INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 2, Pages 350–373

© 2013 Institute for Scientific
Computing and Information

LOCAL ERROR ESTIMATES OF THE LDG METHOD FOR 1-D SINGULARLY PERTURBED PROBLEMS

HUIQING ZHU AND ZHIMIN ZHANG

Abstract. In this paper local discontinuous Galerkin method (LDG) was analyzed for solving 1-D convection-diffusion equations with a boundary layer near the outflow boundary. Local error estimates are established on quasi-uniform meshes with maximum mesh size h. On a subdomain with $O(h\ln(1/h))$ distance away from the outflow boundary, the L^2 error of the approximations to the solution and its derivative converges at the optimal rate $O(h^{k+1})$ when polynomials of degree at most k are used. Numerical experiments illustrate that the rate of convergence is uniformly valid and sharp. The numerical comparison of the LDG method and the streamline-diffusion finite element method are also presented.

Key words. Local discontinuous Galerkin method, singularly perturbed, local error estimates.

1. Introduction

We are interested in the convection-diffusion problem

$$\begin{aligned} -\epsilon u'' + au' + bu &= f \qquad \text{in } \mathcal{I} = (0,1), \\ u &= 0 \qquad \text{on } \partial\mathcal{I} = \{0,1\}, \end{aligned} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is the diffusion parameter, $a = a(x) \geq \alpha > 0$ accounts for the convection, and $b = b(x)$ accounts for the reaction term. The function $f = f(x)$ is a given source term. We assume that α is a constant; a, b, and f are sufficiently smooth on $\overline{\mathcal{I}}$.

When ϵ is small, the solution to Problem (1.1) typically has a boundary layer with width $O(\epsilon \ln \frac{1}{\epsilon})$ at $x = 1$. The standard finite element method produces numerical solutions that exhibits nonphysical oscillation on uniform mesh unless the mesh size is comparable with ϵ. Many techniques have been developed to eliminate the nonphysical oscillation (c.f. [1, 11, 12, 15, 16]). Among these techniques is the streamline-diffusion finite element method (SDFEM) proposed in eighties by Hughes et.al. (c.f. [12]) by adding an appropriate amount of artificial diffusion in the streamline direction to stabilize the conforming finite element method. The SDFEM is quite satisfactory for practical situations, but may lead to large artificial layers near boundaries and discontinuities. There has been many theoretical results published up to now (c.f. [6, 14, 16]). Another technique is to employ a layer-adapted mesh based on the a priori knowledge of Problem (1.1), such as Shishkin-type meshes, Bakhvalov-type meshes (c.f. [15, 16, 20, 21]).

Starting from 1970's, discontinuous Galerkin methods has been intensively studied and applied to hyperbolic and convection-dominated elliptic problems with great success (c.f. [7, 8, 9, 13]). Recently, the superconvergence of the numerical traces and the L^2 convergence of DG methods have been discussed for one-dimensional convection-diffusion problems (c.f. [4, 5, 18, 19, 21]). It has been reported in the

Received by the editors November 21, 2011 and, in revised form, February 7, 2012.
1991 *Mathematics Subject Classification.* 65L10, 65L20, 65L60,65M50.
This research was supported in part by the US National Science Foundation through grants DMS-0612908 and DMS-1115530 .

numerical experiments of [18] that the error curves of numerical traces didn't show any any oscillation even on uniform meshes if mesh size h is comparable with ϵ, or if $\epsilon \ll h$ is extremely small. It implies that the local discontinuous Galerkin method (LDG) seems not to produce a large artificial layers as SDFEM did outside the boundary layer region of Problem (1.1). Motivated by this finding, we are interested in investigating the LDG method for Problem (1.1) on uniform or quasi-uniform meshes to see how efficient it could be.

In this work, we proved that the L^2 errors of $u' - \epsilon^{-1}Q$ and $u - U$ converge at the optimal rate $O(h^{k+1})$ on a subdomain $\mathcal{I}_0 \subset \mathcal{I}$ where $\partial\mathcal{I}_0$ is $O(h\ln(1/h))$ distance away from the outflow boundary of $\mathcal{I}$, i.e., $x = 1$. Here (U, Q) denotes the LDG approximation of $(u, \epsilon u')$; h denotes the maximum mesh size; and approximation space consists of piecewise polynomials of degree at most k. These rates of convergence are uniformly valid in terms of the singular perturbation parameter ϵ, as verified by our numerical experiments. The numerical comparison of LDG and SDFEM are also presented in this paper. The numerical results in Section 4 illustrate that the L^2 errors of the LDG approximations to the exact solution and its derivative on $\mathcal{I}_0$ are smaller comparing with the L^2 errors of SDFEM on the same subdomain $\mathcal{I}_0$. For a fixed uniform mesh, the subdomain $\mathcal{I}_0$ of the LDG method expands and contains more mesh elements as the parameter $\epsilon \to 0$. If $\epsilon \ll h$ is extremely small, the error curves of numerical traces will not show any oscillation. Furthermore, numerical results shows that a small artificial layer does exist for small ϵ if the mesh size h is not very large.

On the other hand, the subdomain $\mathcal{I}_0$ of SDFEM expands slower than LDG and the artificial layer always contains $(k+1)\ln N$ mesh elements as $\epsilon \to 0$. Therefore, its nodal error curves will always show an oscillation near the outflow boundary $x = 1$ even if ϵ is extremely small. This finding, then, seems to support the former view in [18] that the DG method is more 'local' than finite element method.

The outline of this article is as follows: In Section 2, we present the LDG discretization and state our main results, which give some local error estimates. The proof of the main results is carried out in details in Section 3. In section 4, we present several numerical experiments testing our theoretical results. We end in Section 5 with some concluding remarks.

Notations. Throughout this article, the letter C will denote a generic constant not necessarily the same at each occurrence. It might depend on the coefficient functions a, b, the right-hand side function f, and the polynomial degree k, but is independent of the singular perturbation parameter ϵ and the mesh. For any measurable subdomain $D \subseteq \mathcal{I}$, we use the standard Sobolev spaces $L^2(D)$, $H^1(D)$, $H^s(D) = W_2^s(D)$ for some nonnegative integer s.

2. The LDG discretization and main results

In this section, we present the LDG discretization and state our main results. We begin with partitioning the domain $\mathcal{I}$. If $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$, we denote by $\mathcal{I}_h = \{I_j = (x_{j-1}, x_j), j = 1, 2, \cdots, N\}$ a quasi-uniform partition of domain $\mathcal{I}$, and by $h_j = x_j - x_{j-1}$ the length of the j-th element. Let $h = \max_{j=1,\cdots,N} h_j$. For any $j = 1, 2, \cdots, N$, there exists a constant C_q such that $h_j \geq C_q h$. Define $v(x_j^{\pm}) = \lim_{\delta\to 0} v(x_j \pm \delta)$ as in [13]. For each element $I_j \in \mathcal{I}_h$, we set its outward unit normal $n_{I_j}(x_j) = 1$ and $n_{I_j}(x_{j-1}) = -1$. We denote $v_j = v(x_j)$, $v_j^{\pm} = v(x_j^{\pm})$, $[\![v_0]\!] = -v_0^+$ and $[\![v_N]\!] = v_N^-$, $[\![v_j]\!] = v_j^- n_{I_j}(x_j) + v_j^+ n_{I_{j+1}}(x_j) = v_j^- - v_j^+$ for $j = 1, \cdots, N-1$.

