Convergence Analysis of the LDG Method Applied to Singularly Perturbed Problems

Huiqing Zhu,[1] **Zhimin Zhang**[2]

[1] *Department of Mathematics, The University of Southern Mississippi, Hattiesburg, Mississippi 39406*

[2] *Department of Mathematics, Wayne State University, Detroit, Michigan 48202*

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Considering a two-dimensional singularly perturbed convection–diffusion problem with exponential boundary layers, we analyze the local discontinuous Galerkin (DG) method that uses piecewise bilinear polynomials on Shishkin mesh. A convergence rate $O(N^{-1}\ln N)$ in a DG-norm is established under the regularity assumptions, while the total number of mesh points is $O(N^2)$. The rate of convergence is uniformly valid with respect to the singular perturbation parameter ϵ. Numerical experiments indicate that the theoretical error estimate is sharp. Numer Methods Partial Differential Eq 29: 396–421, 2013

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I. INTRODUCTION

We will consider the singularly perturbed convection–diffusion problem given by

$$\begin{cases} -\epsilon\Delta u + \boldsymbol{\beta}\cdot\nabla u + cu = f, & \text{in } \Omega = (0,1)^2, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

under the assumptions that $0 < \epsilon \ll 1$ is a small perturbation parameter, $\boldsymbol{\beta}(x,y) \geq (\alpha,\alpha) > (0,0)$, $c(x,y) \geq 0$, and

$$c(x,y) - \frac{1}{2}\text{div}\boldsymbol{\beta}(x,y) \geq c_0 > 0, \quad \text{on } \overline{\Omega}, \tag{1.2}$$

where α and c_0 are constants independent of ϵ. We also assume that $\boldsymbol{\beta}(x,y)$, $c(x,y)$, and $f(x,y)$ are sufficiently smooth. These hypotheses ensure that (1.1) has a unique solution in $H_0^1(\Omega)\cap H^2(\Omega)$ (cf. [1,2]). A further smoothness of the solution u can be ensured if f satisfies certain compatibility

Correspondence to: Huiqing Zhu, Department of Mathematics, The University of Southern Mississippi, Hattiesburg, Mississippi 39406 (e-mail: Huiqing.Zhu@usm.edu)
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at the corners of the square domain. Note that for sufficiently small parameter ϵ, condition (1.2) can be ensured by the other hypotheses and a transformation $v(x, y) = e^{-t(x+y)}u(x, y)$ when t is chosen suitably.

The solution u of (1.1) exhibits exponential boundary layers near $x = 1$ and $y = 1$. Most traditional numerical methods, such as the finite element method and the finite difference method, fail to capture the rapid change of the solution inside the boundary layers, which in turn pollutes the numerical results throughout the whole domain. Many numerical methods, such as up-winding scheme, special trail/test functions, and layer-adapted meshes, have been developed to overcome the difficulty caused by the boundary layers (cf. [3–6]).

Using conforming finite element method(FEM) with bilinear approximation space and the Shishkin mesh, Stynes and O'Riordan (cf. [1]) proved an optimal error estimate as follows:

$$\sqrt{\epsilon}|u - u_N|_1 \leq CN^{-1} \ln N,$$

where piecewise bilinear trial functions were used. Here, the total number of mesh points is $O(N^2)$, u_N denotes the finite element solution. Similar conclusion has been derived for streamline-diffusion FEM in Ref. [7]. Supercolse property of the FEM using bilinear elements and the Shishkin mesh for (1.1) was presented in Ref. [2].

Starting from 1970's, discontinuous Galerkin (DG) methods have been intensively studied and applied to hyperbolic and convection-dominated elliptic problems with great success (cf. [8–13]). Recently, Arnold et al. [14] provided a framework for the analysis of a large class of DG methods for elliptic problems. This framework allows for the understanding of most of the DG methods that have been proposed for the numerical treatment of the elliptic problems (cf. [15–17], etc.). Among these DG methods, the local DG (LDG) method is a typical one with a symmetric primal formulation. The flexibility of the discontinuous finite element space is expected to enhance the the accuracy when solving singularly perturbed problems. A nonsymmetric DG method with internal penalties (the NIPG method) was discussed by Zarin and Roos for singularly perturbed problems in Ref. [18]. A convergence rate of order $O(\ln^{3/2} N/N)$ for the DG approximation was obtained. Optimal convergence analysis of the LDG method for one-dimensional singularly perturbed problems of convection-diffusion type and reaction-diffusion type has also been established in Ref. [19].

As the optimal and uniform convergence of the LDG method on layer-adapted meshes has not been discussed so far, our main concern here is to investigate the convergence of the LDG approximation to the gradient and the potential. Here is the outline of the article. We introduce in Section 2 the regularity results, the Shishkin mesh, and the LDG discrtization. In Section 3, we present the interpolation error estimates. Section 4 presents the main result, which is a uniform and optimal convergence rate of order $O(\ln N/N)$. In Section 5, numerical experiments are presented, which indicate that our theoretical result is sharp.

Notations. Throughout the article, C denotes a generic positive constant that is always independent ϵ and the mesh used. Vectors and scalars are denoted by bold and plain letters, respectively. For any measurable subdomain $D \subseteq \Omega$, we denote the inner products and norms of $L^2(D)$ by $(\cdot,\cdot)_D$ and $\|\cdot\|_D$. $H^1(D)$ stands for the space $W_2^2(D)$, whose norm and seminorm are denoted by $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$, respectively.

II. THE LDG DISCRETIZATION ON THE SHSHKIN MESH

In this section, we shall describe the regularities of the exact solution, the Shishkin mesh and the LDG formulation. We refer to Ref. [14] for more details concerning the formulation of the LDG method.

A. The Regularities of the Exact Solution

If we assume that the right side term f satisfies $f \in C^4(\bar{\Omega})$ and certain compatibility conditions (cf. Lemma 2.1, [2], and Theorem 1.26, Part III, [6]), the Problem (1.1) has a classical solution $u \in C^3(\bar{\Omega})$ which can be decomposed into

$$u = \bar{u} + w, \quad w = w_0 + w_1 + w_2, \tag{2.1}$$

and for all $(x, y) \in \Omega$,

$$\left| \frac{\partial^{i+j} \bar{u}}{\partial^i x \partial^j y}(x, y) \right| \leq C \tag{2.2}$$

for $0 \leq i + j \leq 2$ and

$$\left| \frac{\partial^{i+j} w_1}{\partial^i x \partial^j y}(x, y) \right| \leq C\epsilon^{-i} e^{-\alpha(1-x)/\epsilon}, \tag{2.3}$$

$$\left| \frac{\partial^{i+j} w_2}{\partial^i x \partial^j y}(x, y) \right| \leq C\epsilon^{-j} e^{-\alpha(1-y)/\epsilon}, \tag{2.4}$$

$$\left| \frac{\partial^{i+j} w_0}{\partial^i x \partial^j y}(x, y) \right| \leq C\epsilon^{-(i+j)} e^{-\alpha(1-x)/\epsilon} e^{-\alpha(1-y)/\epsilon}, \tag{2.5}$$

for $0 \leq i + j \leq 2$, and constant C only depends on $\boldsymbol{\beta}$, c and f (cf. Theorem 5.1, [20] for more details).

Remark 2.1. Conditions that ensure these regularities are also presented in Ref. [21] and its references. For the upper bounds of higher derivatives, it requires more compatibility of right side term f (cf. [6, 22]). It is a common practice to assume that the solution of Problem (1.1) on the unit square has the above regularities when investigating the uniform convergence of a numerical method on layer-adapted meshes (cf. [6, 18, 23], etc.).

B. The Shishkin Mesh

The tensor product Shishkin mesh $\mathcal{T}_N$ that we shall construct in $\bar{\Omega}$ is the same as in Ref. [1, 2]. Define the transition parameter

$$\tau = \min\left(\frac{1}{2}, \frac{\kappa}{\alpha}\epsilon \ln N\right)$$

with constant $\kappa = 2$. We divide the domain Ω into four subdomains:

$$\Omega_0 = (0, 1-\tau)^2, \qquad \Omega_x = (1-\tau, 1) \times (0, 1-\tau),$$

$$\Omega_y = (0, 1-\tau) \times (1-\tau, 1), \qquad \Omega_{xy} = (1-\tau, 1)^2.$$

Each subdomain is equally decomposed into $N \times N$ rectangles with $N \geq 2$. Therefore there are $(2N+1)^2$ nodes (x_i, y_j), $i, j = 0, \ldots, 2N$. $\mathcal{T}_N$ is composed of $4N^2$ elements and

$$\Omega_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j), \qquad i, j = 1, 2, \ldots, 2N.$$

Let $H = \frac{1-\tau}{N}$, $h = \frac{\tau}{N}$. We always assume that ϵ is so small that $\tau = \frac{\kappa}{\alpha}\epsilon \ln N$, since otherwise N^{-1} is much less than ϵ and the problem can be analyzed by using uniform mesh. Throughout the article, we make a practical assumption that $\epsilon \leq N^{-1}$.

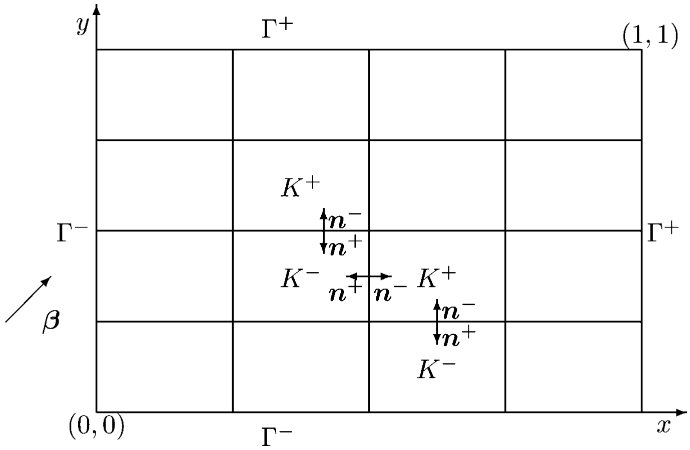

FIG. 1. Notations about the domain and the rectangular elements.

C. The LDG Method

For convenience, we list some of the most commonly used notations (see Fig. 1). We denote by $\mathcal{E}_N$ the set of all edges of the mesh $\mathcal{T}_N$. Let $\mathcal{E}_N^0 \subset \mathcal{E}_N$ be the set of all interior edges. Let $\mathcal{E}_N^\partial \subset \mathcal{E}_N$ be the set of all edges on the boundary $\partial\Omega$ with $\mathcal{E}_N^{\partial-} = \{(x, y) \in \mathcal{E}_N^\partial | \boldsymbol{\beta}(x, y) \cdot \boldsymbol{n}(x, y) < 0\}$, $\mathcal{E}_N^{\partial+} = \{(x, y) \in \mathcal{E}_N^\partial | \boldsymbol{\beta}(x, y) \cdot \boldsymbol{n}(x, y) > 0\}$. Γ denotes the union of the boundaries of all elements K of $\mathcal{T}_N$. $\Gamma^0 = \Gamma \setminus \partial\Omega$. e denotes either a boundary edge or an interior edge shared by elements K^+ and K^-. We denote by $v^\pm$ the function defined on $K^\pm$. $\boldsymbol{n}^\pm$ denote unit normal vectors on e pointing exterior to $K^\pm$ ($\boldsymbol{n}$ denotes the outer normal vector of e). An adjacent perpendicular edge of $e \in K^+ \cap K^-$ was denoted by $e_p^\pm \in \partial K^\pm$. We also use $e_p \in \partial K^+ \cup \partial K^-$ to indicate a general adjacent perpendicular edge of $e \in K^+ \cap K^-$. $|e|$ denotes the length of the edge e. For function v and vector function $\boldsymbol{r}$, we define jumps and averages as follows:

$$
\begin{aligned}
[\![\boldsymbol{r}]\!]_{|e} &= \boldsymbol{r}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{r}^- \cdot \boldsymbol{n}^-, && \text{for all } e \in \mathcal{E}_N^0,\\
[\![v]\!]_{|e} &= v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-, && \text{for all } e \in \mathcal{E}_N^0,\\
[\![v]\!]_{|e} &= v\boldsymbol{n}, && \text{for all } e \in \mathcal{E}_N^\partial,\\
\{\boldsymbol{r}\}_{|e} &= \frac{1}{2}(\boldsymbol{r}^+ + \boldsymbol{r}^-), && \text{for all } e \in \mathcal{E}_N^0,\\
\{\boldsymbol{r}\}_{|e} &= \boldsymbol{r}, && \text{for all } e \in \mathcal{E}_N^\partial,\\
\{v\}_{|e} &= \frac{1}{2}(v^+ + v^-), && \text{for all } e \in \mathcal{E}_N^0,\\
\{v\}_{|e} &= v, && \text{for all } e \in \mathcal{E}_N^\partial.
\end{aligned}
$$

