by a factor N or N^2, as we will see later. Similarly, at the $N - 1$ special points $\omega_{N+1}''(x) = 0$, we have only the second and third terms left in the error equation (2.16). Again, we may gain a factor N in the error bounds.

We consider the four sets of interpolation points in the previous subsection.

The exponential decay of the error is provided by the value of $\omega_{N+1}(z)$ on the ellipse E_ρ in the denominators of (2.14)–(2.16). In all four sets of interpolation points, $\omega_{N+1}(z)$ involves the Chebyshev polynomials of the first kind. We have the following characteristic expressions. The proof is elementary by the definition of $T_{N+1}(z)$ and therefore is omitted.

LEMMA 2.1. *For $z \in E_\rho$, we have*

$$|T_{N+1}(z)| = \frac{1}{2}\sqrt{\rho^{2N+2} + \rho^{-2N-2} + 2\cos 2(N+1)\theta}; \tag{2.17}$$

$$|T_{N+1}(z) - T_{N-1}(z)| = \frac{1}{2}\sqrt{\rho^2 + \rho^{-2} - 2\cos 2\theta}\sqrt{\rho^{2N} + \rho^{-2N} - 2\cos 2N\theta}; \tag{2.18}$$

$$|T_{N+1}(z) \pm T_N(z)| = \frac{1}{2}\sqrt{1 + \rho^{-2} \pm 2\rho^{-1}\cos\theta}\sqrt{\rho^{2N+1} + \rho^{-2N} \pm 2\rho\cos(2N+1)\theta}. \tag{2.19}$$

We also need to bound the derivatives of $\omega_{N+1}(x)$ for $x \in [-1, 1]$.

LEMMA 2.2.

$$\max_{x\in[-1,1]} |T'_{N+1}(x)| = (N+1)^2, \tag{2.20}$$

$$\max_{x\in[-1,1]} |T'_{N+1}(x) - T'_{N-1}(x)| = 4N, \tag{2.21}$$

$$\max_{x\in[-1,1]} |T'_{N+1}(x) \pm T'_{N}(x)| = 2N^2 + 2N + 1; \tag{2.22}$$

$$\max_{x\in[-1,1]} |T''_{N+1}(x)| = \frac{1}{3}N(N+1)^2(N+2), \tag{2.23}$$

$$\max_{x\in[-1,1]} |T''_{N+1}(x) - T''_{N-1}(x)| = \frac{4}{3}N(2N^2+1), \tag{2.24}$$

$$\max_{x\in[-1,1]} |T''_{N+1}(x) \pm T''_{N}(x)| = \frac{2}{3}N(N+1)(N^2+N+1). \tag{2.25}$$

The proof is also elementary and is omitted. To establish the error bounds, we define some constants, which include $C_\rho(u) = \max_{z\in E_\rho} |u(z)|$, D_ρ, the shortest distance from E_ρ to $[-1,1]$, and L_ρ, the arc length of the ellipse E_ρ. We have

$$D_\rho = \frac{1}{2}(\rho + \rho^{-1}) - 1, \qquad L_\rho \le \pi\sqrt{\rho^2 + \rho^{-2}}. \tag{2.26}$$

The latter is the Euler estimate, which overestimates the perimeter by less than 12 percent.

THEOREM 2.1. *Assume that u is analytic on $[-1,1]$ and can be extended analytically to the region bounded by an ellipse E_ρ. Let $u_N \in \mathcal{P}_N[-1,1]$ be the interpolant of u at $N+1$ zeros of $T_{N+1}(x)$ on $[-1,1]$. Then*

$$\max_{x\in[-1,1]} |u(x) - u_N(x)| \le \frac{C_\rho(u)L_\rho}{\pi D_\rho} \frac{1}{\rho^{N+1} - \rho^{-N-1}}, \tag{2.27}$$

$$\max_{x\in[-1,1]} |u'(x) - u'_N(x)| \le \frac{C_\rho(u)L_\rho}{\pi D_\rho} \left((N+1)^2 + \frac{1}{D_\rho}\right) \frac{1}{\rho^{N+1} - \rho^{-N-1}}, \tag{2.28}$$

$$\max_{x\in[-1,1]} |u''(x) - u''_N(x)| \le \frac{C_\rho(u)L_\rho}{\pi} \left(\frac{N(N+1)^2(N+2)}{3D_\rho} + \frac{2(N+1)^2}{D_\rho^2} + \frac{2}{D_\rho^3}\right) \frac{1}{\rho^{N+1} - \rho^{-N-1}}. \tag{2.29}$$

Furthermore, at those special points where $T'_{N+1}(x) = 0$, we have

$$\max_{1\le j\le N} |u'(t_j) - u'_N(t_j)| \le \frac{C_\rho(u)L_\rho}{\pi D_\rho^2} \frac{1}{\rho^{N+1} - \rho^{-N-1}}, \quad t_j = \cos\frac{j\pi}{N+1}, \quad j = 1, 2, \ldots, N, \tag{2.30}$$

and at those special points where $T''_{N+1}(x) = 0$, we have

$$\max_{1\le j\le N-1} |u''(\tau_j) - u''_N(\tau_j)| \le \frac{2C_\rho(u)L_\rho}{\pi D_\rho^2} \left((N+1)^2 + \frac{1}{D_\rho}\right) \frac{1}{\rho^{N+1} - \rho^{-N-1}}, \tag{2.31}$$

where $\tau_j = \cos\theta_j$ with θ_j satisfying (2.4).