We denote by $H^m(\mathcal{I}_h)$ the space of functions on $\mathcal{I}$ whose restriction to each element I belongs to the Sobolev space $H^m(I)$. V_h denotes the finite dimensional space of functions that are polynomial of degree degree at most k on each element. For any $\mathcal{D} \subseteq \mathcal{I}_h$ the Sobolev seminorm on $H^s(\mathcal{D})$ is defined as

$$|v|_{s,\mathcal{D}} := (v^{(s)}, v^{(s)})_{\mathcal{D}}^{1/2}.$$

Accordingly, the Sobolev norm on $H^r(\mathcal{D})$ is defined as

$$\|v\|_{r,\mathcal{D}} := \left(\sum_{s=0}^{r} |v|_{s,D}^2\right)^{1/2}.$$

We drop the first subscript whenever $r = 0$, and drop the second subscript if $\mathcal{D} = \mathcal{I}_h$.

Introducing a new variable $q = \epsilon u'$, we rewrite the problem (1.1) as a system of one-order equations

$$\begin{aligned} q &= \epsilon u' && \text{in } \mathcal{I} = (0,1), \\ -q' + au' + bu &= f && \text{in } \mathcal{I} = (0,1), \\ u &= 0 && \text{on } \partial\mathcal{I} = \{0,1\}. \end{aligned}$$

Then we will search for the approximate solution (Q, U) of the LDG method from a finite-dimensional subspace of $H^1(\mathcal{I}_h) \times H^1(\mathcal{I}_h)$, $V_h \times V_h$. We consider the following problem (see [4]): Find $(Q, U) \in V_h \times V_h$, such that

(2.1a) $$(Q, w)_{\mathcal{I}_h} = -\epsilon\,(U, w')_{\mathcal{I}_h} + \langle \epsilon \hat{u}^{\epsilon}, w \rangle_{\partial \mathcal{I}_h},$$

(2.1b) $$(Q - aU, v')_{\mathcal{I}_h} - \langle \hat{q}, v \rangle_{\partial \mathcal{I}_h} + \langle a \hat{u}^c, v \rangle_{\partial \mathcal{I}_h} + ((b - a')U, v)_{\mathcal{I}_h} = (f, v)_{\mathcal{I}_h},$$

for all $(w, v) \in V_h \times V_h$. Here we have used the notations

$$(\varphi, \psi)_{\mathcal{I}_h} = \sum_{I_j \in \mathcal{I}_h} (\varphi, \psi)_{I_j} = \sum_{I_j \in \mathcal{I}_h} \int_{I_j} \varphi(x)\psi(x)dx,$$

and

$$\langle \varphi, \psi \rangle_{\partial \mathcal{I}_h} = \sum_{I_j \in \mathcal{I}_h} \langle \varphi, \psi \rangle_{\partial I_j} = \sum_{j=1}^{N} \left[\varphi_j^- \psi_j^- - \varphi_{j-1}^+ \psi_{j-1}^+\right].$$

For simplicity, we always write the above two inner products as (φ, ϕ) and $\langle \varphi, \psi \rangle$ for functions $\varphi, \phi \in H^1(\mathcal{I}_h)$ without the subscripts.

To completely define the LDG scheme, we take the following numerical traces:

(2.2) $$\begin{aligned} &\hat{q}(x_j) = Q_j^+, && \text{for } j = 0, 1, \cdots, N-1, \\ &\hat{q}(1) = Q(1^-) - \lambda U(1^-), && \\ &\hat{u}^{\epsilon}(x_j) = U_j^-, && \text{for } j = 1, \cdots, N-1, \\ &\hat{u}^{\epsilon}(0) = 0;\ \hat{u}^{\epsilon}(1) = 0, && \\ &\hat{u}^c(x_j) = U_j^-, && \text{for } j = 1, \cdots, N, \\ &\hat{u}^c(0) = 0, && \end{aligned}$$

where the stabilization parameter $\lambda \geq 0$ will be determined later. The LDG method defined by these numerical traces was called the md-LDG method in [4]. Substituting the numerical traces (2.2) into (2.1), we have

$$\epsilon^{-1}(Q, w) + (U, w') - \sum_{j=1}^{N-1} U_j^- [\![w_j]\!] = 0,$$

$$(Q - aU, v') + ((b - a')U, v) + Q_0^+ v_0^+ - \sum_{j=1}^{N-1} Q_j^+ [\![v_j]\!] \tag{2.3}$$

$$- (Q_N^- - \lambda U_N^-) v_N^- + \sum_{j=1}^{N} a_j^- U_j^- [\![v_j]\!] = (f, v).$$

The above system leads to

$$a(Q, w) + b_1(U, w) = 0, \tag{2.4a}$$

$$b_2(Q, v) + c(U, v) = f(v). \tag{2.4b}$$

where these bilinear forms are defined as

$$a(Q, w) = \epsilon^{-1}(Q, w), \tag{2.5a}$$

$$b_1(U, w) = (U, w') - \sum_{j=1}^{N-1} U_j^- [\![w_j]\!], \tag{2.5b}$$

$$b_2(Q, v) = (Q, v') - \sum_{j=0}^{N-1} Q_j^+ [\![v_j]\!] - Q_N^- v_N^-, \tag{2.5c}$$

$$c(U, v) = -(aU, v') + ((b - a')U, v) + \sum_{j=1}^{N} a_j^- U_j^- [\![v_j]\!] + \lambda U_N^- v_N^-, \tag{2.5d}$$

$$f(v) = (f, v). \tag{2.5e}$$

In this article, we assume that $a(x)$ is continuous so that $a_j^+ = a_j^- = a_j$ for any $j = 0, 1, \cdots, N$. By the integration by parts, we can verify that $b_1(v, w) = -b_2(w, v)$.

The following proposition guarantees the existence and uniqueness of the numerical solution defined by (2.1) and (2.2). The proof is provided in Section 3.1.

Proposition 2.1. *Suppose $\lambda \geq 0$ and $b - a'/2 \geq 0$, then the LDG solution determined by (2.1) and numerical traces (2.2) exists and is unique.*

Define the compact form of our LDG discretization

$$\mathcal{A}(\phi, \psi; w, v) = a(\phi, w) + b_1(\psi, w) + b_2(\phi, v) + c(\psi, v), \tag{2.6}$$

for any $(\phi, \psi),\ (w, v) \in V_h \times V_h$. By the consistency of the numerical traces, it is straightforward to verify the orthogonality property

$$\mathcal{A}(q - Q, u - U; w, v) = 0 \tag{2.7}$$

for any $(w, v) \in V_h \times V_h$.

We are now ready to state out main results, which give some local error estimates on a subdomain $\mathcal{I}_0 \subset \mathcal{I}$.