Let $\mathcal{Q}^1(K)$ be the bilinear polynomial space on K. Define finite element spaces $\boldsymbol{V}_N \times M_N$ on the Shishkin mesh as

$$
\begin{aligned}
M_N &= \{v \in L^2(\Omega)|\ v_{|K} \in \mathcal{Q}^1(K),\ \ \forall K \in \mathcal{T}_N\},\\
\boldsymbol{V}_N &= \{\boldsymbol{r} \in (L^2(\Omega))^2|\ \boldsymbol{r}_{|K} \in (\mathcal{Q}^1(K))^2,\ \ \forall K \in \mathcal{T}_N\}.
\end{aligned}
$$

We will use the notations

$$(\varphi,\psi)=\sum_{K\in\mathcal{T}_N}(\varphi,\psi)_K=\sum_{K\in\mathcal{T}_N}\int_K\varphi\psi\,dxdy$$

and

$$\langle\varphi,\psi\rangle_{\mathcal{E}_N}=\sum_{e\in\mathcal{E}_N}\langle\varphi,\psi\rangle_e=\sum_{e\in\mathcal{E}_N}\int_e\varphi\psi\,ds.$$

Recalling the definition of lifting operators $\boldsymbol{\gamma}:\ [L^2(\Gamma)]^2\to\boldsymbol{V}_N$ and $l:\ L^2(\Gamma)\to\boldsymbol{V}_N$ from Ref. [14], we have

$$\int_\Omega\boldsymbol{\gamma}(\boldsymbol{\varphi})\cdot\boldsymbol{r}\,dxdy=-\int_\Gamma\boldsymbol{\varphi}\cdot\{\boldsymbol{r}\}ds,\quad\int_\Omega l(v)\cdot\boldsymbol{r}\,dxdy=-\int_{\Gamma^0}v[\![\boldsymbol{r}]\!]ds\ \ \forall\boldsymbol{r}\in\boldsymbol{V}_N.\qquad(2.6)$$

Furthermore, these two lifting operators could be written piecewisely by defining $\boldsymbol{\gamma}_e:\ [L^1(e)]^2\to\boldsymbol{V}_N$ such that

$$\int_\Omega\boldsymbol{\gamma}_e(\boldsymbol{\varphi})\cdot\boldsymbol{r}\,dxdy=-\int_e\boldsymbol{\varphi}\cdot\{\boldsymbol{r}\}ds,\ \ \forall\boldsymbol{r}\in\boldsymbol{V}_N,\ \ \boldsymbol{\varphi}\in[L^1(e)]^2.\qquad(2.7)$$

Therefore, $\boldsymbol{\gamma}_e(\varphi)$ vanishes outside the union of one or two triangles containing e because $\boldsymbol{r}$ is arbitrary, and $\boldsymbol{\gamma}(\boldsymbol{\varphi})=\sum_{e\in\mathcal{E}_N}\boldsymbol{\gamma}_e(\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi}\in[L^1(\Gamma)]^2$.

LDG Discretization

Introducing a new variable $\boldsymbol{q}=\epsilon\nabla u$, we rewrite the problem (1.1) into a first-order system

$$\begin{cases}\boldsymbol{q}=\epsilon\nabla u, & \text{in }\Omega,\\ -\mathrm{div}\boldsymbol{q}+\boldsymbol{\beta}\cdot\nabla u+cu=f, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases}\qquad(2.8)$$

Then, we will search for the approximate solution $(\boldsymbol{q}_N,u_N)$ in $\boldsymbol{V}_N\times M_N$. We consider the LDG formulation: Find $(\boldsymbol{q}_N,u_N)\in\boldsymbol{V}_N\times M_N$, such that

$$\begin{cases}(\boldsymbol{q}_N,\boldsymbol{r})=-\epsilon(u_N,\nabla\cdot\boldsymbol{r})+\sum\limits_{K\in\mathcal{T}_N}\langle\hat{u},\boldsymbol{r}\cdot\boldsymbol{n}_K\rangle_{\partial K},\\ (\boldsymbol{q}_N,\nabla v)+(u_N,(c-\mathrm{div}\boldsymbol{\beta}-\boldsymbol{\beta}\cdot\nabla)v)\\ \quad-\sum\limits_{K\in\mathcal{T}_N}\langle\widehat{\boldsymbol{q}}\cdot\boldsymbol{n}_K,v\rangle_{\partial K}+\sum\limits_{K\in\mathcal{T}_N}\langle\hat{u}^c,v(\boldsymbol{\beta}\cdot\boldsymbol{n}_K)\rangle_{\partial K}=(f,v)\end{cases}\qquad(2.9)$$

for all $(\boldsymbol{r},v)\in\boldsymbol{V}_N\times M_N$. Here, $\hat{u}^c$ is the upwinding trace, $\widehat{\boldsymbol{q}}$ and $\hat{u}$ are defined as:

$$\begin{aligned}&\hat{u}:=\{u_N\}+\boldsymbol{C}_{12}\cdot[\![u_N]\!], && \text{on }\ \mathcal{E}_N^0,\\ &\widehat{\boldsymbol{q}}:=\{\boldsymbol{q}_N\}-\boldsymbol{C}_{12}[\![\boldsymbol{q}_N]\!]-C_{11}[\![u_N]\!], && \text{on }\ \mathcal{E}_N^0,\\ &\hat{u}:=0, && \text{on }\ \mathcal{E}_N^\partial,\\ &\widehat{\boldsymbol{q}}\cdot\boldsymbol{n}:=\boldsymbol{q}_N\cdot\boldsymbol{n}-C_{11}u_N, && \text{on }\ \mathcal{E}_N^\partial,\end{aligned}\qquad(2.10)$$

where parameter $\boldsymbol{C}_{12} \geq (0, 0)$ are piecewisely defined on $\mathcal{E}_N$. Parameter $C_{11}(e) = \rho\epsilon \min\{|e_p|\}^{-1}$ for all $e \in \mathcal{E}_N$, where the constant ρ is independent of ϵ and N.

Notice that $\hat{u}$, $\hat{u}^c$, $\widehat{\boldsymbol{q}}$ are conservative, i.e., $[\![\hat{u}]\!] = 0$ on $\mathcal{E}_N$, $[\![\hat{u}^c]\!] = 0$, $[\![\widehat{\boldsymbol{q}}]\!] = 0$ on $\mathcal{E}_N^0$. Substituting the definition of numerical fluxes into (2.9), we obtain the primal formulation by eliminating $\boldsymbol{q}_N$:

$$B_N(u_N, v) = f(v), \quad \forall v \in M_N, \tag{2.11}$$

where

$$\begin{aligned} B_N(u_N, v) &= \epsilon(\nabla u_N, \nabla v) - \epsilon\langle [\![u_N]\!] \cdot \{\nabla v\} + \{\nabla u_N\} \cdot [\![v]\!], 1\rangle_{\mathcal{E}_N} \\ &\quad + \epsilon\langle (\boldsymbol{C}_{12} \cdot [\![u_N]\!])[\![\nabla v]\!] + [\![\nabla u_N]\!](\boldsymbol{C}_{12} \cdot [\![v]\!]), 1\rangle_{\mathcal{E}_N^0} \\ &\quad + \epsilon(\boldsymbol{\gamma}([\![u_N]\!]) + l(\boldsymbol{C}_{12} \cdot [\![u_N]\!]), \boldsymbol{\gamma}([\![v]\!]) + l(\boldsymbol{C}_{12} \cdot [\![v]\!])) + c(u_N, v), \\ c(u_N, v) &= (u_N, (c - \operatorname{div}\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \nabla)v) + \langle C_{11}[\![u_N]\!], [\![v]\!]\rangle_{\mathcal{E}_N} + \langle u_N^-, \boldsymbol{\beta} \cdot [\![v]\!]\rangle_{\mathcal{E}_N^0 \cup \mathcal{E}_N^{\partial+}}, \\ f(v) &= (f, v). \end{aligned} \tag{2.12}$$

Orthogonality of $B_N(\cdot, \cdot)$

Recalling the definition of numerical fluxes, we see that all of them are consistent, i.e., $\hat{u}(u) = u|_\Gamma$, $\hat{u}^c(u) = u|_\Gamma$, $\widehat{\boldsymbol{q}}(u, \boldsymbol{q}) = \nabla u|_\Gamma$. Therefore, it is straightforward to show that the bilinear form $B_N(\cdot, \cdot)$ satisfies Galerkin orthogonality:

$$B_N(u - u_N, v) = 0\,, \quad \forall v \in M_N\,. \tag{2.13}$$

Stability of $B_N(\cdot, \cdot)$

Using the scaling argument and direct calculations, we have the imbedding inequality for any $v \in M_N$ (cf. Lemma 3.1, [24]):

$$\|v\|_e \leq C|e_p|^{-\frac{1}{2}}\|v\|_K, \quad e, e_p \in \partial K. \tag{2.14}$$

For any $v \in M_N$, define the DG-norm

$$\|v\|_{DG}^2 = \epsilon|v|_{1,\mathcal{T}_N}^2 + \epsilon|v|_*^2 + \|v\|_{\mathcal{T}_N}^2\,, \tag{2.15}$$

where

$$|v|_{1,\mathcal{T}_N}^2 := \sum_{K \in \mathcal{T}_N} |v|_{1,K}^2\,, \quad \|v\|_{\mathcal{T}_N}^2 := \sum_{K \in \mathcal{T}_N} \|v\|_K^2, \quad |v|_*^2 := \sum_{e \in \mathcal{E}_N} \|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega^2.$$

To prove the stability of B_N, we need the following lemma.

Lemma 2.2. *For any $v \in M_N$, $K \in \mathcal{T}_N$, there exists a constant C independent of ϵ and N such that*

$$\|[\![v]\!]\|_{e\in\partial K} \le 2\sqrt{2}|e_p|^{\frac{1}{2}}\|\boldsymbol{\gamma}_e([\![v]\!])\|_K\,, \qquad (e_p \in \partial K), \tag{2.16a}$$

$$\|[\![v]\!]\|_e \le 2\sqrt{2}\max\{|e_p|\}^{\frac{1}{2}}\|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega\,, \tag{2.16b}$$

$$\|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega \le 3\sqrt{2}\min\{|e_p|\}^{-\frac{1}{2}}\|[\![v]\!]\|_e\,, \tag{2.16c}$$

where $\max\{|e_p|\}$ *(or* $\min\{|e_p|\}$*) is the maximum (or minimum) length of adjacent perpendicular edges to e in the union of the one or two elements containing e.*

Proof. For any $e \in \partial K$, $\varphi \in \mathcal{P}_1(e)$, define $\pi_e(\varphi) \in \mathcal{P}_1(K)$ such that $\pi_e(\varphi)$ is an extension of φ on K (cf. [15]), which is a constant along the line orthogonal to e. Then,

$$\|\pi_e(\varphi)\|_K = \left[\int_K |\pi_e(\varphi)|^2 dxdy\right]^{\frac{1}{2}} = \left[(2|e_p|)\int_e |\varphi|^2 ds\right]^{\frac{1}{2}} = \sqrt{2}|e_p|^{\frac{1}{2}}\|\varphi\|_e.$$

For any $\boldsymbol{\varphi} \in [\mathcal{P}_1(e)]^2$, $\pi_e(\boldsymbol{\varphi})$ is defined as the extension in $[\mathcal{P}_1(K)]^2$. Let $\boldsymbol{\varphi} = [\![v]\!]$, $\boldsymbol{r}_{|K} = \pi_e([\![v]\!])$, $\boldsymbol{r} = 0$ elsewhere in (2.7). Using the above identity we obtain

$$\begin{aligned}
\|[\![v]\!]\|_e^2 &\le 2\int_K |\boldsymbol{\gamma}_e([\![v]\!]) \cdot \pi_e([\![v]\!])|dxdy \\
&\le 2\|\boldsymbol{\gamma}_e([\![v]\!])\|_K\|\pi_e([\![v]\!])\|_K \\
&\le 2\sqrt{2}|e_p|^{\frac{1}{2}}\|[\![v]\!]\|_e\|\boldsymbol{\gamma}_e([\![v]\!])\|_K.
\end{aligned}$$

The inequality (2.16a) follows the above inequality after cancelling $\|[\![v]\!]\|_e$. Noticing that e_p has two different choices if e is an interior edge, (2.16b) holds true automatically.