Proof. Since $u_N \in \mathcal{P}_N[-1,1]$ interpolates u at zeros of $T_{N+1}(x)$, we have $\omega_{N+1} = T_{N+1}$ in (2.14)–(2.16). By the identity (2.17), we can derive the lower bound

$$|T_{N+1}(z)| \geq \frac{1}{2}(\rho^{N+1} - \rho^{-N-1}).$$

Substituting this into (2.14) and using $\max_{x\in[-1,1]} |T_{N+1}(x)| = 1$, we derive

$$|u(x) - u_N(x)| \leq \frac{1}{\pi}\int_{E_\rho} \frac{|u(z)|}{|z-x|} d|z| \frac{1}{\rho^{N+1} - \rho^{-N-1}} \quad \forall x \in [-1,1]. \tag{2.32}$$

Note that $|u(z)| \leq C_\rho(u)$ and $|z - x|^{-1} \leq D_\rho^{-1}$ for $z \in E_\rho$ and $x \in [-1,1]$, and we obtain (2.27) from (2.32).

Using (2.20) in (2.15) and following the same procedure as above, we derive (2.28). Similarly, we establish (2.29) by applying (2.20) and (2.23) in (2.16).

At the special points when $T'_{N+1}(x) = (N+1)U_N(x) = 0$, the first term on the right-hand side of (2.15) disappears, and we then obtain (2.30) following the same argument as for deriving (2.27).

At the special points when $T''_{N+1}(x) = (N+1)U'_N(x) = 0$, the first term on the right-hand side of (2.16) disappears, and we then obtain (2.31) following the same argument as for deriving (2.28). Since $U_N(x)$ has N simple roots in $(-1,1)$, it is guaranteed that $U'_N(x)$ has $N-1$ simple roots in between and they can be expressed as $x_k = \cos\theta_k$ with θ_k satisfying (2.4). ∎

Remark 2.4. In an early work [6], Boyd developed a theory for interpolation error by a different argument. The conclusion is that the error oscillates like $T_N(x)$ multiplied by a slowly varying "envelope." Differentiation of the envelope shows that the dominant contribution to the error for the first derivative is $T'_N = NU_{N-1}$. Results here are derived more rigorously, bounds rather than approximations, and are considerably more extensive.

THEOREM 2.2. *Under the same assumption as in Theorem* 2.1, *let* $u_N \in \mathcal{P}_N[-1,1]$ *be the interpolant of* u *at* $N+1$ *zeros of* $T_{N+1}(x) - T_{N-1}(x)$ *on* $[-1,1]$. *Then*

$$\max_{x\in[-1,1]} |u(x) - u_N(x)| \leq \frac{C_\rho(u)L_\rho}{\pi D_\rho}\frac{(\rho-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.33}$$

$$\max_{x\in[-1,1]} |u'(x) - u'_N(x)| \leq \frac{C_\rho(u)L_\rho}{\pi D_\rho}\left(4N + \frac{1}{D_\rho}\right)\frac{(\rho-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.34}$$

$$\max_{x\in[-1,1]} |u''(x) - u''_N(x)| \leq \frac{C_\rho(u)L_\rho}{\pi}\left(\frac{4N(2N^2+1)}{3D_\rho} + \frac{8N}{D_\rho^2} + \frac{2}{D_\rho^3}\right)\frac{(\rho-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}. \tag{2.35}$$

Furthermore, at those special points where $T'_{N+1}(x) - T'_{N-1}(x) = 0$, *we have*

$$\max_{1\leq j\leq N} |u'(t_j) - u'_N(t_j)| \leq \frac{C_\rho(u)L_\rho}{\pi D_\rho^2}\frac{(\rho-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.36}$$

where $t_j = \cos\theta_j$ *with* θ_j *satisfying* (2.6); *at those special points where* $T''_{N+1}(x) - T''_{N-1}(x) = 0$, *we have*

$$\max_{1\leq j\leq N-1} |u''(\tau_j) - u''_N(\tau_j)| \leq \frac{2C_\rho(u)L_\rho}{\pi D_\rho^2}\left(4N + \frac{1}{D_\rho}\right)\frac{(\rho-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.37}$$

where $\tau_j = \cos\theta_j$ *with* θ_j *satisfying* (2.7).

Proof. Since $u_N \in \mathcal{P}_N[-1,1]$ interpolates u at zeros of $(T_{N+1} - T_{N-1})(x)$, we have $\omega_{N+1} = T_{N+1} - T_{N-1}$ in (2.14)–(2.16). By the identity (2.18), we can derive the lower bound

$$|(T_{N+1} - T_{N-1})(z)| \geq \frac{1}{2}(\rho - \rho^{-1})(\rho^N - \rho^{-N}).$$

Using (2.21) and (2.24), the rest is then similar to the proof of Theorem 2.1. ∎

THEOREM 2.3. *Under the same assumption as in Theorem* 2.1, *let* $u_N \in \mathcal{P}_N[-1,1]$ *be the interpolant of* u *at* $N+1$ *zeros of* $T_{N+1}(x) \pm T_N(x)$ *on* $[-1,1]$. *Then*

$$\max_{x\in[-1,1]} |u(x) - u_N(x)| \leq \frac{C_\rho(u)L_\rho}{\pi D_\rho} \frac{(1-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.38}$$

$$\max_{x\in[-1,1]} |u'(x) - u'_N(x)| \leq \frac{C_\rho(u)L_\rho}{\pi D_\rho} \left(4N + \frac{1}{D_\rho}\right) \frac{(1-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.39}$$

$$\max_{x\in[-1,1]} |u''(x) - u''_N(x)| \leq \frac{C_\rho(u)L_\rho}{\pi} \left(\frac{4N(2N^2+1)}{3D_\rho} + \frac{8N}{D_\rho^2} + \frac{2}{D_\rho^3}\right) \frac{(1-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}. \tag{2.40}$$

Furthermore, at those special points where $T'_{N+1}(x) \pm T'_N(x) = 0$, *we have*

$$\max_{1\leq j\leq N} |u'(t_j) - u'_N(t_j)| \leq \frac{C_\rho(u)L_\rho}{\pi D_\rho^2} \frac{(1-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.41}$$

where $t_j = \cos\theta_j$ *with* θ_j *satisfying* (2.12) *in the case* $T_{N+1} + T_N$ *and* (2.9) *for* $T_{N+1} - T_N$; *at those special points where* $T''_{N+1}(x) \pm T''_N(x) = 0$, *we have*

$$\max_{1\leq j\leq N-1} |u''(\tau_j) - u''_N(\tau_j)| \leq \frac{2C_\rho(u)L_\rho}{\pi D_\rho^2} \left(4N + \frac{1}{D_\rho}\right) \frac{(1-\rho^{-1})^{-1}}{\rho^N - \rho^{-N}}, \tag{2.42}$$

where $\tau_j = \cos\theta_j$ *with* θ_j *satisfying* (2.13) *in the case* $T_{N+1} + T_N$ *and* (2.10) *for* $T_{N+1} - T_N$.