Theorem 2.2. *Let* $\mathcal{I}_0 = (0, A)$ *and* $\mathcal{I}_s^+ = (0, A + s\ln(1/h)\rho)$*, where* ρ *and* A *are defined in (3.21);* A *is chosen such that* $A + s\ln(1/h)\rho$ *is an interior mesh point. Let* m *be a positive constant such that* $h^{2m} \le C\epsilon$*. Suppose that the positive constant* L *defined in (3.32) is sufficiently small; the exact solution of problem (1.1) satisfies*

$$\|u\|_{L^\infty(\mathcal{I})} \le C, \qquad \|u\|_{W^1_\infty(\mathcal{I})} \le C\epsilon^{-1} \tag{2.8}$$

for some constant C *independent of* ϵ*.*
(1) If $b(x) - a'(x)/2 > 0$ *for all* $x \in \mathcal{I}$ *and* $\|u\|_{k+2,\mathcal{I}^+_{m+k+1}} \le C$*, then there exists a constant* C *such that*

$$\|u - U\|_{\mathcal{I}_0} + \sqrt{\epsilon}\|u' - \epsilon^{-1}Q\|_{\mathcal{I}_0} \le Ch^{k+1}. \tag{2.9}$$

(2) If $a(x) = constant$*,* $b(x) = 0$ *for all* $x \in \mathcal{I}$*, and* $\|u\|_{k+2,\mathcal{I}^+_{m+k+1}} \le C$*, then there exists a constant* C *such that*

$$\|u - U\|_{\mathcal{I}_0} + \sqrt{\epsilon}\|u' - \epsilon^{-1}Q\|_{\mathcal{I}_0} \le C\ln N h^{k+1}. \tag{2.10}$$

(3) If $a(x) = constant$*,* $b(x) = 0$ *for all* $x \in \mathcal{I}$*, and* $\|u\|_{k+2,\mathcal{I}^+_{2m+k+1}} \le C$*, then there exists a constant* C *such that*

$$\|u - U\|_{\mathcal{I}_0} + \|u' - \epsilon^{-1}Q\|_{\mathcal{I}_0} \le Ch^{k+1}. \tag{2.11}$$

Remark 2.3. Note that all these three convergence rates are independent of the singular perturbation parameter ϵ. The error bounds of (2.9) and (2.10) indicate that when $s = m + k + 1$, the L^2 errors of $u - U$ and $\sqrt{\epsilon}(u' - \epsilon^{-1}Q)$ converge at the optimal rate on subdomain $\mathcal{I}_0$, which is $O(h\ln(1/h))$ distance away from the outflow boundary $x = 1$. On the other hand, (2.11) shows that the L^2 error of $u' - \epsilon^{-1}Q$ converges at the optimal rate on a smaller subdomain, when $b(x) = 0$, $a(x) = constant$, and $s = 2m + k + 1$.

Remark 2.4. Since the specially chosen numerical traces (2.2) significantly simplifies the proof, the error estimate may not hold for a different choice of numerical traces. Similar local error estimates could be derived for reaction-diffusion problem if the position of the boundary layer is known.

3. Proofs

This section is devoted to the proofs of Proposition 2.1 and Theorem 2.2. To prove Theorem 2.2, we first establish interpolation error estimates on quasi-uniform meshes. Then we introduce the cut-off function and Lemma 3.7, which is the foundation of the error analysis. Theorem 2.2 will follow after a direct application of Lemma 3.4 and Lemma 3.7.

3.1. Proof of Theorem 2.1.

Proof. We only need to verify that $Q = 0$, $U = 0$ in (2.4) if $f = 0$. Taking $w = Q$ and $v = U$, $f = 0$ in (2.4) and adding (2.4a) and (2.4b) together we get

$$\begin{aligned} &a(Q, Q) + c(U, U) \\ &= \epsilon^{-1}\|Q\|^2_{\mathcal{I}_h} - (aU, U') + ((b - a')U, U) + \sum_{j=1}^{N} a_j U_j^- [\![U_j]\!] + \lambda U_N^- v_N^- = 0. \end{aligned} \tag{3.1}$$

By an integration by parts,

$$-(aU, U') = (aU, U') + (a'U, U) - \sum_{j=1}^{N} \left[a_j (U_j^-)^2 - a_{j-1}(U_{j-1}^+)^2\right].$$

Solving the above equation for (aU, U') one has

$$-(aU, U') = \frac{1}{2}(a'U, U) - \frac{1}{2}\left[-a_0(U_0^+)^2 + \sum_{j=1}^{N-1} a_j [\![U_j^2]\!] + a_N(U_N^-)^2\right]. \tag{3.2}$$

Therefore, the sum of the second term, the third term and the fourth term of (3.1) can be simplified as

$$\begin{aligned}
&((b - a'/2)U, U) + \frac{1}{2}a_0 \left(U_0^+\right)^2 - \frac{1}{2}a_N \left(U_N^-\right)^2 \\
&\quad + \sum_{j=1}^{N-1} a_j \left[\left(U_j^-\right)^2 - U_j^- U_j^+ - \frac{1}{2}\left(U_j^-\right)^2 + \frac{1}{2}\left(U_j^+\right)^2\right] + a_N \left(U_N^-\right)^2 \\
&= \left\|(b - a'/2)^{\frac{1}{2}} U\right\|_{\mathcal{I}_h}^2 + \sum_{j=0}^{N} \frac{1}{2} a_j [\![U_j]\!]^2.
\end{aligned} \tag{3.3}$$

By substituting (3.3) into (3.1), the left side of (3.1) becomes

$$a(Q,Q) + c(U,U) = \epsilon^{-1}\|Q\|_{\mathcal{I}_h} + \left\|(b - a'/2)^{\frac{1}{2}} U\right\|_{\mathcal{I}_h}^2 + \sum_{j=0}^{N} \frac{1}{2} a_j [\![U_j]\!]^2 + \lambda \left(U_N^-\right)^2, \tag{3.4}$$

which implies that

$$U = 0, Q = 0 \qquad \text{if } b - a'/2 > 0, \tag{3.5a}$$

$$U \in C^0(\mathcal{I}), Q = 0 \qquad \text{if } b - a'/2 = 0. \tag{3.5b}$$

When $b - a'/2 = 0$ holds true, (2.4a) can be written by using an integration by parts for (2.5b) as

$$\epsilon^{-1}(Q, w) - (U', w) - U_0^+ w_0^+ + U_N^- w_N^- + \sum_{j=1}^{N-1} [\![U_j]\!] w_j^+ = 0.$$

It follows from (3.5b) that $(U', w) = 0$ for all $w \in V_h$, which implies that U is a piecewise constant function on $\overline{\mathcal{I}}$ by taking $w = U'$. This, together with the implementation of (3.5b), prove that $U = 0$. The existence and uniqueness of the LDG solution follow as a consequence. □

3.2. Proof of Theorem 2.2. To provide a detailed proof for Theorem 2.2, we proceed in several steps. First, we introduce interpolation operators and two preliminary lemmas. In step 2, a cut-off function and some properties are introduced. For more details of the cut-off function, we refer the reader to [14, 10]. This function is the one-dimensional case of the cut-off function used in [10]. Then, in Step 3, local error estimates on a subdomain $\mathcal{I}_0 \in \mathcal{I}$ are established from the combination of interpolation error estimates and properties of the cut-off function.