Conversely, set $\boldsymbol{r} = \boldsymbol{\gamma}_e([\![v]\!])$, $\boldsymbol{\phi} = [\![v]\!]$ in (2.7). By using (2.14),

$$\begin{aligned}
\|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega^2 &\le \|\{\boldsymbol{\gamma}_e([\![v]\!])\}\|_e\|[\![v]\!]\|_e \\
&\le (\|\boldsymbol{\gamma}_e([\![v]\!])\|_{e\in K^+} + \|\boldsymbol{\gamma}_e([\![v]\!])\|_{e\in K^-})\|[\![v]\!]\|_e \\
&\le 3\sqrt{2}\min\{|e_p|\}^{-\frac{1}{2}}\|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega\|[\![v]\!]\|_e,
\end{aligned}$$

which implies (2.16c) after cancelling $\|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega$. ∎

Now, we consider the stability of the primal formulation $B_N(\cdot,\cdot)$ defined in (2.12).

Theorem 2.3. *For any given $v \in M_N$, we have*

$$B_N(v,v) \ge C_D\|v\|_{DG}^2\,, \tag{2.17}$$

where $C_D = \min\{\frac{1}{2}, c_0\}$.

Proof. For any $v \in M_N$, the first term of $B_N(v,v)$ defined in (2.12) gives

$$\epsilon(\nabla v, \nabla v) = \epsilon|v|_{1,\mathcal{T}_N}^2. \tag{2.18}$$

Using Schwarz's inequality, (2.14) and (2.16b) we find an upper bound for the second term of $B_N(v, v)$

$$
\begin{aligned}
&2\epsilon\langle [\![v]\!] \cdot \{\nabla v\}, 1\rangle_{\mathcal{E}_N} \\
&= \epsilon \sum_{e\in\mathcal{E}_N^0} \left[\int_{e\in\partial K^+} (\nabla v)^+ \cdot [\![v]\!] ds - \int_{e\in\partial K^-} (\nabla v)^- \cdot [\![v]\!] ds \right] + 2\epsilon \sum_{e\in\mathcal{E}_N^\partial} \int_{e\in\partial K} \nabla v \cdot [\![v]\!] ds \\
&\le C\epsilon \sum_{e\in\mathcal{E}_N^0} \left[|v|_{1,K^+} \left(|e_p^+|^{-1} \|[\![v]\!]\|^2_{e\in\partial K^+}\right)^{\frac{1}{2}} + |v|_{1,K^-} \left(|e_p^-|^{-1} \|[\![v]\!]\|^2_{e\in\partial K^-}\right)^{\frac{1}{2}} \right] \\
&\quad + C\epsilon \sum_{e\in\mathcal{E}_N^\partial} |v|_{1,K} \left[|e_p|^{-1} \|[\![v]\!]\|^2_{e\in\partial K}\right]^{\frac{1}{2}} \\
&\le C\epsilon \sum_{e\in\mathcal{E}_N^0} [|v|_{1,K^+} \|\gamma_e([\![v]\!])\|_{K^+} + |v|_{1,K^-} \|\gamma_e([\![v]\!])\|_{K^-}] + C\epsilon \sum_{e\in\mathcal{E}_N^\partial} |v|_{1,K} \|\gamma_e([\![v]\!])\|_K \\
&\le C\epsilon |v|_{1,\mathcal{T}_N} |v|_* \le \delta\epsilon |v|^2_{1,\mathcal{T}_N} + C_1\delta^{-1}\epsilon |v|^2_*. \qquad (2.19)
\end{aligned}
$$

Similarly, the third term of $B_N(v, v)$ gives rise to

$$
\begin{aligned}
2\epsilon\langle [\![\nabla v]\!] \boldsymbol{C}_{12} \cdot [\![v]\!], 1\rangle_{\mathcal{E}_N^0} &= 2\epsilon \sum_{e\in\mathcal{E}_N^0} \left[\int_{e\in\partial K^+} \left(\frac{\partial v}{\partial n}\right)^+ \boldsymbol{C}_{12} \cdot [\![v]\!] ds - \int_{e\in\partial K^-} \left(\frac{\partial v}{\partial n}\right)^- \boldsymbol{C}_{12} \cdot [\![v]\!] ds \right] \\
&\le C\epsilon \sum_{e\in\mathcal{E}_N^0} \left[|v|_{1,K^+} \left(\left|e_p^+\right|^{-1} \|[\![v]\!]\|^2_{e\in\partial K^+}\right)^{\frac{1}{2}} + |v|_{1,K^-} \left(\left|e_p^-\right|^{-1} \|[\![v]\!]\|^2_{e\in\partial K^-}\right)^{\frac{1}{2}} \right] \\
&\le C\epsilon \sum_{e\in\mathcal{E}_N^0} [|v|_{1,K^+} \|\gamma_e([\![v]\!])\|_{K^+} + |v|_{1,K^-} \|\gamma_e([\![v]\!])\|_{K^-}] \\
&\le C\epsilon |v|_{1,\mathcal{T}_N} |v|_* \le \delta\epsilon |v|^2_{1,\mathcal{T}_N} + C_2\delta^{-1}\epsilon |v|^2_*, \qquad (2.20)
\end{aligned}
$$

where we assume that $|\boldsymbol{C}_{12}| \sim O(1)$.

The fourth term of $B_N(v, v)$ is always positive because of the symmetry, i.e.,

$$\epsilon(\boldsymbol{\gamma}([\![v]\!]) + l(\boldsymbol{C}_{12} \cdot [\![v]\!]), \boldsymbol{\gamma}([\![v]\!]) + l(\boldsymbol{C}_{12} \cdot [\![v]\!])) \ge 0.$$

Then, we consider the last term $c(v, v)$ of bilinear form $B_N(\cdot, \cdot)$. Applying integration by parts yields

$$
\begin{aligned}
c(v, v) &= \left(v, \left(c - \frac{1}{2}\mathrm{div}\boldsymbol{\beta}\right) v\right) + \langle C_{11}[\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_N} - \frac{1}{2} \sum_{K\in\mathcal{T}_N} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n})v, v\rangle_{\partial K} \\
&\quad + \sum_{e\in\mathcal{E}_N^0} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n})v^-, (v^- - v^+)\rangle_e + \sum_{e\in\mathcal{E}_N^{\partial+}} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n})v, v\rangle_e \\
&= \left\| \left(c - \frac{1}{2}\mathrm{div}\boldsymbol{\beta}\right)^{\frac{1}{2}} v \right\|^2_{\mathcal{T}_N} + \left\| \left(C_{11} + \frac{1}{2}|\boldsymbol{\beta}\cdot\boldsymbol{n}|\right)^{\frac{1}{2}} [\![v]\!] \right\|^2_{\mathcal{E}_N}. \qquad (2.21)
\end{aligned}
$$

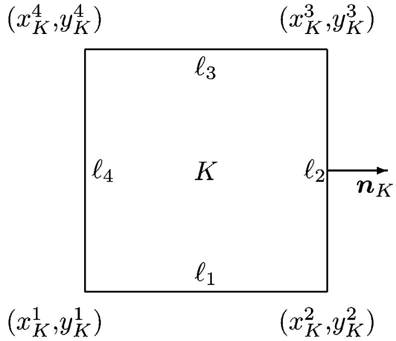

FIG. 2. Notations in a rectangular element.

Combining the error bounds of all terms of $B_N(v, v)$ from (2.18)–(2.21) and Lemma (2.2), we arrive at

$$\begin{aligned}
B_N(v, v) &\geq \epsilon(\nabla v, \nabla v) - 2\epsilon\langle [\![v]\!] \cdot \{\nabla v\}, 1\rangle_{\mathcal{E}_N} + 2\epsilon\langle [\![\nabla v]\!]\boldsymbol{C}_{12} \cdot [\![v]\!], 1\rangle_{\mathcal{E}_N^0} + c(v, v) \\
&\geq (1 - 2\delta)\epsilon|v|_{1,\mathcal{T}_N}^2 + \|c_0 v\|_{\mathcal{T}_N}^2 + \left\| \left(C_{11} + \frac{1}{2}|\boldsymbol{\beta} \cdot \boldsymbol{n}|\right)^{\frac{1}{2}} [\![v]\!] \right\|_{\mathcal{E}_N}^2 - (C_1 + C_2)\delta^{-1}\epsilon|v|_*^2 \\
&\geq (1 - 2\delta)\epsilon|v|_{1,\mathcal{T}_N}^2 + \|c_0 v\|_{\mathcal{T}_N}^2 + \sum_{e\in\mathcal{E}_N} \left(\frac{C_{11}(e)}{18} \min\{|e_p|\} - (C_1 + C_2)\delta^{-1}\epsilon\right) \|\boldsymbol{\gamma}_e([\![v]\!])\|_\Omega^2 \\
&\geq (1 - 2\delta)\epsilon|v|_{1,\mathcal{T}_N}^2 + \|c_0 v\|_{\mathcal{T}_N}^2 + \left(\frac{1}{18}\rho - (C_1 + C_2)\delta^{-1}\right) \epsilon|v|_*^2.
\end{aligned}$$

The proof of (2.17) was completed by taking $\delta = \frac{1}{4}$ and a sufficiently large ρ such that $\frac{1}{18}\rho - (C_1 + C_2)\delta^{-1} \geq \frac{1}{2}$ for the constants C_1, C_2 in the estimate (2.19) and (2.20). ∎

Remark 2.4. As a consequence of Theorem 2.3, the LDG method (2.9) defines a unique approximation solution $(u_N, \boldsymbol{q}_N) \in M_N \times \boldsymbol{V}_N$.

III. INTERPOLATION ERROR ESTIMATES

This section includes all interpolation error estimates needed for the proof of our main result in Section IV. We usually treat the regular term $\bar{u}$ and singular terms w_0, w_1, w_2 separately in the estimation of the errors. In dealing with the singular terms, we use the exponential decay property outside the boundary region (cf. [2, 7, 18]). By the symmetric nature of the problem, we only provide a detailed proof for w_1, and omit the proof of w_2, which can be done in the same way by exchanging x and y; for w_0, we only show the proof in one direction.