Proof. Since $u_N \in \mathcal{P}_N[-1,1]$ interpolates u at zeros of $(T_{N+1} \pm T_N)(x)$, we have $\omega_{N+1} = T_{N+1} \pm T_N$ in (2.14)–(2.16). By the identity (2.19), we can derive the lower bound

$$|(T_{N+1} \pm T_N)(z)| \geq \frac{1}{2}(1 - \rho^{-1})(\rho^N - \rho^{-N}).$$

Using (2.22) and (2.25), the rest is then similar to the proof of Theorem 2.1. ∎

We see from the above that superconvergent points involve extremals of $\omega_{N+1}(x)$ for $x \in [-1,1]$. It would be interesting to see the distribution and magnitudes of those points for different cases. Since the data is clear for the case $\omega_{N+1}(x) = T_{N+1}(x)$, we consider the other three cases.

THEOREM 2.4. *The envelope for the extremals of* $T_{N+1} - T_{N-1}$ *on* $[-1,1]$ *forms an ellipse* $x^2 + \frac{y^2}{2^2} = 1$ *(see Figure* 1*).*

Proof. We need to demonstrate that the envelope of $(x, T_{N+1}(x) - T_{N-1}(x))$ or

$$(\cos\theta, \cos(N+1)\theta - \cos(N-1)\theta) \quad \text{or} \quad (\cos\theta, -2\sin N\theta \sin\theta)$$

is an ellipse. Note that $\sin N\theta$ has extremals ± 1 at $\theta_j = \frac{2j-1}{2N}\pi$ and points $(\cos\theta_j, \pm 2\sin\theta_j)$ are on the indicated ellipse. Indeed,

$$(\cos\theta_j)^2 + \frac{(\pm 2\sin\theta_j)^2}{2^2} = 1. \qquad ∎$$

THEOREM 2.5. *The envelope for the extremals of* $T_{N+1} \pm T_N$ *on* $[-1,1]$ *forms a parabola* $2(1 \pm x) = y^2$ *(see Figure* 2*).*

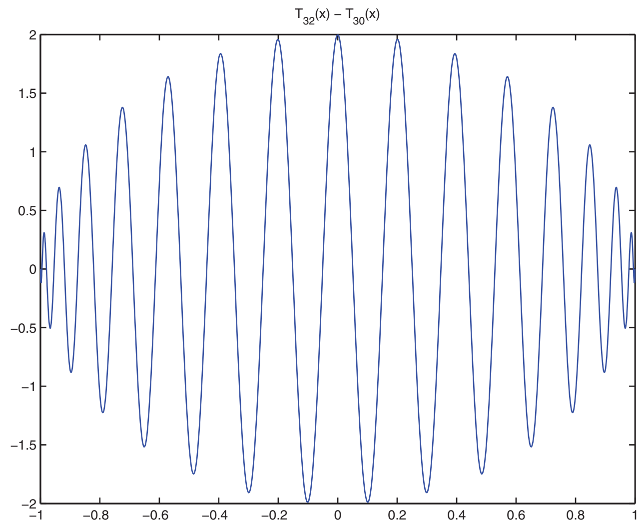

FIG. 1. *Profile of the Chebyshev–Lobatto polynomial.*

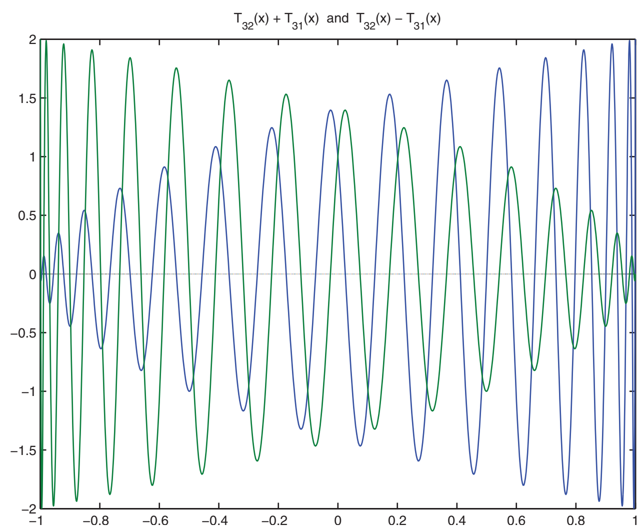

FIG. 2. *Profile of the Chebyshev–Radau polynomials.*

Proof. We need to demonstrate that the envelope of $(x, T_{N+1}(x) \pm T_N(x))$ or

$$(\cos\theta, \cos(N+1)\theta \pm \cos N\theta)$$

or

$$\left(\cos\theta, 2\cos\left(N+\frac{1}{2}\right)\theta\cos\frac{\theta}{2}\right) \quad \text{and} \quad \left(\cos\theta, -2\sin\left(N+\frac{1}{2}\right)\theta\sin\frac{\theta}{2}\right)$$

is on top of the parabola. Note that

$$\cos\left(N+\frac{1}{2}\right)\theta \quad \text{and} \quad \sin\left(N+\frac{1}{2}\right)\theta$$

have extremals at

$$\theta_j = \frac{2j\pi}{2N+1} \quad \text{and} \quad \theta_j = \frac{2j+1}{2N+1}\pi,$$

respectively, and points

$$\left(\cos\theta_j, \pm 2\cos\frac{\theta_j}{2}\right) \quad \text{and} \quad \left(\cos\theta_j, \pm 2\sin\frac{\theta_j}{2}\right)$$

are on the indicated parabola, respectively. Indeed,

$$2(1+\cos\theta) = \left(2\cos\frac{\theta}{2}\right)^2 \quad \text{and} \quad 2(1-\cos\theta) = \left(2\sin\frac{\theta}{2}\right)^2. \qquad \square$$