Step 1: *Interpolations.* We use polynomial interpolation of degree $k > 0$. Let $I = (a^+, a^-)$ be an arbitrary interval and $\mathcal{P}^k(I)$ the space of the polynomials of degree at most k on I.

For $v \in C(\overline{I})$, we define the projection $\pi^\pm v \in \mathcal{P}^k(I)$ with the following two conditions:

$$\pi^\pm v(a^\pm) = v(a^\pm), \qquad \int_I [v(x) - \pi^\pm v(x)] p'(x) dx = 0 \tag{3.6}$$

for any $p(x) \in \mathcal{P}^k(I)$. Let $\xi_u := u - \pi^- u$, $\eta_u := \pi^- u - U$, $e_u := u - U$, $\xi_q := q - \pi^+ q$, $\eta_q := \pi^+ q - Q$, and $e_q := q - Q$. As a consequence, we have $e_u = \xi_u + \eta_u$, $e_q = \xi_q + \eta_q$.

In the following proof, we will estimate (ξ_q, ξ_u) and (η_q, η_u) separately. To estimate the interpolation errors ξ_u and ξ_q, we need two preliminary lemmas and some frequently used inequalities. The first lemma was proven in [17].

Lemma 3.1. *(Lemma 3.7,* [17]*) For any $v \in C(\overline{I})$, the interpolation operators $\pi^{\pm}$ satisfy*

$$\left\|\pi^{\pm} v\right\|_I^2 \leq C\left(\|v\|_I^2 + |v(\pm 1)|^2\right) \tag{3.7}$$

on the reference element $I = (-1, 1)$.

The second lemma gives elementwise error bounds of the interpolation [3].

Lemma 3.2. *(Lemma 3.3,* [3]*) If $v \in H^{k+1}(I_j)$, there exists a constant C such that*

$$\left|(v - \pi^{\pm} v)(x_j^{\pm})\right| \leq C h_j^{s-\frac{1}{2}} \|u\|_{s,I_j}, \quad j = 0, \cdots, N-1, \tag{3.8a}$$

$$\left\|v - \pi^{\pm} v\right\|_{I_j} \leq C h_j^{s} \|u\|_{s,I_j}, \qquad j = 1, \cdots, N, \tag{3.8b}$$

for $s = 0, 1, \cdots, k+1$.

A direct application of Lemma 3.2 on mesh $\mathcal{I}_h$ produces the following lemma.

Lemma 3.3. *Suppose that the exact solution u of (1.1) satisfies*

$$\|u\|_{W^1_\infty(\mathcal{D})} \leq C \tag{3.9}$$

for some subdomain $\mathcal{D} \subset \mathcal{I}$. If $\mathcal{D}_h = \{I_j \in \mathcal{I}_h \mid I_j \subset \mathcal{D}\}$, then there exists a constant C such that

$$\epsilon^{-1} \|\xi_q\|_{\mathcal{D}_h} + \|\xi_u\|_{\mathcal{D}_h} \leq C h^{k+1}. \tag{3.10}$$

Applying the definition of the interpolation (3.6) and Lemma 3.1 yields the following conclusion.

Lemma 3.4. *Suppose that u is the exact solution of (1.1) and satisfies (2.8), then there exists a constant C such that*

$$\epsilon^{-\frac{1}{2}} \|\xi_q\|_{\mathcal{I}_h} + \|\xi_u\|_{\mathcal{I}_h} + |(\xi_q)_N^-| \leq C \epsilon^{-\frac{1}{2}}. \tag{3.11}$$

Proof. By (3.6) and (3.7), we have

$$\|\xi_q\|_{\mathcal{I}_h}^2 \leq \|q\|_{\mathcal{I}_h}^2 + \left\|\pi^+ q\right\|_{\mathcal{I}_h}^2 \leq C \|q\|_{\mathcal{I}_h}^2 + Ch \sum_{i=0}^{N-1} (q_j^+)^2.$$

Using (2.8) and $q = \epsilon u'$

$$\epsilon^{-\frac{1}{2}} \|\xi_q\|_{\mathcal{I}_h} \leq C \epsilon^{-\frac{1}{2}} (1 + hN)^{\frac{1}{2}} \leq C \epsilon^{-\frac{1}{2}}. \tag{3.12}$$

Similarly, we use (3.6), (3.7) and (2.8) to prove

$$\|\xi_u\|_{\mathcal{I}_h} \leq C \left(\|u\|_{\mathcal{I}_h}^2 + h \sum_{i=1}^{N} (u_j^-)^2 \right)^{\frac{1}{2}} \leq C. \tag{3.13}$$

To estimate $\left|(\xi_q)_N^-\right|$, we first apply the trace inequality to obtain

$$\left|(\xi_q)_N^-\right| \leq \left|q_N^-\right| + \left|(\pi^+ q)_N^-\right| \leq \left|q_N^-\right| + h^{-\frac{1}{2}} \left\|\pi^+ q\right\|_{I_N}. \tag{3.14}$$

Since the assumption(2.8) implies $\left|q_N^-\right| \leq C$ and $\|q\|_{I_N} \leq C h^{\frac{1}{2}}$, then using (3.7) we have

$$\left\|\pi^+ q\right\|_{I_N} \leq |q|_{I_N} + h^{\frac{1}{2}} |q_N^-| \leq C h^{\frac{1}{2}}. \tag{3.15}$$

Substituting (3.15) into (3.14) yields $\left|q_N^-\right| \leq C$, which, combined with (3.12) and (3.13) establishes (3.11). □

Step 2: *The cut-off function.* In this step we introduce a *cut-off function* which will be instrumental throughout the proof of Theorem 2.2. Let $\varphi(t)$ be function satisfying the following conditions. We suppose that there exist positive constants C_1 and C_2,which are independent of ϵ and h, such that

$$\begin{aligned} &C_1 \leq \varphi(t) \leq C_2, && \text{for } t \leq 1,\\ &\varphi(t) = e^{-t}, && \text{for } t \geq 0,\\ &\varphi(t) = 3 - \frac{1}{\ln(|t|)+1}, && \text{for } t \leq -1; \end{aligned} \tag{3.16}$$

$$\varphi'(t) < 0, \qquad \text{for } t \in (-\infty,\infty); \tag{3.17}$$

$$\begin{aligned} &|\varphi^{\ell}(t)| \leq C_2|\varphi(t)|, \quad (1 \leq \ell \leq k+1) && \text{for } t \in (-\infty,\infty),\\ &|\varphi^{\ell}(t)| \leq C_2|\varphi'(t)|, \quad (2 \leq \ell \leq k+1) && \text{for } t \in (-\infty,\infty),\\ &|\varphi(t)| \leq C_2(|t|+1)(\ln(|t|+1)+1)^2|\varphi'(t)|, && \text{for } t \in (-\infty,\infty). \end{aligned} \tag{3.18}$$

Finally, setting

$$RO(\widehat{\mathcal{I}},\varphi) = \frac{\max\limits_{x\in\widehat{\mathcal{I}}}|\varphi(x)|}{\min\limits_{x\in\widehat{\mathcal{I}}}|\varphi(x)|},$$

we assume that

$$RO(\widehat{\mathcal{I}},\varphi) + RO(\widehat{\mathcal{I}},\varphi') \leq C_2. \tag{3.19}$$

for any interval $\widehat{\mathcal{I}}$ of length 1.