A. Preliminaries

On an individual rectangular element $K \in \mathcal{T}_N$ (Fig. 2), we denote by (x_K, y_K) the center of K. (x_K^j, y_K^j), ℓ_j and ϕ_K^j $(j = 1, 2, 3, 4)$ denote four vertices of K, four edges, and bilinear interpolation basis functions, respectively. Let $h_K = x_K^2 - x_K$, $\hbar_K = y_K^3 - y_K$. Then, we can express a function $v \in M_N$ by

$$v(x,y) = \sum_{j=1}^{4} v_K^j \phi_K^j(x,y) = \sum_{j=1}^{4} v_K^j \widehat{\phi}_j(\xi,\eta), \quad (\xi,\eta) \in \widehat{K} = [-1,1]^2, \tag{3.1}$$

where

$$\begin{aligned} v_K^j &= v\big(x_K^j, y_K^j\big), & j &= 1,2,3,4;\\ x &= x_K + h_K\xi, & y &= y_K + \hbar_K\eta,\\ \widehat{\phi}_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta), & \widehat{\phi}_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta),\\ \widehat{\phi}_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta), & \widehat{\phi}_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta). \end{aligned}$$

The Inverse Inequality (cf.* [2]*). One dimensional (1D) If $\varphi(x)$ is a linear function defined on interval $\ell = [-t,t]$, then,

$$\|\varphi_x\|_\ell \le \sqrt{3}t^{-1}\|\varphi\|_\ell. \tag{3.2}$$

(2D) If $v \in M_N$ and $K \in \mathcal{T}_N$, then,

$$\|v_x\|_K \le \frac{3}{h_K}\|v\|_K, \quad \|v_y\|_K \le \frac{3}{\hbar_K}\|v\|_K. \tag{3.3}$$

The Trace Theorem (cf. Theorem 1.6.6,* [25]*). If a domain D has a Lipschitz boundary, then there is a constant C such that

$$\|v\|_{\partial D} \le C\left(\|v\|_D^{\frac{1}{2}}|v|_{1,D}^{\frac{1}{2}} + \|v\|_D\right). \tag{3.4}$$

Anisotropic Interpolation Error Estimates. It is for the bilinear nodal interpolant φ_I of function φ on any $K \in \mathcal{T}_N$ (cf. [26]):

$$\begin{aligned} \|(\theta_\varphi)_x\|_K &\le C(h_K\|\varphi_{xx}\|_K + \hbar_K\|\varphi_{xy}\|_K),\\ \|(\theta_\varphi)_y\|_K &\le C(h_K\|\varphi_{yx}\|_K + \hbar_K\|\varphi_{yy}\|_K). \end{aligned} \tag{3.5}$$

Some inequalities regarding the exponential boundary layer functions:

$$\begin{aligned} & h\sum_{i=N+1}^{2N} e^{-2\alpha(1-x_i)/\epsilon} < \frac{\epsilon}{\alpha}, & & H\sum_{i=1}^{N} e^{-2\alpha(1-x_i)/\epsilon} < \left(\frac{\epsilon}{\alpha} + 2H\right)\frac{1}{N^4},\\ & e^{-2\alpha\tau/\epsilon} = \frac{1}{N^4}, & & H\sum_{i=0}^{N-1} e^{-2\alpha(1-x_i)/\epsilon} < \frac{\epsilon}{N^4}. \end{aligned} \tag{3.6}$$

Some notations for subsets of $\mathcal{E}_N$:

$$\Theta^v = \{e \in \mathcal{E}_N \mid e \text{ is a vertical edge}\},\quad \Theta^h = \{e \in \mathcal{E}_N \mid e \text{ is a horizontal edge}\},$$
$$\Theta_0^v = \{e \subset \overline{\Omega}_0 \mid e \text{ is a vertical edge}\},\quad \Theta_0^h = \{e \subset \overline{\Omega}_0 \mid e \text{ is a horizontal edge}\},$$
$$\Theta_x^v = \{e \subset \overline{\Omega_0 \cup \Omega_x} \setminus \overline{\Omega}_0 \mid e \text{ is vertical}\},\quad \Theta_x^h = \{e \subset \overline{\Omega_0 \cup \Omega_x} \setminus \overline{\Omega}_0 \mid e \text{ is horizontal}\},$$
$$\Theta_y^v = \{e \subset \overline{\Omega_0 \cup \Omega_y} \setminus \overline{\Omega}_0 \mid e \text{ is vertical}\},\quad \Theta_y^h = \{e \subset \overline{\Omega_0 \cup \Omega_y} \setminus \overline{\Omega}_0 \mid e \text{ is horizontal}\},$$
$$\Theta_{xy}^v = \Theta^v \setminus \left(\Theta_0^v \cup \Theta_x^v \cup \Theta_y^v\right),\quad \Theta_{xy}^h = \Theta^h \setminus \left(\Theta_0^h \cup \Theta_x^h \cup \Theta_y^h\right),$$
$$\Theta_1^v = \Theta_0^v \cup \Theta_y^v,\quad \Theta_1^h = \Theta_0^h \cup \Theta_y^h,\quad \Theta_2^v = \Theta_x^v \cup \Theta_{xy}^v,\quad \Theta_2^h = \Theta_x^h \cup \Theta_{xy}^h,$$
$$\Theta_3^v = \Theta_0^v \cup \Theta_x^v,\quad \Theta_3^h = \Theta_0^h \cup \Theta_x^h,\quad \Theta_4^v = \Theta_y^v \cup \Theta_{xy}^v,\quad \Theta_4^h = \Theta_y^h \cup \Theta_{xy}^h,$$
$$L^v = \left\{e \in \Theta_1^v \mid e \subset \{1-\tau\} \times (0,1) \subset \Omega\right\},\quad L^h = \left\{e \in \Theta_3^v \mid e \subset (0,1) \times \{1-\tau\} \subset \Omega\right\}.$$

B. Interpolation Error Estimates on $\mathcal{E}_N$

We shall prove some error estimates for the interpolation on the element boundaries. These results are essential for the proof of the main result.

Lemma 3.1. *There exists a constant C independent of ϵ and N such that*

$$\|\{(\theta_{w_1})_y\}\|_{\Theta_1^h} \le C\frac{1}{N^2},\qquad \|\{(\theta_{w_1})_y\}\|_{\Theta_2^h} \le C,$$
$$\left\|\theta_{w_1}^-\right\|_{(\Theta_0^h \setminus \mathcal{E}_N^{\partial -}) \setminus L^h} \le C\sqrt{\frac{\epsilon}{N}},\qquad \left\|\theta_{w_1}^-\right\|_{\Theta_y^h \cup (\Theta_0^h \cap L^h)} \le C\frac{1}{N^2},$$
$$\left\|\theta_{w_1}^-\right\|_{\Theta_2^h} \le C\sqrt{\epsilon}\frac{(\ln N)^2}{N},$$

and

$$\|\{(\theta_{w_1})_x\}\|_{\Theta_1^v \setminus L^v} \le C\frac{1}{\sqrt{\epsilon}N^{1.5}},\qquad \|\{(\theta_{w_1})_x\}\|_{L^v} \le C\frac{1}{\epsilon N \ln N},$$
$$\|\{(\theta_{w_1})_x\}\|_{\Theta_2^v} \le C\epsilon^{-1}\sqrt{\frac{\ln N}{N}},\qquad \left\|\theta_{w_1}^-\right\|_{(\Theta_1^v \setminus \mathcal{E}_N^{\partial}) \setminus L^v} \le C\frac{\sqrt{\epsilon}}{N^{1.5}},$$
$$\left\|\theta_{w_1}^-\right\|_{L^v} \le C\frac{1}{N^2},\qquad \left\|\theta_{w_1}^-\right\|_{\Theta_2^v} \le C\frac{1}{N^{1.5}\sqrt{\ln N}}.$$

Proof. Recall the decomposition $\Theta^h = \Theta_1^h \cup \Theta_2^h$ and $\Theta^v = \Theta_1^v \cup \Theta_2^v$. We consider the error of $\{(\theta_{w_1})_y\}$ on Θ_1^h and Θ_2^h in the first part of the proof. Then, we move on to the estimation on Θ_1^v and Θ_2^v. We will frequently use the assumption $\epsilon < N^{-1}$.

1. For any edge $e \in \Theta^h$, the average $\{(\theta_{w_1})_y\}$ can be written as

$$\{(\theta_{w_1})_y\} = \{(w_1)_y\} - \{(w_{1_I})_y\} = (w_1)_y - \{(w_{1_I})_y\}.$$

Consider $(w_1)_y$ on Θ_1^h. From the regularity (2.3), we have

$$\|(w_1)_y\|_{\Theta_1^h} \le C\left(2N\int_0^{1-\tau} e^{-2\alpha(1-x)/\epsilon}dx\right)^{\frac{1}{2}} \le C\sqrt{\frac{\epsilon}{2\alpha N^3}} \le C\frac{1}{N^2}. \tag{3.7}$$

To estimate $\{(w_{1_I})_y\}$ on Θ_1^h, we shall analyze $(w_{1_I})_y^+$ on $\Theta_1^h \setminus \mathcal{E}_N^{\partial+}$ and $(w_{1_I})_y^-$ on $\Theta_1^h \setminus \mathcal{E}_N^{\partial-}$. For simplicity, we assume that $K = K^+$, and $\ell_1 \subset \partial K$. From (3.1), we have

$$\begin{aligned}
(w_{1_I})_y|_{\ell_1} &= \big(w_1\big(x_K^4, y_K^4\big) - w_1\big(x_K^1, y_K^1\big)\big)\frac{(x_K + h_K) - x}{4h_K\hbar_K} \\
&\quad + \big(w_1\big(x_K^3, y_K^3\big) - w_1\big(x_K^2, y_K^2\big)\big)\frac{x - (x_K - h_K)}{4h_K\hbar_K} \\
&= \frac{\partial w_1}{\partial y}\big(x_K^1, \eta_1\big)\frac{(x_K + h_K) - x}{2h_K} + \frac{\partial w_1}{\partial y}\big(x_K^2, \eta_2\big)\frac{x - (x_K - h_K)}{2h_K} \\
&\le 2e^{-\alpha(1-x_K^2)/\epsilon},
\end{aligned} \tag{3.8}$$

where $\eta_1, \eta_2 \in (y_K - \hbar, y_K + \hbar)$. It follows that

$$\|(w_{1_I})_y\|_{\ell_1\in\partial K}^2 \le Ch_K e^{-2\alpha(1-x_K^2)/\epsilon}. \tag{3.9}$$

Collecting (3.9) from $\Theta_1^h \setminus \mathcal{E}_N^{\partial+}$ and $\Theta_1^h \setminus \mathcal{E}_N^{\partial-}$ we get that

$$\big\|\big(w_{1_I}^-\big)_y\big\|_{\Theta_1^h\setminus\mathcal{E}_N^{\partial-}} + \big\|\big(w_{1_I}^+\big)_y\big\|_{\Theta_1^h\setminus\mathcal{E}_N^{\partial+}} \le C\left(2N\sum_{i=1}^N He^{-2\alpha(1-x_i)/\epsilon}\right)^{\frac{1}{2}} \le C\frac{1}{N^2}. \tag{3.10}$$

Here, we have used (3.6) and (2.3). Combining (3.8) and (3.10) yields

$$\|\{(\theta_{w_1})_y\}\|_{\Theta_1^h} \le C\frac{1}{N^2}.$$

Next, consider $\{(\theta_{w_1})_y\}\Theta_2^h$. From the definition of the average, it requires to estimate $(\theta_{w_1})_y^+$ on $\Theta_2^h \setminus \mathcal{E}_N^{\partial+}$ and $(\theta_{w_1})_y^-$ on $\Theta_2^h \setminus \mathcal{E}_N^{\partial-}$. We estimate $(\theta_{w_1})_y^+$ on $\Theta_2^h \setminus \mathcal{E}_N^{\partial+}$, but omit the similar proof for $(\theta_{w_1})_y^-$ on $\Theta_2^h \setminus \mathcal{E}_N^{\partial-}$. Let $K = K^+$ and $e \in \partial K$. Using the trace theorem (3.4), (3.5) and the scaling argument we obtain

$$\begin{aligned}
\|(\theta_{w_1})_y\|_e^2 &\le C\big[\hbar_K^{-2}\|(\theta_{w_1})_y\|_K(h_K\hbar_K\|(\theta_{w_1})_{yx}\|_K + \hbar_K^2\|(w_1)_{yy}\|_K) + \hbar_K^{-1}\|(\theta_{w_1})_y\|_K^2\big] \\
&\le C\hbar_K^{-2}[h_K\|(w_1)_{yx}\|_K + \hbar_K\|(w_1)_{yy}\|_K]\big[h_K\hbar_K\|(\theta_{w_1})_{yx}\|_K + \hbar_K^2\|(w_1)_{yy}\|_K^2\big] \\
&\quad + C\hbar_K^{-1}\|(\theta_{w_1})_y\|_K^2 \\
&\le C\hbar_K^{-2}\big[h_K^2\hbar_K\|(w_1)_{yx}\|_K^2 + h_K^2\hbar_K\|(w_{1_I})_{yx}\|_K + \hbar_K^3\|(w_1)_{yy}\|_K^2\big] \\
&\quad + C\hbar_K^{-1}\|(\theta_{w_1})_y\|_K^2.
\end{aligned} \tag{3.11}$$