3. Derivative interpolation. In this part, we consider different interpolants to the first derivative of a smooth function and identify superconvergent points for the function value approximation. To be more precise, we construct polynomial $u_N \in \mathcal{P}_N$ such that

(3.1)
$$u_N(-1) = u(-1), \quad u_N'(x_k) = u'(x_k), \quad k = 1, 2, \dots, N, \quad -1 \le x_1 < \cdots < x_N \le 1,$$

and locate the point y_k, where $(u - u_N)(y_k)$ is superconvergent.

The interpolant discussed in this paper is different from the traditional "Hermite interpolant" in that the function value is not being "interpolated," except at one point, namely, the left end point $x = -1$. The procedure is equivalent to solving a first-order ODE with an initial condition. Therefore, the supeconvergence knowledge can be utilized in a spectral collocation method for solving ODEs. We shall demonstrate this point later in our numerical examples.

To fix the idea, we consider only the case $u \in \mathcal{P}_{N+1}$, since superconvergent property may be narrowed down to the capability of a polynomial space to approximate polynomials of one order higher [3].

In addition to Chebyshev, Chebyshev–Lobatto, and Chebyshev–Radau interpolants, we also consider Gauss, Gauss–Lobatto, and Gauss–Radau interpolants.

We begin with two technical lemmas.

Lemma 3.1.

$$N(L_N + L_{N-1})(x) = (x+1)(L_N - L_{N-1})'(x). \tag{3.2}$$

$$N(L_N - L_{N-1})(x) = (x-1)(L_N + L_{N-1})'(x). \tag{3.3}$$

Proof. We only prove (3.2). Multiplying both sides of (3.2) by $x - 1$ and using the identity

$$(x^2-1)L_N'(x) = (N+1)L_{N+1}(x) - (N+1)xL_N(x),$$

we have

$$\begin{aligned} &N(x-1)(L_N + L_{N-1})(x) \\ &\quad = (N+1)L_{N+1}(x) - (N+1)xL_N(x) - NL_N(x) + NxL_{N-1}(x). \end{aligned}$$

Canceling and collecting the same terms on two sides, we obtain

$$(2N+1)xL_N(x) - NL_{N-1}(x) = (N+1)L_{N+1}(x),$$

which is the three-term recurrence relation. $\square$

Remark 3.1. We see that zeros of the derivatives of the right (left) Legendre–Radau polynomials are zeros of the left (right) Legendre–Radau polynomials.

LEMMA 3.2.

$$(T_N + T_{N-1})(x) = (x+1)(U_{N-1} - U_{N-2})(x). \tag{3.4}$$

$$(T_N - T_{N-1})(x) = (x-1)(U_{N-1} + U_{N-2})(x). \tag{3.5}$$

Proof. By the definition,

$$(T_N + T_{N-1})(x) = \cos N\theta + \cos(N-1)\theta = 2\cos\left(N - \frac{1}{2}\right)\theta\cos\frac{\theta}{2},$$

$$(U_{N-1} - U_{N-2})(x) = \frac{\sin N\theta}{\sin\theta} - \frac{\sin(N-1)\theta}{\sin\theta} = \frac{\cos(N-\frac{1}{2})\theta}{\cos\frac{\theta}{2}}.$$

Therefore,

$$\begin{aligned}(T_N + T_{N-1})(x) &= 2\cos^2\frac{\theta}{2}(U_{N-1} - U_{N-2})(x)\\ &= (1+\cos\theta)(U_{N-1} - U_{N-2})(x) = (1+x)(U_{N-1} - U_{N-2})(x).\end{aligned}$$

This establishes (3.4). The proof of (3.5) is similar and hence is omitted. ∎

Remark 3.2. There is an interesting similarity between (3.4) and (3.2), as well as between (3.5) and (3.3). Note that $T_N'(x) = NU_{N-1}(x)$, and therefore, zeros of the derivatives of the right (left) Chebyshev–Radau polynomials are almost zeros of the left (right) Chebyshev–Radau polynomials.

THEOREM 3.1. *Let* $u \in \mathcal{P}_{N+1}$ *and* $u_N \in \mathcal{P}_N$ *satisfy* (3.1) *with the collocation points* $x_k s$ *being the roots of* L_N. *Then we have (up to a multiplicative constant)*

$$(u - u_N)(x) = \int_{-1}^{x} L_N(t)dt = \frac{1}{2N+1}(L_{N+1} - L_{N-1})(x) = \frac{x^2-1}{N(N+1)}L_N'(x). \tag{3.6}$$

Proof. We see that $(u - u_N)' \in \mathcal{P}_N$. By the definition of the interpolation points, we have, up to a constant, $(u - u_N)'(x) = L_N(x)$. Using the initial condition $u_N(-1) = u(-1)$, the conclusion follows by integration. ∎

Remark 3.3. Theorem 3.1 says that when interpolating the derivative at the N Gauss points, the function value approximation is superconvergent at the $N-1$ interior Lobatto points, i.e., roots of L_N'.