Then the cut-off function ω is defined as

$$\omega(x) = \varphi\left(\frac{x-A}{\rho}\right), \tag{3.20}$$

where

$$A \in \mathcal{I} \qquad \text{and} \qquad \rho = \theta h, \tag{3.21}$$

for some positive constant $\theta \geq 1$. Parameter θ will be specified in the proof of Lemma 3.7 to produce a sufficiently small constant L.

Remark 3.5. We note that $\omega(x)$ is a one-dimensional version of the function introduced by Guzmán in [10] which was a variation of an analogous function employed by Johnson *et al.* in [14]. Guzmán's modification was necessary in order to handle the case where $b(x)$ is not bounded away from zero from below, i.e., in the absence of the reaction term.

Some of the basic properties of $\omega(x)$ which follow directly from the properties (3.16)–(3.19) are gathered in the following lemma.

Lemma 3.6. *The cut-off function $\omega(x)$ has the following regularity properties*

$$|\omega^{(\alpha)}(x)| \leq C\rho^{-\alpha}|\omega(x)|, \quad 1 \leq \alpha \leq k+1, \quad x \in \mathcal{I}_h, \tag{3.22a}$$

$$|\omega^{(\alpha)}(x)| \leq C\rho^{-\alpha+1}|\omega'(x)|, \quad 1 \leq \alpha \leq k+1, \quad x \in \mathcal{I}_h, \tag{3.22b}$$

$$|\omega(x)| \leq C(\ln N)^2|\omega'(x)|, \qquad x \in (0,1-A), \tag{3.22c}$$

$$|\omega(x)| = \rho|\omega'(x)|, \qquad x \in (1-A,1), \tag{3.22d}$$

$$RO(I_j,\omega) + RO(I_j,\omega') \leq C, \;\; j = 1,2,\cdots,N \tag{3.22e}$$

for some constant C.

Step 3: *A weighted norm.*

For any $(\tau, v) \in H^1(\mathcal{I}_h) \times H^1(\mathcal{I}_h)$ we define the weighted norm

$$
\begin{aligned}
|(\tau, v)|_{\mathcal{W}}^2 := \epsilon^{-1} \|\omega\tau\|_{\mathcal{I}_h}^2 + \left\|(b - a'/2)^{\frac{1}{2}}\, \omega v\right\|_{\mathcal{I}_h}^2 \\
+ \left\|(a\omega|\omega'|)^{\frac{1}{2}}\, v\right\|_{\mathcal{I}_h}^2 + \frac{1}{2}\sum_{j=0}^{N} a_j\, [\![\omega_j v_j]\!]^2 + \lambda \left(\omega_N^- v_N^-\right)^2 .
\end{aligned}
\tag{3.23}
$$

Step 4: *The estimation of* $|(\eta_q, \eta_u)|_{\mathcal{W}}$*.*

The following lemma (Lemma 3.7) is essential in the proof of local error estimates. It shows that the error η_u and η_q in the weighted norm can be bounded by interpolation errors. Some supplementary error estimates are provided in Lemma 3.8 after Lemma 3.7.

Lemma 3.7. *Suppose that the positive constant L defined in (3.32) is sufficiently small. If $b - a'/2 > 0$, there exists a constant C such that*

$$
|\,(\eta_q, \eta_u)\,|_{\mathcal{W}} \le C\left[\epsilon^{-\frac{1}{2}}\|\omega\xi_q\|_{\mathcal{I}_h} + \|\omega\xi_u\|_{\mathcal{I}_h} + \left|(w\xi_q)_N^-\right|\right]. \tag{3.24}
$$

Otherwise, if a = constant and b = 0, there exists a constant C such that

$$
|\,(\eta_q, \eta_u)\,|_{\mathcal{W}} \le C\left[\epsilon^{-\frac{1}{2}}\|\omega\xi_q\|_{\mathcal{I}_h} + \left|(w\xi_q)_N^-\right|\right]. \tag{3.25}
$$

Proof. The proof includes three parts. The first part is devoted to transform the expression of the error in the weighted-norm to an expression suitable for the estimation. Then, we estimate each terms of this expression in the second part and the third part of the proof.

Part I. Since (3.17) and (3.20) imply $\omega' < 0$, the definition of the weighted norm (3.23) gives

$$
\begin{aligned}
|(\eta_q, \eta_u)|_{\mathcal{W}}^2 = \epsilon^{-1}\left(\eta_q, \omega^2\eta_q\right) + \left((b - a'/2)\eta_u, \omega^2\eta_u\right) - \left(a\omega\eta_u^2, \omega'\right) \\
+ \frac{1}{2}\sum_{j=0}^{N} a_j [\![(\omega\eta_u)_j]\!]^2 + \lambda\left(\omega^-\eta_u^-\right)_N^2 .
\end{aligned}
\tag{3.26}
$$

To simplify the right side of the equation, we need to investigate the term $\left(a\omega\eta_u^2, \omega'\right)$. By an integration by parts, we have

$$
\begin{aligned}
(a\omega\eta_u^2, \omega') = \sum_{j=1}^{N}\left[a_j\left(\omega^-\eta_u^-\right)_j^2 - a_{j-1}\left(\omega^+\eta_u^+\right)_{j-1}^2\right] \\
- (a'\omega\eta_u^2, \omega) - (a\omega'\eta_u^2, \omega) - 2(a\omega^2\eta_u, \eta_u'),
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
-(a\omega^2\eta_u, \eta_u') = -\sum_{j=1}^{N}\left[a_j\left(\omega^-\eta_u^-\right)_j^2 - a_{j-1}\left(\omega^+\eta_u^+\right)_{j-1}^2\right] \\
+ (a'\omega\eta_u^2, \omega) + (a\eta_u, (\omega^2\eta_u)').
\end{aligned}
\tag{3.28}
$$

Substituting (3.28) into (3.27) yields

$$
\begin{aligned}
(a\omega\eta_u^2, \omega') = -\frac{1}{2}\sum_{j=1}^{N}\left[a_j\left(\omega^-\eta_u^-\right)_j^2 - a_{j-1}\left(\omega^+\eta_u^+\right)_{j-1}^2\right] \\
+ \frac{1}{2}(a'\omega\eta_u^2, \omega) + (a\eta_u, (\omega^2\eta_u)'),
\end{aligned}
$$

which, combined with (3.26), gives

$$|(\eta_q,\eta_u)|^2_{\mathcal{W}} = \epsilon^{-1}\left(\eta_q,\omega^2\eta_q\right) + \left((b-a')\eta_u,\omega^2\eta_u\right) - \left(a\eta_u,(\omega^2\eta_u)'\right) + \sum_{j=1}^{N-1} a_j\left(\omega^-\eta_u^-\right)_j [\![(\omega\eta_u)_j]\!] + (a_N+\lambda)(\eta_u)^-_N(\omega^2\eta_u)^-_N.$$