Collecting (3.11) for all $e \in \Theta_2^h \setminus \mathcal{E}_N^{\partial+}$ yields

$$\begin{aligned}
\|(\theta_{w_1})_y\|^2_{\Theta_2^h\setminus\mathcal{E}_N^{\partial+}} &\le C \sum_{e\in\Theta_2^h\setminus\mathcal{E}_N^{\partial+}} \|(\theta_{w_1})_y\|^2_{e\in\partial K}\\
&\le C \sum_{K\in\Omega_x\cup\Omega_{xy}} \hbar_K^{-2}\big[h_K^2\hbar_K\|(w_1)_{yx}\|_K^2 + h_K^2\hbar_K\|(w_{1_I})_{yx}\|_K + \hbar_K^3\|(w_1)_{yy}\|_K^2\big]\\
&\quad + C\sum_{K\in\Omega_x\cup\Omega_{xy}} \hbar_K^{-1}\|(\theta_{w_1})_y\|_K^2\\
&\le C\big[h\|(w_1)_{yx}\|^2_{\Omega_x\cup\Omega_{xy}} + h\|(w_{1_I})_{yx}\|^2_{\Omega_x\cup\Omega_{xy}} + H\|(w_1)_{yy}\|^2_{\Omega_x\cup\Omega_{xy}}\big]\\
&\quad + C\big[H^{-1}\|(\theta_{w_1})_y\|^2_{\Omega_x} + h^{-1}\|(\theta_{w_1})_y\|^2_{\Omega_{xy}}\big]. \qquad (3.12)
\end{aligned}$$

All these terms in (3.12) can be bounded by applying (3.5) and (2.3):

$$\begin{aligned}
\|(\theta_{w_1})_y\|_{\Omega_x} &\le C\epsilon^{\frac12}, & \|(\theta_{w_1})_y\|_{\Omega_{xy}} &\le C\epsilon,\\
\|(w_1)_{yx}\|_{\Omega_x\cup\Omega_{xy}} &\le C\epsilon^{-\frac12}, & \|(w_1)_{yy}\|_{\Omega_x\cup\Omega_{xy}} &\le C\epsilon^{\frac12},\\
\|(w_{1_I})_{yx}\|_{\Omega_x\cup\Omega_{xy}} &\le C\epsilon^{-\frac12}, & &\text{the proof is similar to (3.8).}
\end{aligned}$$

Therefore, it directly follows from (3.12) that

$$\big\|(\theta_{w_1})_y^+\big\|_{\Theta_2^h} \le C. \qquad (3.13)$$

Similarly, we can prove the same upper bound for $\|(\theta_{w_1})_y^-\|_{\Theta_2^h}$ and conclude that

$$\big\|\{(\theta_{w_1})_y^+\}\big\|_{\Theta_2^h} \le C.$$

Next, consider $\theta_{w_1}^-$ on $(\Theta_0^h \setminus \mathcal{E}_N^{\partial-}) \setminus L^h$. Recall that $\theta_{w_1}^- = w_1 - w_{1_I}^-$. As in (3.7), the error bound of $\|w_1\|_{\Theta_0^h\setminus L^h}$ is derived from

$$\|w_1\|_{\Theta_0^h\setminus L^h} \le C\left(N\int_0^{1-\tau} e^{-\frac{2\alpha(1-x)}{\epsilon}}dx\right)^{\frac12} \le C\frac{\sqrt{\epsilon}}{N^{1.5}}. \qquad (3.14)$$

Notice that if $\epsilon \le N^{-2}$, which implies $N\sqrt{\epsilon} \le 1$, then we have $e^{-\frac{\alpha}{\epsilon N}} \le e^{\frac{-\alpha}{\sqrt{\epsilon}}} \le C\sqrt{\epsilon}$. Otherwise, if $N^{-2} < \epsilon < N^{-1}$, then $e^{-\frac{\alpha}{\epsilon N}} \le e^{-\alpha} \le C$ and $N^{-1} < \sqrt{\epsilon}$. Therefore, using similar argument as for (3.7) we arrive at

$$\begin{aligned}
\|w_{1_I}^-\|_{\Theta_0^h\setminus L^h} &\le C\left(N\sum_{i=1}^{N-1} He^{-\frac{2\alpha(1-x_i)}{\epsilon}}\right)^{\frac12}\\
&\le C\sqrt{N}e^{-\frac{\alpha[1-(1-\tau-H)]}{\epsilon}}\\
&\le C\sqrt{N}e^{-\frac{\alpha\tau}{\epsilon}}e^{-\frac{\alpha}{\epsilon N}} \le C\frac{1}{N^{1.5}}e^{-\frac{\alpha}{\epsilon N}} \le C\sqrt{\frac{\epsilon}{N}}. \qquad (3.15)
\end{aligned}$$

The combination of (3.14) and (3.15) gives rise to the estimation of $\theta_{w_1}^-$ on $(\Theta_0^h \setminus \mathcal{E}_N^{\partial-}) \setminus L^h$. The estimation of $\theta_{w_1}^-$ on $\Theta_y^h \cup (\Theta_0^h \cap L^h)$ and Θ_2^h can be done by (3.5) and the same technique used for $(\theta_{w_1})_y^-$.

2. For any edge $e \in \Theta^v$, the average $\{(\theta_{w_1})_x\}$ can be written as

$$\{(\theta_{w_1})_x\} = \{(w_1)_x\} - \{(w_{1_I})_x\} = (w_1)_x - \{(w_{1_I})_x\}.$$

Consider $(w_1)_x$. Using (3.1) and (2.3), we have

$$\|(w_1)_x\|_{\Theta_1^v \setminus L^v} \le C\frac{1}{\epsilon}\left(\sum_{i=0}^{N-1} e^{-2\alpha(1-x_i)/\epsilon}\right)^{\frac{1}{2}} \le C\frac{1}{\sqrt{\epsilon}N^{1.5}}, \tag{3.16}$$

$$\|(w_1)_x\|_{L^v} \le C\frac{1}{\epsilon}e^{-\alpha\tau/\epsilon} \le C\frac{1}{\epsilon N^2}. \tag{3.17}$$

To estimate $\{(w_{1_I})_x\}$, we use the same technique as in the proof of (3.8). Take a vertical edge $e = \ell_4 \in \partial K^+$. For simplicity, we use $K = K^+$. Then, we have

$$\begin{aligned}(w_{1_I})_x|_{\ell_4} &= \left(w_1\left(x_K^2, y_K^2\right) - w_1\left(x_K^1, y_K^1\right)\right)\frac{(y_K + \hbar_K) - y}{4h_K\hbar_K} \\ &\quad + \left(w_1\left(x_K^3, y_K^3\right) - w_1\left(x_K^4, y_K^4\right)\right)\frac{y - (y_K - \hbar_K)}{4h_K\hbar_K} \\ &\le Ch_K^{-1}e^{-\alpha(1-x_K^2)/\epsilon},\end{aligned}$$

which implies that

$$\|(w_{1_I})_x\|_{\ell_4 \in \partial K}^2 \le Ch_K^{-2}\hbar_K e^{-2\alpha(1-x_K^2)/\epsilon}.$$

As a consequence, using (3.6) and (2.3) we have

$$\begin{aligned}\left\|\left(w_{1_I}^+\right)_x\right\|_{\Theta_1^v \setminus L^v} &\le C\left(H^{-2}\sum_{i=1}^{N} e^{-2\alpha(1-x_i)/\epsilon}\right)^{\frac{1}{2}} \le C\left(\sqrt{\frac{\epsilon}{N}} + \frac{1}{N}\right), \\ \left\|\left(w_{1,I}^-\right)_x\right\|_{(\Theta_1^v \setminus \mathcal{E}_N^{\partial}) \setminus L^v} &\le C\left(H^{-2}\sum_{i=0}^{N-1} e^{-2\alpha(1-x_i)/\epsilon}\right)^{\frac{1}{2}} \le C\sqrt{\frac{\epsilon}{N}}, \\ \left\|\left(w_{1,I}^+\right)_x\right\|_{L^v} &\le Ch^{-1}e^{-\alpha\tau/\epsilon} \le C\frac{1}{\epsilon N\ln N}, \\ \left\|\left(w_{1,I}^-\right)_x\right\|_{L^v} &\le CH^{-1}e^{-\alpha\tau/\epsilon} \le C\frac{1}{N}.\end{aligned} \tag{3.18}$$

Combining (3.16), (3.17), and (3.18) yields

$$\|\{(\theta_{w_1})_x\}\|_{\Theta_1^v \setminus L^v} \le C\frac{1}{\sqrt{\epsilon}N^{1.5}}, \quad \|\{(\theta_{w_1})_x\}\|_{L^v} \le C\frac{1}{\epsilon N\ln N}.$$

Next, consider $\{(\theta_{w_1})_x\}$ on Θ_2^v. According to the definition, it requires to estimate $(\theta_{w_1})_x^+$ on $\Theta_2^v \setminus \mathcal{E}_N^{\partial+}$ and $(\theta_{w_1})_x^-$ on Θ_2^v. The estimation of $(\theta_{w_1})_x^+$ on $\Theta_2^v \setminus \mathcal{E}_N^{\partial+}$ can be bounded in a similar way to the proof of (3.11), (3.12), and (3.13). The estimate of $\theta^-_{w_1}$ on Θ^v can be done by using similar techniques.

∎

Remark 3.2. By the symmetric nature of w_1 and w_2, the estimation of θ_{w_2} on $\mathcal{E}_N$ can be shown by exchanging x and y in the proof of Lemma 3.1.

Lemma 3.3. *There exists a constant C independent of ϵ and N such that*

$$\|\{(\theta_{w_0})_x\}\|_{\Theta_1^v\backslash L^v} \leq C\frac{1}{N^{1.5}}, \qquad \|\{(\theta_{w_0})_x\}\|_{L^v} \leq C\left(\frac{1}{\sqrt{\epsilon}N}+\frac{1}{\epsilon N^{3.5}}\right),$$

$$\|\{(\theta_{w_0})_x\}\|_{\Theta_x^v} \leq C\frac{1}{\epsilon N^2\sqrt{\ln N}}, \qquad \|\{(\theta_{w_0})_x\}\|_{\Theta_{xy}^v} \leq C\sqrt{\frac{\ln N}{\epsilon N}},$$

$$\left\|\theta^-_{w_0}\right\|_{(\Theta_1^v\backslash\mathcal{E}_N^\partial)\backslash L^v} \leq C\left(\frac{\epsilon}{N^{1.5}}+\frac{\sqrt{\epsilon}}{N^4}\right), \qquad \left\|\theta^-_{w_0}\right\|_{L^v} \leq C\left(\frac{\sqrt{\epsilon}}{N^2}+\frac{1}{N^{4.5}}\right),$$

$$\left\|\theta^-_{w_0}\right\|_{\Theta_x^v} \leq C\left(\frac{\sqrt{\epsilon}}{N^{1.5}}+\frac{1}{N^2}\right), \qquad \left\|\theta^-_{w_0}\right\|_{\Theta_{xy}^v} \leq C\sqrt{\epsilon}\left(\frac{\ln N}{N}\right)^{1.5}.$$

Proof. Recalling the decomposition $\Theta^v = \Theta_1^v \cup \Theta_x^v \cup \Theta_{xy}^v$, we have $\{(\theta_{w_0})_x\} = (w_0)_x - \{(w_{0_I})_x\}$ for any edge $e \in \Theta^v$.