THEOREM 3.2. *Let* $u \in \mathcal{P}_{N+1}$ *and* $u_N \in \mathcal{P}_N$ *satisfy* (3.1) *with the collocation points* $x_k s$ *being the roots of* $L_N - L_{N-2}$. *Then we have (up to a multiplicative constant)*

(3.7)

$$u(x) - u_N(x) = \frac{(x^2-1)(2N-1)L_{N-1}(x)}{N(N+1)} - \frac{4N-2}{N(N+1)(2N-3)}(L_{N-1} - L_{N-3})(x).$$

Proof. From $(u - u_N)' \in \mathcal{P}_N$, we have (up to a constant) $(u - u_N)'(x) = L_N(x) - L_{N-2}(x)$. By the identity

$$[(1-x^2)L_n'(x)]' + n(n+1)L_n(x) = 0,$$

we have

$$u'(x) - u_N'(x) = \frac{[(1-x^2)L_{N-2}'(x)]'}{(N-2)(N-1)} - \frac{[(1-x^2)L_N'(x)]'}{N(N+1)}.$$

Integrating with the initial condition and using the identity

$$(2n+1)L_n(x) = (L_{n+1} - L_{n-1})'(x),$$

we derive

$$\begin{aligned} u(x) - u_N(x) &= \frac{(1-x^2)L'_{N-2}(x)}{(N-2)(N-1)} - \frac{(1-x^2)L'_N(x)}{N(N+1)} \\ &= \frac{(x^2-1)(2N-1)L_{N-1}(x)}{N(N+1)} \\ &\quad + \frac{4N-2}{(N+1)N(N-1)(N-2))}(1-x^2)L'_{N-2}(x), \end{aligned}$$

which is the right-hand side of (3.7) by the identity

$$\frac{1}{2n+1}(L_{n+1}(x) - L_{n-1}(x)) = \frac{1}{n(n+1)}(x^2-1)L'_n(x). \qquad \square$$

Remark 3.4. We see that the magnitude of the first term on the right-hand side of (3.7) is larger than that of the second term by a factor of about N. Therefore, the function value approximation reaches its best at roots of L_{N-1}. In other words, Theorem 3.2 says that when interpolating the derivative at N Lobatto points, the function value approximation is superconvergent at the $N-1$ Gauss points.

THEOREM 3.3. *Let $u \in \mathcal{P}_{N+1}$ and $u_N \in \mathcal{P}_N$ satisfy* (3.1), *and let the collocation points $x_k s$ be the roots of $L_N \mp L_{N-1}$. Then we have (up to a multiplicative constant)*

$$(3.8)\quad (u-u_N)(x) = \frac{N^2}{N^2-1}(L_N \pm L_{N-1})(x)(x\mp 1) - \frac{N}{N^2-1}(L_N \mp L_{N-1})(x)(x\pm 1).$$

Proof. By (3.3), we have (up to a constant)

$$(u-u_N)'(x) = N(L_N \mp L_{N-1})(x) = (L_N \pm L_{N-1})'(x)(x\mp 1).$$

Integrating both sides with the initial condition and using (3.2)–(3.3), we have

$$\begin{aligned} (u-u_N)(x) &= \int_{-1}^{x} (L_N \pm L_{N-1})'(t)(t\mp 1)dt \\ &= (L_N \pm L_{N-1})(x)(x\mp 1) - \frac{1}{N}(L_N \mp L_{N-1})(x)(x\pm 1) \\ &\quad + \frac{1}{N}\int_{-1}^{x}(L_N \mp L_{N-1})(t)dt. \end{aligned}$$

Multiplying both sides by N and moving the last term on the right-hand side to the left-hand side, we obtain (3.8). $\square$

Remark 3.5. We see that the magnitude of the first term on the right-hand side of (3.8) is larger than that of the second term by a factor N. Therefore, function value approximation reaches its best at the roots of $L_N \pm L_{N-1}$ for (3.8). In other words, Theorem 3.3 says that when interpolating the derivative at N left (right) Radau points, the function value approximation is superconvergent at the N right (left) Radau points.

Now we turn to the Chebyshev polynomials. We need the following two identities:

$$(\sqrt{1-x^2}T_n'(x))' + \frac{n^2}{\sqrt{1-x^2}}T_n(x) = 0, \tag{3.9}$$

$$\frac{1}{2}(T_{n-1}(x) - T_{n+1}(x)) = (x^2-1)U_{n-1}(x). \tag{3.10}$$

THEOREM 3.4. *Let $u \in \mathcal{P}_{N+1}$ and $u_N \in \mathcal{P}_N$ satisfy* (3.1), *and let the collocation points x_ks be the N roots of T_N. Then we have (up to a multiplicative constant)*

$$(u-u_N)(x) = \frac{N}{N^2-1}(x^2-1)U_{N-1}(x) - \frac{xT_N(x)+(-1)^N}{N^2-1}. \tag{3.11}$$

Proof. We have (up to a constant) $(u-u_N)'(x) = T_N(x)$. By (3.9), we write

$$(u-u_N)'(x) = -\frac{\sqrt{1-x^2}}{N^2}(\sqrt{1-x^2}T_N'(x))'.$$

Integrating and using the initial condition, we derive

$$\begin{aligned}(u-u_N)(x) &= \int_{-1}^x T_N(t)dt = -\int_{-1}^x \frac{\sqrt{1-t^2}}{N^2}(\sqrt{1-t^2}T_N'(t))'dt \\ &= -\frac{1-x^2}{N^2}T_N'(x) - \frac{1}{N^2}\int_{-1}^x tT_N'(t)dt \\ &= \frac{x^2-1}{N}U_{N-1}(x) - \frac{xT_N(x)+(-1)^N}{N^2} + \frac{1}{N^2}\int_{-1}^x T_N(t)dt.\end{aligned}$$

Moving the last term to the left and multiplying the resultant by $\frac{N^2}{N^2-1}$, we obtain (3.11). ∎

Remark 3.6. We see that the magnitude of the first term on the right-hand side of (3.11) is larger than that of the second term by a factor of N. Therefore, the function value approximation reaches its best at roots of U_{N-1}. In other words, Theorem 3.4 says that when interpolating the derivative at the N Chebyshev points of the first kind, the function value approximation is superconvergent at $N-1$ Chebyshev points of the second kind.