Noticing that from (2.5) we have

$$\mathcal{A}(\eta_q,\eta_u;\omega^2\eta_q,\omega^2\eta_u) = a(\eta_q,\omega^2\eta_q) + b_1(\eta_u,\omega^2\eta_q) - b_1(\omega^2\eta_u,\eta_q) + c(\eta_u,\omega^2\eta_u).$$

By (2.5) and the fact $\omega\in C(\overline{\mathcal{I}})$, it is straightforward to verify that

$$|(\eta_q,\eta_u)|^2_{\mathcal{W}} = \mathcal{A}(\eta_q,\eta_u;\omega^2\eta_q,\omega^2\eta_u) - (\eta_u,2\omega\omega'\eta_q). \tag{3.29}$$

Part II. Next, we are going to estimate the fist term of (3.29). Let $E_q := \omega^2\eta_q - \pi^+(\omega^2\eta_q)$, $E_u := \omega^2\eta_u - \pi^-(\omega^2\eta_u)$. The estimation of E_q and E_u are listed in Lemma 3.8 at the end of this section. Using the orthogonality of the compact form (2.7), we have

$$\begin{aligned}\mathcal{A}(\eta_q,\eta_u;\omega^2\eta_q,\omega^2\eta_u) &= \mathcal{A}(\eta_q,\eta_u;E_q,E_u) + \mathcal{A}(\eta_q,\eta_u;\pi^+(\omega^2\eta_q),\pi^-(\omega^2\eta_u))\\ &= \mathcal{A}(\eta_q,\eta_u;E_q,E_u) + \mathcal{A}(\xi_q,\xi_u;\pi^+(\omega^2\eta_q),\pi^-(\omega^2\eta_u))\\ &= R_1+R_2.\end{aligned} \tag{3.30}$$

(1) Consider R_1. Define $\bar a|_{I_j} = \dfrac{1}{h_j}\displaystyle\int_{I_j} a(x)dx$ for all $j=1,2,\cdots,N$. Therefore, standard approximation theory implies that $\|a-\bar a\|_{L^\infty(I_j)} \le Ch_j\|a\|_{W^1_\infty(I_j)}$ for any $j=1,2,\cdots,N$. Instead of the expression $\mathcal{A}(\eta_q,\eta_u;E_q,E_u)$ as in (2.6), we use

$$\mathcal{A}(\eta_q,\eta_u;E_q,E_u) = a(\eta_q,E_q) - b_2(E_q,\eta_u) - b_1(\eta_q,E_u) + c(\eta_u,E_u),$$

where

$$\begin{aligned} a(\eta_q,E_q) &= \epsilon^{-1}\left(\eta_q,E_q\right),\\ -b_2(E_q,\eta_u) &= -\left(E_q,\eta_u'\right) - (E_q\eta_u)^+_0 + \sum_{j=1}^{N-1}(E_q)^+_j[\![(\eta_u)_j]\!] + (E_q\eta_u)^-_N\\ &= (E_q\eta_u)^-_N,\\ -b_1(\eta_q,E_u) &= -(E_u,\eta_q') + \sum_{j=1}^{N-1}(E_u)^-_j[\![(\eta_q)_j]\!],\\ c(\eta_u,E_u) &= (b\eta_u,E_u) + (aE_u,\eta_u') - \sum_{j=1}^{N-1} a_j(E_u)^+_j[\![(\eta_u)_j]\!] + a_0(E_u\eta_u)^+_0 \end{aligned}$$

where we apply (3.6) and an integration by parts to $c(\eta_u,E_u)$. Adding these four terms give rise to

$$\begin{aligned} R_1 &= \underline{\epsilon^{-1}(\eta_q,E_q) + (b\eta_u,E_u) + ((a-\bar a)E_u,\eta_u')} + \underline{(E_q\eta_u)^-_N} - \underline{\sum_{j=0}^{N-1} a_j(E_u)^+_j[\![(\eta_u)_j]\!]}\\ &= S_1+S_2+S_3. \end{aligned}$$

Here we inserted one expression $(E_u,\bar a\eta_u')$, which equals zero in view of (3.6).

Now we turn to the estimate of S_1. We first apply the Schwarz's inequality, the inverse inequality and (3.22e) to obtain

$$\begin{aligned}
\epsilon^{-1}(\eta_q, E_q)_{I_j} &\le C\epsilon^{-1}\|\omega\eta_q\|_{I_j}\|\omega^{-1}E_q\|_{I_j} \le C\theta^{-1}\epsilon^{-1}\left\|\omega\eta_q\right\|_{I_j}^2,\\
(b\eta_u, E_u)_{I_j} &\le C\|\omega\eta_u\|_{I_j}\|\omega^{-1}E_u\|_{I_j} \le C\ln N h^{\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_j}^2,\\
((a-\bar{a})E_u, \eta_u')_{I_j} &\le Ch\|a\|_{W^1_\infty, I_j}\|E_u\|_{I_j}\|\eta_u'\|_{I_j} \le C\|\omega\eta_u\|_{I_j}\|\omega^{-1}E_u\|_{I_j}\\
&\le C\ln N h^{\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_j}^2,
\end{aligned}$$

for any $j = 1, 2, \cdots, N$. Here we also used (3.22a)-(3.22e), and error estimates (3.40a)-(3.40b). Combining the above inequalities for all $I_j \in \mathcal{I}_h$ yields

$$\begin{aligned}
S_1 &\le C\left[\theta^{-1}\epsilon^{-1}\left\|\omega\eta_q\right\|_{\mathcal{I}_h}^2 + \theta^{-\frac{1}{2}}h^{\frac{1}{2}}\ln N\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{\mathcal{I}_h}^2\right]\\
&\le CL|(\eta_q,\eta_u)|_{\mathcal{W}}^2,
\end{aligned} \tag{3.31}$$

where the constant L is defined as

$$L = \max\left\{\sqrt{\frac{1}{\theta}}, \sqrt{\frac{h}{\theta}}\ln N, \sqrt{\frac{\epsilon}{\lambda h}}\frac{1}{\theta}, \frac{\epsilon}{\theta h}\right\}. \tag{3.32}$$

Consider S_2 and S_3. From definition of (E_q, E_u) and (3.40c)-(3.40d),

(3.33a)

$$\left|\left(\omega^{-1}E_q\right)_N^-\right| = \left|\int_{I_N}\omega^{-1}E_q' dx\right| \le C\sqrt{h_N}\left\|\omega^{-1}E_q'\right\|_{I_N} \le Ch^{-\frac{1}{2}}\theta^{-1}\left\|\omega\eta_q\right\|_{I_N},$$