Consider $(w_0)_x$. Using (3.1) and (2.5) we have

$$\|(w_0)_x\|^2_{\Theta_1^v\backslash L^v} \leq C\epsilon^{-2}\sum_{i=0}^{N-1} e^{-2\alpha(1-x_i)/\epsilon}\int_0^1 e^{-2\alpha(1-y)/\epsilon}dy \leq C\frac{1}{N^3}, \tag{3.19}$$

$$\|(w_0)_x\|^2_{L^v} \leq C\epsilon^{-2}e^{-2\alpha\tau/\epsilon}\int_0^1 e^{-2\alpha(1-y)/\epsilon}dy \leq C\frac{1}{\epsilon N^4}, \tag{3.20}$$

$$\|(w_0)_x\|^2_{\Theta_x^v} \leq C\epsilon^{-2}\sum_{i=N+1}^{2N} e^{-2\alpha(1-x_i)/\epsilon}\int_0^{1-\tau} e^{-2\alpha(1-y)/\epsilon}dy \leq C\frac{1}{N^3\epsilon\ln N}. \tag{3.21}$$

To estimate $\{(w_{0_I})_x\}$, we use the same technique as in the proof of (3.8). Taking a vertical edge $e = \ell_4 \in \partial K^+$ (for simplicity, let $K = K^+$), we have

$$\begin{aligned}(w_{0_I})_x|_{\ell_4} &= \left(w_0\left(x_K^2, y_K^2\right) - w_0\left(x_K^1, y_K^1\right)\right)\frac{(y_K+\hbar_K)-y}{4h_K\hbar_K} \\ &\quad + \left(w_0\left(x_K^3, y_K^3\right) - w_0\left(x_K^4, y_K^4\right)\right)\frac{y-(y_K-\hbar_K)}{4h_K\hbar_K} \\ &\leq Ch_K^{-1}e^{-\alpha(1-x_K^3)/\epsilon}e^{-\alpha(1-y_K^3)/\epsilon},\end{aligned}$$

which implies

$$\|(w_{0_I})_x\|^2_{\ell_4\in\partial K} \leq Ch_K^{-2}\hbar_K e^{-2\alpha(1-x_K^3)/\epsilon}e^{-\alpha(1-y_K^3)/\epsilon}.$$

Using the above inequality, (3.6) and (2.5) we have

$$\left\|(w_{0_I}^+)_x\right\|^2_{\Theta_1^v\setminus L^v} \le CH^{-2}\sum_{i=1}^{N} e^{-2\alpha(1-x_i)/\epsilon}\left(H\sum_{i=1}^{N} e^{-2\alpha(1-y_i)/\epsilon} + h\sum_{i=N+1}^{2N} e^{-2\alpha(1-y_i)/\epsilon}\right)$$

$$\le C\left(\frac{\epsilon}{N}+\frac{1}{N^2}\right)\left(\epsilon+\frac{1}{N^5}\right) \le C\frac{1}{N^3}, \tag{3.22}$$

$$\left\|(w_{0_I}^-)_x\right\|^2_{(\Theta_1^v\setminus\mathcal{E}_N^{\partial})\setminus L^v} \le CH^{-2}\sum_{i=1}^{N-1} e^{-2\alpha(1-x_i)/\epsilon}\left(H\sum_{i=1}^{N} e^{-2\alpha(1-y_i)/\epsilon} + h\sum_{i=N+1}^{2N} e^{-2\alpha(1-y_i)/\epsilon}\right)$$

$$\le C\left(\frac{\epsilon^2}{N}+\frac{\epsilon}{N^6}\right), \tag{3.23}$$

$$\left\|(w_{0_I}^+)_x\right\|^2_{L^v} \le Ch^{-2}e^{-\alpha\tau/\epsilon}H\sum_{i=1}^{N} e^{-2\alpha(1-y_i)/\epsilon} + h\sum_{i=N+1}^{2N} e^{-2\alpha(1-y_i)/\epsilon}$$

$$\le C\left[\frac{1}{\epsilon(N\ln N)^2}+\frac{1}{\epsilon^2 N^7(\ln N)^2}\right], \tag{3.24}$$

$$\left\|(w_{0_I}^-)_x\right\|^2_{L^v} \le CH^{-2}e^{-\alpha\tau/\epsilon}H\sum_{i=1}^{N} e^{-2\alpha(1-y_i)/\epsilon} + h\sum_{i=N+1}^{2N} e^{-2\alpha(1-y_i)/\epsilon} \le C\left(\frac{\epsilon}{N^2}+\frac{1}{N^7}\right).$$

Combining (3.19), (3.20), and (3.22)–(3.24) gives

$$\|\{(\theta_{w_0})_x\}\|_{\Theta_1^v\setminus L^v} \le C\frac{1}{N^{1.5}}, \quad \|\{(\theta_{w_0})_x\}\|_{L^v} \le C\left(\frac{1}{\sqrt{\epsilon}\ln NN}+\frac{1}{\epsilon N^{3.5}\ln N}\right).$$

Next, consider $\{(\theta_{w_0})_x\}$ on Θ_x^v. By using a different expression of $(w_{0_I})_x$, one has

$$\begin{aligned}(w_{0_I})_x|_{\ell_4} &= \left(w_0(x_K^2, y_K^2) - w_0(x_K^1, y_K^1)\right)\frac{(y_K+\hbar_K)-y}{4h_K\hbar_K}\\ &\quad + \left(w_0(x_K^3, y_K^3) - w_0(x_K^4, y_K^4)\right)\frac{y-(y_K-\hbar_K)}{4h_K\hbar_K}\\ &\le C\epsilon^{-1}e^{-\alpha(1-x_K^3)/\epsilon}e^{-\alpha(1-y_K^3)/\epsilon},\end{aligned}$$

for some $\xi_1, \xi_2 \in [x_K^1, x_K^2]$. Using the above inequality and (2.5) we have

$$\left\|(w_{0_I}^+)_x\right\|^2_{\Theta_x^v\setminus\mathcal{E}_N^{\partial+}} \le C\epsilon^{-2}\sum_{i=N+2}^{2N} e^{-2\alpha(1-x_i)/\epsilon}\sum_{i=1}^{N} He^{-2\alpha(1-y_i)/\epsilon}$$

$$\le C\frac{1}{\ln N}\left(\frac{1}{\epsilon N^3}+\frac{1}{\epsilon^2 N^4}\right),$$

$$\left\|(w_{0_I}^-)_x\right\|^2_{\Theta_x^v} \le C\epsilon^{-2}\sum_{i=N+1}^{2N} e^{-2\alpha(1-x_i)/\epsilon}\sum_{i=1}^{N} He^{-2\alpha(1-y_i)/\epsilon}$$

$$\le C\frac{1}{\ln N}\left(\frac{1}{\epsilon N^4}+\frac{1}{\epsilon^2 N^5}\right). \tag{3.25}$$

Combining (3.21) and (3.25) and using the assumption $\epsilon \le N^{-1}$, we obtain

$$\|\{(\theta_{w_0})_x\}\|_{\Theta_x^v} \le C\frac{1}{\epsilon N^2\sqrt{\ln N}}.$$

Next, we move on to the estimation of $\{(\theta_{w_0})_x\}$ on Θ_{xy}^v. Based on the definition of the average, we need to estimate $(\theta_{w_0})_x^+$ on $\Theta_{xy}^v \setminus \mathcal{E}_N^{\partial +}$ and $(\theta_{w_0})_x^-$ on Θ_{xy}^v. We only estimate $(\theta_{w_0})_x^+$ on $\Theta_{xy}^v \setminus \mathcal{E}_N^{\partial +}$, because the same error bound for $(\theta_{w_0})_x^-$ on Θ_{xy}^v can be shown similarly. Let $K = K^+$. Using (3.4), (3.5) and the same technique as in (3.13), we obtain

$$\|(\theta_{w_0})_x\|_{\Theta_{xy}^v\setminus\mathcal{E}_N^{\partial +}} \le C\sum_{e\in\Theta_{xy}^v\setminus\mathcal{E}_N^{\partial +}} \|(\theta_{w_0})_x\|_{e\in K} \le C\sqrt{\frac{\ln N}{\epsilon N}}, \tag{3.26}$$

where we have used error bounds for the derivatives of w_0 and w_{0_I},

$$\|(w_0)_{xx}\|_{\Omega_{xy}} \le C\epsilon^{-1}, \quad \|(w_0)_{xy}\|_{\Omega_{xy}} \le C\epsilon^{-1}, \quad \|(w_{0_I})_{xy}\|_{\Omega_{xy}} \le C\epsilon^{-1}. \tag{3.27}$$

Therefore, we obtain

$$\|\{(\theta_{w_0})_x\}\|_{\Theta_{xy}^v} \le C\sqrt{\frac{\ln N}{\epsilon N}}.$$

The estimation for $\theta_{w_0}^-$ on Θ^v can be done in a similar way. ■

IV. THE MAIN RESULT

Let $v_I \in M_N$ be the continuous nodal bilinear interpolant of a function $v \in H^2(\Omega)$. In view of $v_I \in M_N \cap C(\overline{\Omega})$, we have $|v - v_I|_* = 0$.

For any $v \in H^1(\mathcal{T}_N)$, we define the ϵ-norm as

$$||v||_\epsilon^2 = \epsilon\|\nabla v\|_{\mathcal{T}_N}^2 + \|v\|_{\mathcal{T}_N}^2,$$

Using the triangle inequality, we have

$$\begin{aligned}\|u - u_N\|_{DG} &\le \|u - u_I\|_{DG} + \|u_I - u_N\|_{DG}\\ &\le ||u - u_I||_\epsilon + \|u_I - u_N\|_{DG}.\end{aligned} \tag{4.1}$$

Now we are going to estimate these two terms individually.

A. The Estimation of $\|u - u_I\|_\epsilon$

The estimate of the interpolation error has been proven in Ref. [1].

Lemma 4.1 ([1]). *If $u_I \in M_N$ is the nodal bilinear interpolant of u, then, there exists a constant C independent of ϵ and N such that*

$$||u - u_I||_\epsilon \le C\frac{\ln N}{N}. \tag{4.2}$$

TABLE I. The estimate of all terms in the primal formulation.

Terms	Estimates	
$\epsilon(\nabla\theta_w, \nabla v) + (\theta_w, (c - \text{div}\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \nabla)v)$	$CN^{-1} \ln N \|v\|_\epsilon$	(4.10)
$\epsilon(\nabla\theta_{\bar{u}}, \nabla v) + (\boldsymbol{\beta} \cdot \nabla\theta_{\bar{u}}, v) + (\theta_{\bar{u}}, cv)$	$CN^{-1}\|v\|_\epsilon$	(4.10)
$-\epsilon < \{\nabla\theta\}, [\![v]\!] >_{\mathcal{E}_N} +\epsilon < [\![\nabla\theta]\!], \boldsymbol{C}_{12} \cdot [\![v]\!] >_{\mathcal{E}_N^0}$	$CN^{-1} \ln N \sqrt{\epsilon}\|v\|_*$	(4.12)
$< \theta_w^-, \boldsymbol{\beta} \cdot [\![v]\!] >_{\mathcal{E}_N^0 \cup \mathcal{E}_N^{\partial +}}$	$CN^{-1}\sqrt{\epsilon}\|v\|_*$	(4.17)
$< (\boldsymbol{\beta} \cdot \boldsymbol{n})\theta_{\bar{u}}, v >_{\mathcal{E}_N^{\partial -}}$	$CN^{-1.5}\|v\|_{DG}$	(4.18)

B. The Estimation of $\|u_I - u_N\|_{DG}$

Lemma 4.2. *If $u_I \in M_N$ is the nodal bilinear interpolant of the solution u, then there exists a constant C independent of ϵ and N such that*

$$\|u_I - u_N\|_{DG} \le C\frac{\ln N}{N}. \tag{4.3}$$

Proof. Using the stability (2.17) and the orthogonality (2.13) yields

$$\|u_I - u_N\|_{DG}^2 \le B_N(u_I - u_N, u_I - u_N) = B_N(u - u_I, u_N - u_I). \tag{4.4}$$

Let $\theta = u - u_I$, then $[\![\theta]\!]_{|e} = 0$ for any $e \in \mathcal{E}_N$. According to (2.12), we get

$$\begin{aligned} B_N(\theta, v) = \epsilon(\nabla\theta, \nabla v) + (\theta, (c - \text{div}\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \nabla)v) - \epsilon\langle\{\nabla\theta\} \cdot [\![v]\!], 1\rangle_{\mathcal{E}_N} \\ + \epsilon\langle[\![\nabla\theta]\!], \boldsymbol{C}_{12} \cdot [\![v]\!]\rangle_{\mathcal{E}_N^0} + \langle\theta^-, \boldsymbol{\beta} \cdot [\![v]\!]\rangle_{\mathcal{E}_N^0 \cup \mathcal{E}_N^{\partial +}}, \end{aligned} \tag{4.5}$$

for any $v \in M_N$. Then, we will decompose this bilinear form into several parts.