THEOREM 3.5. *Let $u \in \mathcal{P}_{N+1}$ and $u_N \in \mathcal{P}_N$ satisfy* (3.1), *and let the collocation points x_ks be ± 1 plus the $N-2$ roots of U_{N-2}. Then we have (up to a multiplicative constant)*

$$\begin{aligned}(u-u_N)(x) = \frac{N(1-x^2)}{N^2-1}T_{N-1}(x) - \frac{N(1-x^2)}{(N^2-1)(N-2)}U_{N-3}(x) + \frac{xT_N(x)+(-1)^N}{2(N^2-1)} \\ + \frac{N^2(xT_{N-2}(x)+(-1)^N)}{2(N-2)^2(N^2-1)} + \frac{N-1}{(N^2-1)(N-2)^2}\int_{-1}^x T_{N-2}(t)dt.\end{aligned} \tag{3.12}$$

Proof. We have (up to a constant) $(u-u_N)'(x) = (x^2-1)U_{N-2}(x)$. By (3.10) and (3.9), we write

$$\begin{aligned}(u-u_N)'(x) &= \frac{1}{2}(T_{N-2}(x) - T_N(x)) \\ &= \frac{\sqrt{1-x^2}}{2}\left(\frac{1}{N^2}(\sqrt{1-x^2}T_N'(x))' - \frac{1}{(N-2)^2}(\sqrt{1-x^2}T_{N-2}'(x))'\right).\end{aligned}$$

Integrating and using the initial condition, we derive

$$\begin{aligned}(u-u_N)(x) &= \frac{1}{2}\int_{-1}^{x}(T_{N-2}(t)-T_N(t))dt\\ &= \frac{1-x^2}{2N^2}T_N'(x)+\frac{1}{2N^2}\int_{-1}^{x}tT_N'(t)dt\\ &\quad -\frac{1-x^2}{2(N-2)^2}T_{N-2}'(x)-\frac{1}{2(N-2)^2}\int_{-1}^{x}tT_{N-2}'(t)dt\\ &= \frac{1-x^2}{2N}(U_{N-1}(x)-U_{N-3}(x))-\frac{1-x^2}{N(N-2)}U_{N-3}(x)+\frac{xT_N(x)+(-1)^N}{2N^2}\\ &\quad +\frac{xT_{N-2}(x)+(-1)^N}{2(N-2)^2}+\frac{N-1}{N^2(N-2)^2}\int_{-1}^{x}T_{N-2}(t)dt\\ &\quad +\frac{1}{2N^2}\int_{-1}^{x}(T_{N-2}(t)-T_N(t))dt.\end{aligned}$$

Moving the last term to the left-hand side and multiplying the resultant by $\frac{N^2}{N^2-1}$, we derive (3.12) when replacing $U_{N-1}-U_{N-3}$ by $2T_{N-1}$. ◻

Remark 3.7. We see that the magnitude of the first term on the right-hand side of (3.12) is larger than that of the other terms by a factor of about N. Therefore, $(u-u_N)(x)$ reaches its best at roots of T_{N-1}. In other words, Theorem 3.5 says that when interpolating the derivative at ± 1 plus the $N-2$ Chebyshev points of the second kind, the function value approximation is superconvergent at $N-1$ Chebyshev points of the first kind.

THEOREM 3.6. *Let $u \in \mathcal{P}_{N+1}$ and $u_N \in \mathcal{P}_N$ satisfy* (3.1), *and let the collocation points x_ks be the roots of $T_N \pm T_{N-1}$. Then we have (up to a multiplicative constant)*

$$\begin{aligned}(u-u_N)(x) &= \frac{N(x\pm 1)}{N^2-1}(T_N \mp T_{N-1})(x) \pm \frac{N(x^2-1)}{(N^2-1)(N-1)}U_{N-2}(x)\\ &\quad \mp\frac{xT_N(x)+(-1)^N}{N^2-1}\\ &\quad \mp\frac{N^2(xT_{N-1}(x)+(-1)^{N-1})}{(N^2-1)(N-1)^2} \pm \frac{2N-1}{(N^2-1)(N-1)^2}\int_{-1}^{x}T_{N-1}(t)dt. \qquad (3.13)\end{aligned}$$

Proof. The proof is similar to that of Theorem 3.5 and hence is omitted. ◻

Remark 3.8. We see that the magnitude of the first term on the right-hand side of (3.13) is larger than that of the other terms by a factor of about N. Therefore, the function value approximation reaches its best at the roots of $T_N \mp T_{N-1}$ for (3.13). In other words, Theorem 3.6 says that when interpolating the derivative at the N left (right) Chebyshev–Radau points, the function value approximation is superconvergent at the N right (left) Chebyshev–Radau points.

4. Numerical tests.

In this section, we perform numerical tests on two typical analytic functions.

Example 1. $f(x) = (1+25x^2)^{-1}$. This is the well-known Runge example [32]. Derivative errors of its interpolants at the Chebyshev points, the second-type Chebyshev points (or Chebyshev–Lobatto points), and the right Chebyshev–Radau points are depicted in Figures 3, 4, and 5, respectively. Derivative superconvergent points

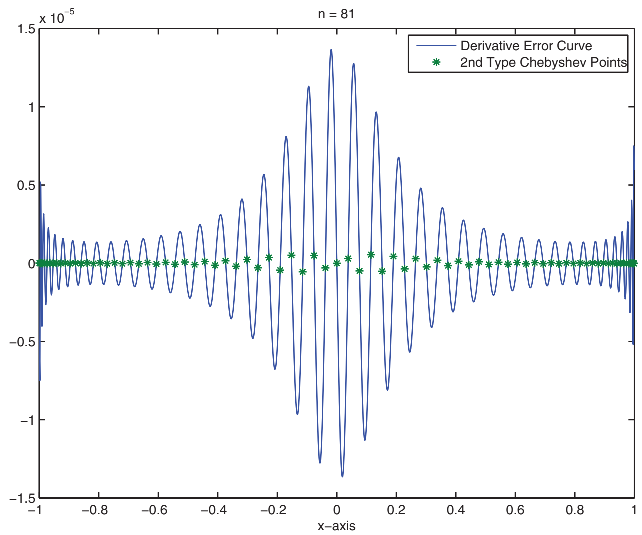

FIG. 3. *Derivative error: Chebyshev collocation* (2.1), *Example* 1.