(3.33b)

$$\left|\left(\omega^{-1}E_u\right)_j^+\right| = \left|\int_{I_{j+1}}\omega^{-1}E_u' dx\right| \le C\sqrt{h_j}\left\|\omega^{-1}E_u'\right\|_{I_{j+1}} \le C\theta^{-\frac{1}{2}}\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_{j+1}},$$

for any $j = 0, 1, \cdots, N-1$. By Schwarz's inequality, (3.33a) and (3.22e), we have

$$\begin{aligned}
S_2 &\le C\lambda^{-\frac{1}{2}}\left|\left(\omega^{-1}E_q\right)_N^-\right|\left(\lambda\left(\omega\eta_u^-\right)_N^2\right)^{\frac{1}{2}}\\
&\le C\lambda^{-\frac{1}{2}}\theta^{-1}h^{-\frac{1}{2}}\left\|\omega\eta_q\right\|_{\mathcal{I}_h}\left(\lambda\left(\omega\eta_u^-\right)_N^2\right)^{\frac{1}{2}} \le CL|(\eta_q,\eta_u)|_{\mathcal{W}}^2.
\end{aligned} \tag{3.34}$$

Here we used the general assumption $\epsilon < h$. Meanwhile, it follows from (3.33b) that

$$\begin{aligned}
S_3 &\le C\sum_{j=0}^{N-1}|a_j[\![(\omega\eta_u)_j]\!]|\cdot\left|\left(\omega^{-1}E_u\right)_j^+\right|\\
&\le C\sum_{j=1}^{N}|a_j[\![(\omega\eta_u)_j]\!]|\cdot\theta^{-\frac{1}{2}}\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_{j+1}}\\
&\le C\theta^{-\frac{1}{2}}\left[\sum_{j=1}^{N}|a_j[\![(\omega\eta_u)_j]\!]|^2\right]^{\frac{1}{2}}\cdot\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_{j+1}}\\
&\le CL|(\eta_q,\eta_u)|_{\mathcal{W}}^2.
\end{aligned} \tag{3.35}$$

The combination of the estimates (3.31)-(3.35) yields

$$R_1 \le CL|(\eta_q,\eta_u)|_{\mathcal{W}}^2. \tag{3.36}$$

(2) The estimation of R_2. The expression of R_2 can be written as

$$\begin{aligned}R_2 &= \mathcal{A}(\xi_q,\xi_u;\pi^+(\omega^2\eta_q),\pi^-(\omega^2\eta_u))\\ &= a(\xi_q,\pi^+(\omega^2\eta_q)) + b_1(\xi_u,\pi^+(\omega^2\eta_q)) + b_2(\xi_q,\pi^-(\omega^2\eta_u)) + c(\xi_u,\pi^-(\omega^2\eta_u))\end{aligned}$$

where these bilinear forms can be simplified by interpolation operators (3.6)

$$\begin{aligned}a(\xi_q,\pi^+(\omega^2\eta_q)) &= \epsilon^{-1}\left(\xi_q,\pi^+(\omega^2\eta_q)\right),\\ b_1(\xi_u,\pi^+(\omega^2\eta_q)) &= \left(\xi_u,\left[\pi^+(\omega^2\eta_q)\right]'\right) - \sum_{j=1}^{N-1}(\xi_u)_j^-[\![\pi^+(\omega^2\eta_q)_j]\!] = 0,\\ b_2(\xi_q,\pi^-(\omega^2\eta_u)) &= \left(\xi_q,\left[\pi^-(\omega^2\eta_u)\right]'\right) - \sum_{j=0}^{N-1}(\xi_q)_j^+[\![\pi^-(\omega^2\eta_u)_j]\!] - (\xi_q)_N^-\left(\pi^-(\omega^2\eta_u)\right)_N^-\\ &= -(\xi_q)_N^-\left(\pi^-(\omega^2\eta_u)\right)_N^-,\end{aligned}$$

$$\begin{aligned}c(\xi_u,\pi^-(\omega^2\eta_u)) &= \left((b-a')\xi_u,\pi^-(\omega^2\eta_u)\right) - \left(a\xi_u,\left[\pi^-(\omega^2\eta_u)\right]'\right)\\ &\quad + \sum_{j=1}^{N-1}a_j(\xi_u)_j^-[\![\pi^-(\omega^2\eta_u)_j']\!] + (a_N+\lambda)(\xi_u)_N^-\left(\pi^-(\omega^2\eta_u)\right)_N^-\\ &= \left((b-a')\xi_u,\pi^-(\omega^2\eta_u)\right) - \left(a\xi_u,\pi^-(\omega^2\eta_u)'\right).\end{aligned}$$

Adding the above four terms yields

$$\begin{aligned}R_2 = \epsilon^{-1}&\left(\xi_q,\pi^+(\omega^2\eta_q)\right) + \left((b-a')\xi_u,\pi^-(\omega^2\eta_u)\right) - \left(a\xi_u,\left[\pi^-(\omega^2\eta_u)\right]'\right)\\ &- (\xi_q)_N^-\left(\pi^-(\omega^2\eta_u)\right)_N^-\\ =\ & T_1+T_2+T_3+T_4.\end{aligned}$$

We will estimate these four terms separately. By the facts $\pi^+(\omega^2\eta_q) = \omega^2\eta_q - E_q$, $\pi^-(\omega^2\eta_u) = \omega^2\eta_u - E_u$, the Schwarz's inequality, the triangle inequality, (3.22e) and (3.40a), we obtain

$$\begin{aligned}T_1 &= \epsilon^{-1}\left(\xi_q,\omega^2\eta_q - E_q\right)\\ &\le C\epsilon^{-1}\|\omega\xi_q\|_{\mathcal{I}_h}^2 + \frac{1}{8}\epsilon^{-1}\|\omega\eta_q\|_{\mathcal{I}_h}^2 + \frac{1}{8}\epsilon^{-1}\left\|\omega^{-1}E_q\right\|_{\mathcal{I}_h}^2\\ &\le C\epsilon^{-1}\|\omega\xi_q\|_{\mathcal{I}_h}^2 + (\frac{1}{8}+CL)|(\eta_q,\eta_u)|_{\mathcal{W}}^2.\end{aligned}$$

If $a = \mathit{constant}$ and $b = 0$, then we have $b-a' = 0$, which implies $T_2 = 0$. Otherwise, if $b-a'/2 > 0$, using Schwarz's inequality, the triangle inequality, (3.22e) and (3.40b) we obtain

$$\begin{aligned}T_2 &= \left((b-a')\xi_u,\omega^2\eta_u - E_u\right)\\ &\le C\|\omega\xi_u\|_{\mathcal{I}_h}^2 + \frac{1}{8}\left\|(b-a'/2)^{\frac{1}{2}}\omega\eta_u\right\|_{\mathcal{I}_h}^2 + \left\|\omega^{-1}E_u\right\|_{\mathcal{I}_h}^2\\ &\le C\|\omega\xi_u\|_{\mathcal{I}_h}^2 + (\frac{1}{8}+CL)|(\eta_q,\eta_u)|_{\mathcal{W}}^2.\end{aligned}$$