Based on the assumption (2.1), we denote by $\bar{u}_I$, w_{0_I}, w_{1_I}, and w_{2_I} the corresponding bilinear interpolants for each component of u. Denote by $\theta_\varphi = \varphi - \varphi_I$ the interpolation error for any $\varphi \in \{u, \bar{u}, w, w_0, w_1, w_2\}$. As a consequence, we obtain

$$B_N(\theta, v) = B_N(\theta_w, v) + B_N(\theta_{\bar{u}}, v), \tag{4.6}$$

where

$$\begin{aligned} B_N(\theta_w, v) = \epsilon(\nabla\theta_w, \nabla v) + (\theta_w, (c - \text{div}\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \nabla)v) - \epsilon\langle\{\nabla\theta_w\}, [\![v]\!]\rangle_{\mathcal{E}_N} \\ + \epsilon\langle[\![\nabla\theta_w]\!], \boldsymbol{C}_{12} \cdot [\![v]\!]\rangle_{\mathcal{E}_N^0} + \langle\theta_w^-, \boldsymbol{\beta} \cdot [\![v]\!]\rangle_{\mathcal{E}_N^0 \cup \mathcal{E}_N^{\partial +}}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} B_N(\theta_{\bar{u}}, v) = \epsilon(\nabla\theta_{\bar{u}}, \nabla v) + (\boldsymbol{\beta} \cdot \nabla\theta_{\bar{u}}, v) + (\theta_{\bar{u}}, cv) - \epsilon\langle\{\nabla\theta_{\bar{u}}\}, [\![v]\!]\rangle_{\mathcal{E}_N} \\ + \epsilon\langle[\![\nabla\theta_{\bar{u}}]\!], \boldsymbol{C}_{12} \cdot [\![v]\!]\rangle_{\mathcal{E}_N^0} - \langle(\boldsymbol{\beta} \cdot \boldsymbol{n})\theta_{\bar{u}}, v\rangle_{\mathcal{E}_N^{\partial -}}. \end{aligned} \tag{4.8}$$

Here, we have applied integration by parts to the second term of $B_N(\theta_{\bar{u}}, v)$.

Therefore, we need to deal with all the terms in (4.7) and (4.8). For simplicity, we move the detailed error estimates of all these terms at the end of this section. The error bounds for these terms are listed in Table I. By (4.6), it follows directly from the combination of the second column of Table I that

$$B_N(u - u_I, v) \le C\frac{\ln N}{N}\|v\|_{DG}, \quad \forall v \in M_N. \tag{4.9}$$

The proof of (4.3) was completed by taking $v = u_N - u_I$ in (4.9) and combining it with (4.4). ■

C. Main Result

Our main result now follows directly from (4.1), (4.2), and (4.3).

Theorem 4.3. *If $u_N \in M_N$ is the LDG solution and u is the solution of (1.1) which satisfies the regularity (2.2)–(2.5), then there exists a constant C independent of ϵ and N such that*

$$\|u - u_N\|_{DG} \le C\frac{\ln N}{N}.$$

Remark 4.4. Comparing the primal formulation to other symmetric primal formulations listed in Ref. [14] (Table 3.2), we can similarly prove that the error estimate of $u - u_N$ also holds for some other discontinuous methods listed in Ref. [14].

Remark 4.5. The error bound in Lemma 4.2 shows that there is no superclose between the LDG solution and the bilinear nodal interpolation, which is different from the finite element method (cf. [2]).

Remark 4.6. The theoretical results depend on the regularity of the solution, which is a very complicated issue. Most of the regularity results for the solution of the Problem (1.1) were proposed on the square domain [6, 7, 27–29]. It is possible to use similar layer-adapted meshes on general polygonal domains, although it seems unlikely to derive a uniform convergence on general polygonal domains in lack of the regularity results.

D. Some Lemmas used in Table I (Section B)

There are four boundary terms in Table I. The error bounds of these terms are proved in the following three lemmas.

Lemma 4.7. *There exists a constant C independent of ϵ and N such that for any $v \in M_N$*

$$\begin{aligned} &\epsilon(\nabla\theta_w, \nabla v) + (\theta_w, (c - \mathrm{div}\boldsymbol{\beta} - \boldsymbol{\beta}\cdot\nabla)v) \le C\frac{\ln N}{N}\|v\|_\epsilon, \\ &\epsilon(\nabla\theta_{\bar{u}}, \nabla v) + (\boldsymbol{\beta}\cdot\nabla\theta_{\bar{u}}, v) + (\theta_{\bar{u}}, cv) \le C\frac{1}{N}\|v\|_\epsilon. \end{aligned} \tag{4.10}$$

Proof. Using the error estimates of Lemma 4.1 we have

$$\epsilon(\nabla\theta_w, \nabla v) \le \sqrt{\epsilon}|\theta_w|_{1,\mathcal{T}_N} \cdot \sqrt{\epsilon}|v|_{1,\mathcal{T}_N} \le C\frac{\ln N}{N}\|v\|_\epsilon,$$

$$\epsilon(\nabla\theta_{\bar{u}}, \nabla v) \le \sqrt{\epsilon}|\theta_{\bar{u}}|_{1,\mathcal{T}_N} \cdot \sqrt{\epsilon}|v|_{1,\mathcal{T}_N} \le C\frac{\sqrt{\epsilon}}{N}\|v\|_\epsilon,$$

$$(\vec{\beta}\cdot\nabla\theta_{\bar{u}}, v) \le CH\|\bar{u}\|_{2,\Omega}\|v\|_{\mathcal{T}_N} \le C\frac{1}{N}\|v\|_\epsilon,$$

$$(\theta_{\bar{u}}, cv) \le CH^2\|\bar{u}\|_{2,\Omega}\|v\|_{\mathcal{T}_N} \le C\frac{1}{N^2}\|v\|_\epsilon. \tag{4.11}$$

The following estimate was already proved in Ref. [2] (Theorem 5.1),

$$(\theta_w, (c - \operatorname{div}\boldsymbol{\beta} - \boldsymbol{\beta}\cdot\nabla)v) \le C\frac{\ln N}{N}\|v\|_\epsilon,$$

which, combined with (4.11), establishes (4.10). ∎

Lemma 4.8. *There exists a constant C independent of ϵ and N such that*

$$\epsilon\langle\{\nabla\theta\}, [\![v]\!]\rangle_{\mathcal{E}_N} + \epsilon < [\![\nabla\theta]\!], \boldsymbol{C}_{12}\cdot[\![v]\!] >_{\mathcal{E}_N^0} \le C\frac{\ln N}{N}\|v\|_\epsilon\ , \tag{4.12}$$

for any $v \in M_N$.

Proof. Recall the decomposition $u = \bar{u} + w_1 + w_2 + w_0$. We shall estimate each component of u individually. Also, the estimate is broken into two subset of ε_N, i.e., Θ^v and Θ^h. Applying Schwarz's inequality and Lemma 3.7 one has

$$\begin{aligned}
\epsilon\sum_{e\in\Theta^v}\langle\{\nabla\theta_{w_1}\}, [\![v]\!]\rangle_e &\le C\epsilon\sqrt{H}\|\{(\theta_{w_1})_x\}\|_{\Theta_1^v\setminus L^v}|v|_{*,\Theta_1^v\setminus L^v} + C\epsilon\sqrt{h}\|\{(\theta_{w_1})_x\}\|_{L^v}|v|_{*,L^v} \\
&\quad + C\epsilon\sqrt{h}\|\{(\theta_{w_1})_x\}\|_{\Theta_2^v}|v|_{*,\Theta_2^v} \\
&\le C\epsilon\left[\frac{1}{\sqrt{N}}\frac{1}{\sqrt{\epsilon}N^{1.5}} + \sqrt{\frac{\epsilon\ln N}{N}}\frac{1}{\epsilon N\ln N} + \frac{\ln N}{N\sqrt{\epsilon}}\right]|v|_{*,\Theta^v} \\
&\le C\frac{\ln N}{N}\sqrt{\epsilon}|v|_{*,\Theta^v}\ ,
\end{aligned}$$

and

$$\begin{aligned}
\epsilon\sum_{e\in\Theta^h}\langle\{\nabla\theta_{w_1}\}, [\![v]\!]\rangle_e &\le C\epsilon\sqrt{H}\|\{(\theta_{w_1})_y\}\|_{\Theta_1^h}|v|_{*,\Theta_1^h} + C\epsilon\sqrt{H}\|\{(\theta_{w_1})_y\}\|_{\Theta_2^h}|v|_{*,\Theta_2^h} \\
&\le C\epsilon\left[\sqrt{\frac{1}{N}}\frac{1}{N^2} + \sqrt{\frac{1}{N}}\right]|v|_{*,\Theta^h} \\
&\le C\frac{1}{N}\sqrt{\epsilon}|v|_{*,\Theta^h}.
\end{aligned}$$

Here, we used the assumption $\epsilon < N^{-1}$. Combining the above two inequalities together yields

$$\epsilon\langle\{\nabla\theta_{w_1}\}, [\![v]\!]\rangle_{\mathcal{E}_N} \le C\frac{\ln N}{N}\|v\|_{DG} . \quad (4.13)$$

The term w_2 has the same upper bound as the term w_1 because of the regularity (2.4) and the symmetric property of these two components of u.

Next, consider θ_{w_0}. Using Lemma 3.3, one has

$$\begin{aligned}\epsilon\sum_{e\in\Theta^v}\langle\{\nabla\theta_{w_0}\}, [\![v]\!]\rangle_e &\le C\epsilon\sqrt{H}\|\{(\theta_{w_0})_x\}\|_{\Theta_1^v}|v|_* + C\epsilon\sqrt{h}\|\{(\theta_{w_0})_x\}\|_{\Theta_x^v\cup\Theta_{xy}^v\cup L^v}|v|_*\\ &\le C\left[\sqrt{\epsilon}\frac{\ln N}{N} + \frac{1}{N^2}\right]\sqrt{\epsilon}|v|_*. \end{aligned} \quad (4.14)$$

The error bound of $\|\{\theta_{w_0}\}\|_{\Theta^h}$ will be the same as in (4.14) because of the symmetric property of the regularity result (2.5). Therefore, we get that

$$\epsilon\langle\{\nabla\theta_{w_0}\}, [\![v]\!]\rangle_{\mathcal{E}_N} \le C\left(\sqrt{\epsilon}\frac{\ln N}{N} + \frac{1}{N^2}\right)\|v\|_\epsilon. \quad (4.15)$$

It remains to estimate the smooth component $\bar{u}$. Using (2.2) and (2.14), we get that

$$\epsilon\langle\{\nabla\theta_{\bar{u}}\}, [\![v]\!]\rangle_{\mathcal{E}_N} \le C\epsilon\sum_{e\in\mathcal{E}_N}\int_e|[\![v]\!]|ds \le C\epsilon H\left(\sum_{e\in\mathcal{E}_N}\|\gamma_e([\![v]\!])\|_e^2\right)^{\frac{1}{2}} \le C\frac{\sqrt{\epsilon}}{N}\|v\|_\epsilon. \quad (4.16)$$

Combining (4.13)–(4.16) completes the estimation of the first term of (4.12). Noticing the definition $[\![\nabla\theta]\!] = \frac{\partial\theta^+}{\partial \boldsymbol{n}^+} + \frac{\partial\theta^-}{\partial \boldsymbol{n}^-}$, we apply the same technique as above to yield the same error bound for the second term of (4.12). ■

Lemma 4.9. *If $|\boldsymbol{C}_{12}| \sim O(1)$, then there is a constant C independent of ϵ and N such that for any $v \in M_N$*

$$\langle\theta_w^-, \boldsymbol{\beta}\cdot[\![v]\!]\rangle_{\mathcal{E}_N^0\cup\mathcal{E}_N^{\partial+}} \le C\frac{1}{N}\sqrt{\epsilon}|v|_* , \quad (4.17)$$

$$\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\theta_{\bar{u}}, v\rangle_{\mathcal{E}_N^{\partial-}} \le C\frac{1}{N^{1.5}}\|v\|_{DG} . \quad (4.18)$$

Proof.

1. To estimate (4.17), we recall the decomposition of $u = \bar{u} + w_1 + w_2 + w_0$. We will estimate each term individually.