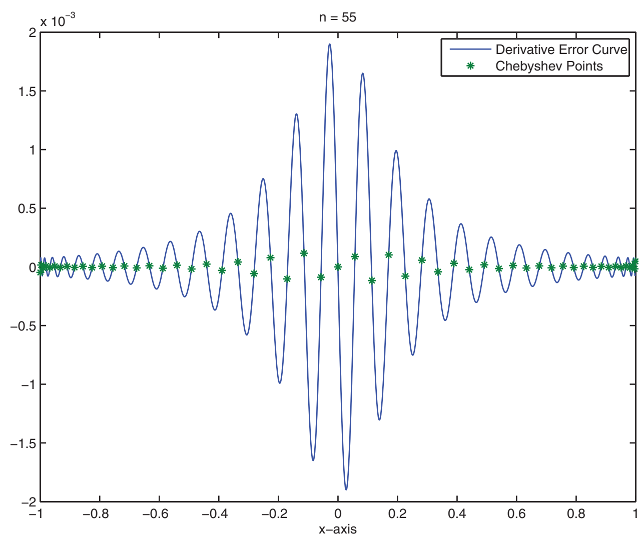

FIG. 4. *Derivative error: Chebyshev–Lobatto collocation* (2.1), *Example* 1.

are marked by *. We see that the errors at the superconvergent points are significantly smaller (by a magnitude) than the maximum error, just as Theorems 2.1–2.3 predicted.

Next, we verify the theoretical results in section 3. As mentioned, the interpolation (3.1) is equivalent to solving an ODE $u'(x) = f(x)$ with initial condition $u(-1)$ given. The numerical scheme is to find $u_N \in \mathcal{P}_N[-1, 1]$ such that $u_N(x_k) = f(x_k)$ and $u_N(-1) = u(-1)$. Here we consider the initial value problem

$$u'(x) = \frac{1}{1 + 25x^2}, \quad u(-1) = \frac{1}{26},$$

by collocating at the Chebyshev points as in (3.1). From Figure 6, we see that when

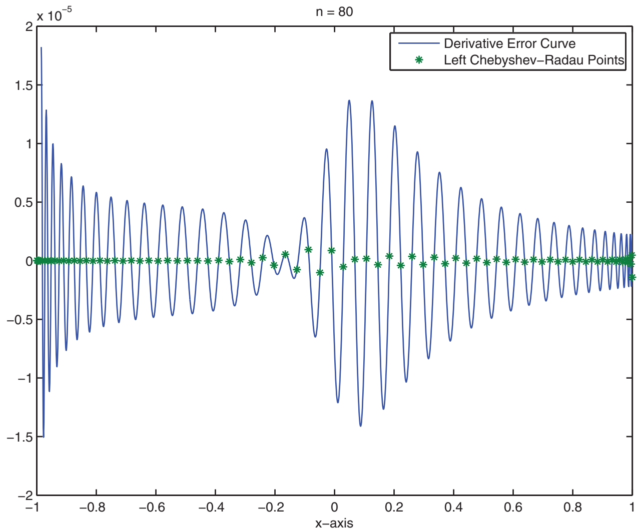

FIG. 5. *Derivative error: right Chebyshev–Radau collocation* (2.1), *Example* 1.

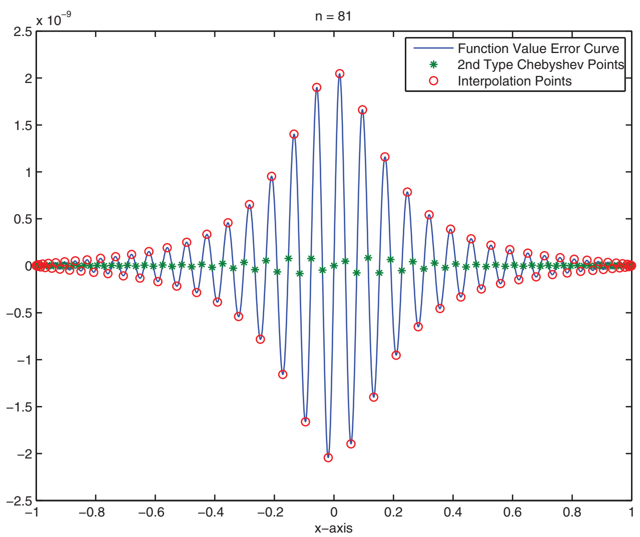

FIG. 6. *Function value error: Chebyshev collocation* 3.1, *Example* 1.

interpolating the derivative at the Chebyshev points, the function value of the interpolant converges much faster at the interior second-type Chebyshev points marked by $*$, which is consistent with Theorem 3.1. We also see that the function value approximation has maximum error at the derivative interpolation points marked by $\circ$.

Analytic function $f(z) = (1 + 25x^2)^{-1}$ has two single poles at $\pm i/5$ and it is straightforward to calculate $\rho = (\sqrt{5^2+1}+1)/5 \approx 1.2198$. Therefore, we expect a slow convergence as we have observed from Figures 3–6.

Example 2. $f(x) = (2 - x)^{-1}$, an analytic function which has a simple pole at $x = 2$ and $\rho = 2 + \sqrt{2^2 - 1} \approx 3.7321$. Therefore, we expect much faster convergence compared with Example 1. It is indeed the case. We plot counterpart graphs in

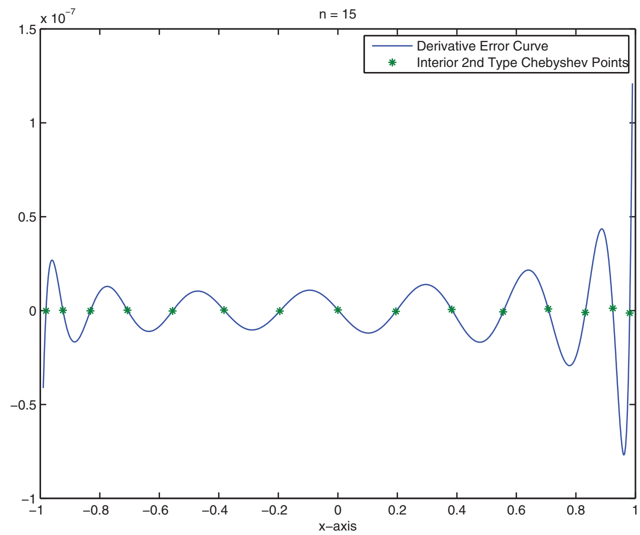

FIG. 7. *Derivative error: Chebyshev collocation* (2.1), *Example* 2.

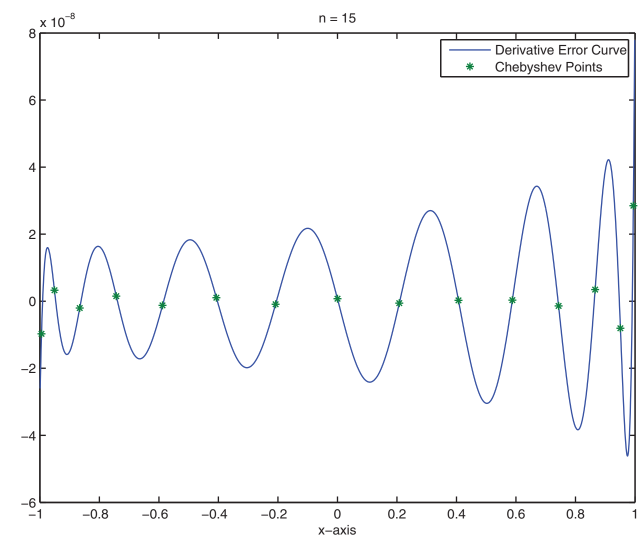

FIG. 8. *Derivative error: Chebyshev–Lobatto collocation* (2.1), *Example* 2.

Figures 7–10, and we see that high accuracy is achieved with a relatively very low polynomial degree n. We observe similar superconvergence phenomena as in Example 1.

Remark 4.1. If an ODE contains both first and second derivatives, the function value superconvergence points are the same as the interpolating points, while the first derivative superconvergence points are the same as those function value superconvergence points of the first-order ODE. Note that the above conclusion is not applied to singularly perturbed differential equations, in which case the coefficient of the second derivative term approaches zero.

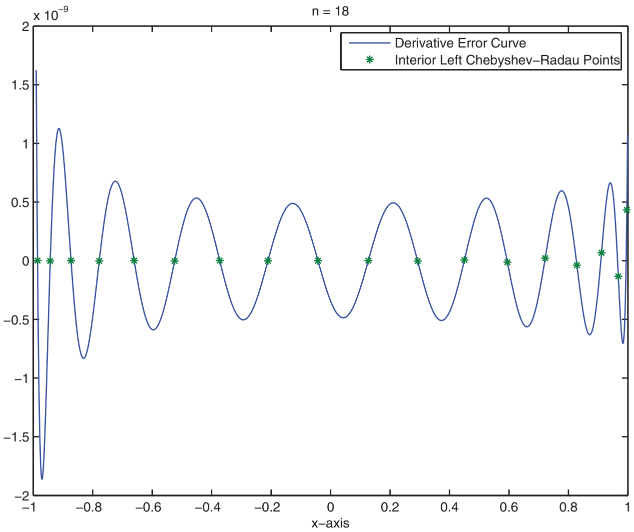

FIG. 9. *Derivative error: right Chebyshev–Radau collocation* (2.1), *Example* 2.

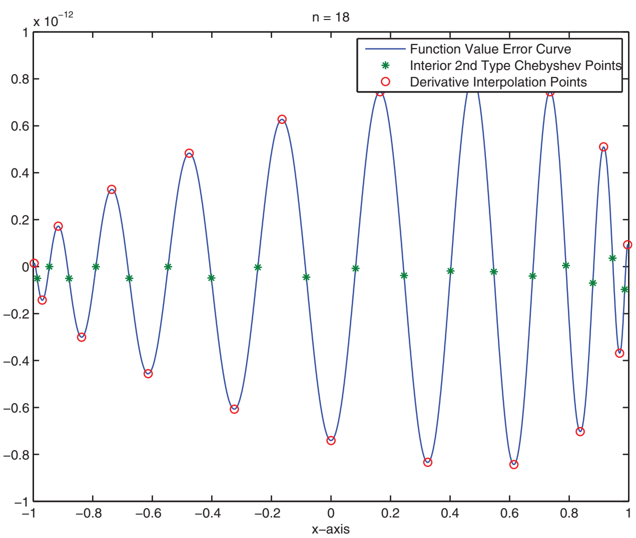

FIG. 10. *Function value error: Chebyshev collocation* 3.1, *Example* 2.

5. Conclusion. The results in section 2 can be extended to all Legendre-based polynomials as in section 3. The reason for using the Chebyshev-based polynomials is that they have simple (trigonometric function) expressions.

Extension of the result to more general Jacobi-type polynomials is feasible. However, the analysis would be much more involved.

The results in this paper can be used to solve differential equations by polynomial-based (instead of Fourier-based) spectral or spectral collocation methods. The data output at superconvergent points as described here could be much more accurate than those at other points.

Another benefit of our superconvergence analysis can be understood from the a posteriori error estimate point of view: One may use the superconvergence knowledge to estimate the error by comparing the superconvergent value with the non-superconvergent one.

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