   By using the same technique as in (4.13) and Lemma 3.1, we have

$$\begin{aligned}\langle\theta_{w_1}^-, \boldsymbol{\beta}\cdot[\![v]\!]\rangle_{\mathcal{E}_N^0\cup\mathcal{E}_N^{\partial+}} &\le C\big[\sqrt{H}\|\theta_{w_1}^-\|_{(\Theta_1^v\backslash\mathcal{E}_N^{\partial-})\backslash L^v} + \sqrt{h}\|\theta_{w_1}^-\|_{L^v} + \sqrt{h}\|\theta_{w_1}^-\|_{\Theta_2^v}\big]|v|_{*,\Theta^v}\\ &+ C\big[\sqrt{H}\|\theta_{w_1}^-\|_{(\Theta_0^h\backslash\mathcal{E}_N^{\partial-})\backslash L^h} + \sqrt{h}\|\theta_{w_1}^-\|_{\Theta_y^h\cup(\Theta_0^h\cap L^h)} + \sqrt{H}\|\theta_{w_1}^-\|_{\Theta_2^h}\big]|v|_{*,\Theta^h}\end{aligned}$$

$$\leq C\left[\frac{1}{\sqrt{N}}\frac{\sqrt{\epsilon}}{N^{1.5}}+\sqrt{\frac{\epsilon\ln N}{N}}\frac{1}{N^2}+\sqrt{\frac{\epsilon\ln N}{N}}\frac{1}{N^{1.5}\sqrt{\ln N}}\right]|v|_{*,\Theta^v}$$
$$+C\left[\sqrt{\frac{1}{N}}\sqrt{\frac{\epsilon}{N}}+\sqrt{\frac{\epsilon\ln N}{N}}\frac{1}{N^2}+\sqrt{\frac{1}{N}}\frac{\sqrt{\epsilon}(\ln N)^2}{N^2}\right]|v|_{*,\Theta^h}$$
$$\leq C\frac{1}{N}\sqrt{\epsilon}|v|_*. \tag{4.19}$$

Similarly, using (2.4) and the decomposition $\mathcal{E}^N = \Theta_3 \cup \Theta_4$, we see that the same error bound holds for w_2.

Next, consider the term θ_{w_0} on $(\mathcal{E}_N^0 \cup \mathcal{E}_N^{\partial+}) \cap \Theta^v$. Using regularity (2.5), Lemma 3.3 and the assumption $\epsilon < N^{-1}$, we obtain

$$\left\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\theta_{w_0}^-, [\![v]\!]\right\rangle_{(\mathcal{E}_N^0\cup\mathcal{E}_N^{\partial+})\cap\Theta^v} \leq C\sqrt{H}\left\|\theta_{w_0}^-\right\|_{(\Theta_1^v\backslash\mathcal{E}_N^{\partial-})\backslash L^v}|v|_{*,(\Theta_1^v\backslash\mathcal{E}_N^{\partial-})\backslash L^v} + C\sqrt{h}\left\|\theta_{w_0}^-\right\|_{L^v}|v|_{*,L^v}$$
$$+C\sqrt{h}\left\|\theta_{w_0}^-\right\|_{\Theta_x^v}|v|_{*,\Theta_x^v}+C\sqrt{h}\left\|\theta_{w_0}^-\right\|_{\Theta_{xy}^v}|v|_{*,\Theta_{xy}^v}$$
$$\leq C\left[\frac{1}{N^{2.5}}+\sqrt{\epsilon}\left(\frac{\ln N}{N}\right)^2\right]\sqrt{\epsilon}|v|_*. \tag{4.20}$$

The estimation for $\theta_{w_0}^-$ on Θ^h can be done similarly because of the symmetric property of the regularity (2.5). Combining (4.19) and (4.20) completes the proof of (4.17).

2. Next, we consider $\theta_{\bar{u}}$ on $\mathcal{E}_N^{\partial-}$. Using the standard approximation theory and (2.14), one has

$$\langle(\boldsymbol{\beta}\cdot\boldsymbol{n})\theta_{\bar{u}}, v\rangle_{\mathcal{E}_N^{\partial-}} \leq C\frac{1}{\sqrt{H}}\|\theta_{\bar{u}}\|_{\mathcal{E}_N^{\partial-}\cap\partial\Omega_0}\|v\|_{\Omega_0} + C\sqrt{H}\|\theta_{\bar{u}}\|_{\mathcal{E}_N^{\partial-}\backslash\partial\Omega_0}|v|_{*,\mathcal{E}_N^{\partial-}\backslash\partial\Omega_0}$$
$$\leq C\sqrt{N}H^2\|v\|_{\Omega_0}+C\sqrt{H}h^2|v|_*$$
$$\leq C\frac{1}{N^{1.5}}\|v\|_{DG}.$$

The proof of (4.18) is completed.

■

V. NUMERICAL EXPERIMENTS

In this section, we present numerical results for the LDG solution of the problem (1.1). Let $\boldsymbol{\beta} = (1, 1)$, $c = 0$ such that the exact solution is then given by

$$u = xy(1-e^{-(1-x)/\epsilon})(1-e^{-(1-y)/\epsilon}).$$

We use two types of numerical fluxes by taking different values of the parameters C_{11} and $\boldsymbol{C}_{12}$ (see Table II). Let $\epsilon = 10^{-1}$, 10^{-4}, and 10^{-7}. The parameter κ in the transition number of the Shishkin mesh is chosen to be $\kappa = 2$. Denote by $E_{DG}^j(N)$ and $E_\epsilon^j(N)$, $j = 1, 2, 3$ the errors in

TABLE II. Two parameters in numerical fluxes.

Parameter	Type 1	Type 2
$C_{11}(e)$	$\epsilon\, \min(h_o^+, h_o^-)^{-1}$	$\epsilon\, \min(h_o^+, h_o^-)^{-1}$ on Γ_+, 0 elsewhere.
$\boldsymbol{C}_{12}(e)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$

TABLE III. The error and convergence rate of $\|u - u_N\|_{DG}$.

	Numerical fluxes type 1					
	$\epsilon = 10^{-1}$		$\epsilon = 10^{-4}$		$\epsilon = 10^{-7}$	
N	$\|\|u - u_N\|\|_{DG}$	Rate	$\|\|u - u_N\|\|_{DG}$	Rate	$\|\|u - u_N\|\|_{DG}$	Rate
4	2.18e − 01	–	2.36e − 01	–	2.36e − 01	–
8	1.67e − 01	0.93	1.82e − 01	0.90	1.82e − 01	0.90
16	1.04e − 01	1.17	1.25e − 01	0.93	1.25e − 01	0.93
32	5.38e − 02	1.40	7.99e − 02	0.95	8.00e − 02	0.95
64	2.74e − 02	1.32	4.89e − 02	0.96	4.89e − 02	0.96
	Numerical fluxes type 2					
4	3.34e − 01	–	3.74e − 01	–	3.74e − 01	–
8	2.61e − 01	0.86	2.90e − 01	0.88	2.90e − 01	0.88
16	1.65e − 01	1.13	1.99e − 01	0.93	1.99e − 01	0.93
32	8.53e − 02	1.40	1.27e − 01	0.96	1.27e − 01	0.96
64	4.33e − 02	1.33	7.75e − 02	0.97	7.75e − 02	097

the DG-norm and ϵ-norm, where the index j indicates three values of $\epsilon = 10^{-1}, 10^{-4}, 10^{-7}$. The rates of convergence are computed from the formula

$$\log_p \frac{E^j_{DG}(N)}{E^j_{DG}(2N)}, \quad \text{and} \quad \log_p \frac{E^j_\epsilon(N)}{E^j_\epsilon(2N)}, \qquad j = 1, 2, 3.$$

TABLE IV. The error and convergence rate of $\|u - u_N\|_\epsilon$.

	Numerical fluxes type 1					
	$\epsilon = 10^{-1}$		$\epsilon = 10^{-4}$		$\epsilon = 10^{-7}$	
N	$\|\|u - u_N\|\|_\epsilon$	Rate	$\|\|u - u_N\|\|_\epsilon$	Rate	$\|\|u - u_N\|\|_\epsilon$	–
4	1.06e − 01	–	1.17e − 01	–	1.16e − 01	–
8	8.03e − 02	0.96	8.73e − 02	0.99	8.73e − 02	0.99
16	4.93e − 02	1.20	5.91e − 02	0.96	5.91e − 02	0.96
32	2.50e − 02	1.44	3.74e − 02	0.97	3.74e − 02	0.97
64	1.26e − 02	1.34	2.26e − 02	0.98	2.26e − 02	0.99
	Numerical fluxes type 2					
4	1.17e − 01	–	1.29e − 01	–	1.29e − 01	–
8	8.88e − 02	0.96	9.71e − 02	0.99	9.71e − 02	0.99
16	5.44e − 02	1.21	6.55e − 02	0.97	6.55e − 02	0.97
32	2.75e − 02	1.45	4.13e − 02	0.98	4.13e − 02	0.98
64	1.38e − 02	1.35	2.49e − 02	0.99	2.49e − 02	0.99

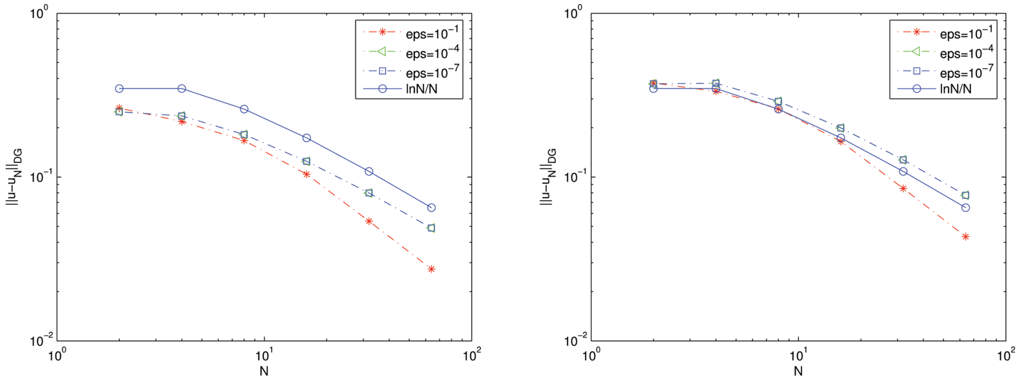

FIG. 3. $||u - u_N||_{DG}$, using numerical fluxes type 1 and type 2. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Here, we used $p = 2(\ln N / \ln 2N)$. Tables III and IV display the data of E^j_{DG} and its convergence rates, $E^j_\epsilon(N)$ and its convergence rates for different values of ϵ. The error curves indicate that the rates of convergence are of $O(\frac{\ln N}{N})$ for these two types of numerical fluxes (see Figs. 3 and 4). In these rates of convergence, the factor $\ln N$ is not removable. Our numerical results illustrate that the rate of convergence of Theorem 4.3 is sharp.

VI. CONCLUDING REMARKS

The LDG method is applied to 2D singularly perturbed convection-diffusion problems with exponential boundary layers. The Shishkin mesh is used to resolve the boundary layers. As a result, we obtained an uniformly valid convergence rate $O(\frac{\ln N}{N})$ for the error, where $2N$ is the number of mesh intervals in each coordinate direction. The error is measured by a DG-norm, which is introduced from the primal formulation of the LDG discretization. The technique used in the error estimate can be extended to the analysis of some other DG methods. The uniform convergence of the LDG approximation to the solution of problem (1.1) using higher-degree approximation polynomial spaces is an ongoing work.

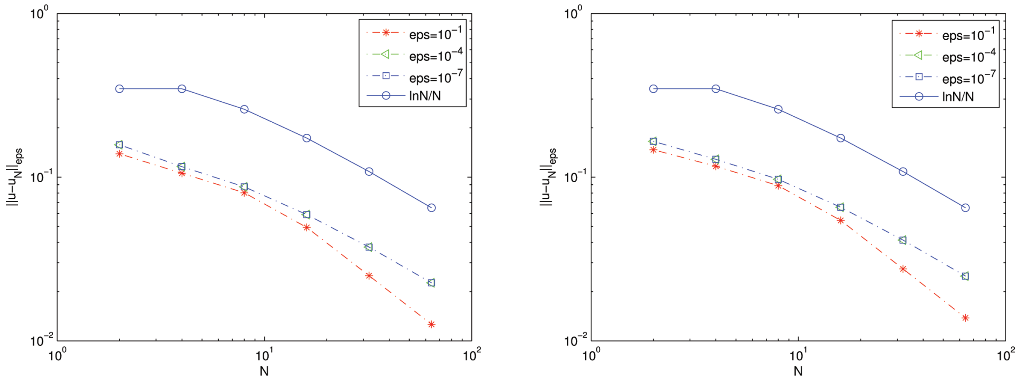

FIG. 4. $||u - u_N||_\epsilon$, using numerical fluxes type 1 and type 2. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

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