Because of (3.6), $T_3 = 0$ if $a(x) = constant$. Otherwise, if $b - a'/2 > 0$ one has

$$\begin{aligned}
T_3 &= \big((a-\bar{a})\xi_u, I^-(\omega^2\eta_u)'\big) + \big(\bar{a}\xi_u, I^-(\omega^2\eta_u)'\big) \\
&= \big((a-\bar{a})\xi_u, (\omega^2\eta_u - E_u)'\big) \\
&= ((a-\bar{a})\xi_u, 2\omega\omega'\eta_u) + \big((a-\bar{a})\xi_u, \omega^2\eta_u'\big) + ((a-\bar{a})\xi_u, E_u') \\
&\le Ch\|a\|_{W^1_\infty(\mathcal{I})}\|\omega\xi_u\|_{\mathcal{I}_h}\left(\|\omega'\eta_u\|_{\mathcal{I}_h} + \|\omega\eta_u'\|_{\mathcal{I}_h} + \left\|\omega^{-1}E_u'\right\|_{\mathcal{I}_h}\right) \\
&\le Ch\|\omega\xi_u\|_{\mathcal{I}_h}\left(\rho^{-\frac{1}{2}}\left\|(a\omega|\omega'|)^{\frac{1}{2}}\eta_u\right\|_{\mathcal{I}_h} + h^{-1}\left\|(b-a'/2)^{\frac{1}{2}}\omega\eta_u\right\|_{\mathcal{I}_h}\right) \\
&\le C\|\omega\xi_u\|^2_{\mathcal{I}_h} + \frac{1}{8}\left\|(b-a'/2)^{\frac{1}{2}}\omega\eta_u\right\|^2_{\mathcal{I}_h} + CL|(\eta_q,\eta_u)|^2_{\mathcal{W}}.
\end{aligned}$$

Here we used the inequality $\|\omega\eta_u'\|_{I_j} \le Cc_0^{-\frac{1}{2}}h^{-1}\left\|(b-a'/2)^{\frac{1}{2}}\omega\eta_u\right\|_{I_j}$ for any $j = 1, 2, \cdots, N$, which is a consequence of inverse inequality and (3.22e). Compared with other two coefficients, the coefficient in the first term could be relatively large.

Note that $\left(\pi^-(\omega^2\eta_u)\right)^-_N = (\omega^2\eta_u)^-_N$. Using (3.6) one gets

$$\begin{aligned}
T_4 = -(\xi_q)^-_N\left(\pi^-(\omega^2\eta_u)\right)^-_N &\le \left|(w\xi_q)^-_N\right|\cdot\left|(\omega\eta_u)^-_N\right| \\
&\le Ca_N^{-1}\left|(w\xi_q)^-_N\right|^2 + \frac{1}{8}\left[\frac{1}{2}a_N(\omega^-_N(\eta_u)^-_N)^2\right].
\end{aligned}$$

If $b(x) = 0$ and $a(x) = constant$, combining error estimates of $T_1 - T_4$ yields

$$R_2 \le C\left[\epsilon^{-1}\|\omega\xi_q\|^2_{\mathcal{I}_h} + \left|(w\xi_q)^-_N\right|^2\right] + \left(\frac{3}{8} + CL\right)|(\eta_q,\eta_u)|^2_{\mathcal{W}}, \tag{3.37}$$

Otherwise, if $b - a'/2 > 0$, we have

$$R_2 \le C\left[\epsilon^{-1}\|\omega\xi_q\|^2_{\mathcal{I}_h} + \|\omega\xi_u\|^2_{\mathcal{I}_h} + \left|(w\xi_q)^-_N\right|^2\right] + \left(\frac{3}{8} + CL\right)|(\eta_q,\eta_u)|^2_{\mathcal{W}}. \tag{3.38}$$

Part III. The estimation of the second term of (3.29) follows (3.22c)-(3.22d)

$$\begin{aligned}
-(\eta_u, 2\omega\omega'\eta_q) &\le C\epsilon^{-\frac{1}{2}}\|\omega\eta_q\|_{\mathcal{I}_h}\cdot\epsilon^{\frac{1}{2}}\|\omega'\eta_u\|_{\mathcal{I}_h} \\
&\le \frac{1}{8}\epsilon^{-1}\|\omega\eta_q\|^2_{\mathcal{I}_h} + C\epsilon\rho^{-1}\left\|(\omega|\omega'|)^{\frac{1}{2}}\eta_u\right\|^2_{\mathcal{I}_h} \\
&\le (\frac{1}{8} + CL)|(\eta_q,\eta_u)|^2_{\mathcal{W}}.
\end{aligned} \tag{3.39}$$

Therefore, when $b(x) = 0$, $a(x) = constant$, we combine (3.36), (3.37) and (3.39) to obtain

$$|(\eta_q,\eta_u)|^2_{\mathcal{W}} \le C\left[\epsilon^{-1}\|\omega\xi_q\|^2_{\mathcal{I}_h} + \left|(w\xi_q)^-_N\right|^2\right] + |(\eta_q,\eta_u)|^2_{\mathcal{W}}.$$

Similarly, if $b - a'/2 > 0$, we combine (3.36), (3.38) and (3.39)

$$|(\eta_q,\eta_u)|^2_{\mathcal{W}} \le C\left[\epsilon^{-1}\|\omega\xi_q\|^2_{\mathcal{I}_h} + \|\omega\xi_u\|^2_{\mathcal{I}_h} + \left|(w\xi_q)^-_N\right|^2\right] + \left(\frac{1}{2} + CL\right)|(\eta_q,\eta_u)|^2_{\mathcal{W}}.$$

For each assumption of the coefficients a and b, local error estimates (3.24) and (3.25) are established by taking sufficiently small L and then moving the last term to the left side of the inequality. $\square$

Lemma 3.8. *Suppose* $E_q := \omega^2\eta_q - \pi^+(\omega^2\eta_q)$ *and* $E_u := \omega^2\eta_u - \pi^-(\omega^2\eta_u)$*. Then there exists a constant* C *such that the estimates*

$$\left\|\omega^{-1}E_q\right\|_{I_j} \le C\theta^{-1}\left\|\omega\eta_q\right\|_{I_j}, \tag{3.40a}$$

$$\left\|\omega^{-1}E_u\right\|_{I_j} \le Ch_j^{\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|(a\omega|\omega'|)^{\frac{1}{2}}\eta_u\right\|_{I_j}, \tag{3.40b}$$

$$\left\|\omega^{-1}E_q'\right\|_{I_j} \le Ch_j^{-1}\theta^{-1}\left\|\omega\eta_q\right\|_{I_j}, \tag{3.40c}$$

$$\left\|\omega^{-1}E_u'\right\|_{I_j} \le Ch_j^{-\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|(a\omega|\omega'|)^{\frac{1}{2}}\eta_u\right\|_{I_j}, \tag{3.40d}$$

hold for any $j = 1, 2, \cdots, N$.

Proof. By Lemma 3.2, the properties of ω (3.22a)-(3.22e), and inverse inequality, we get that for any $j = 1, 2, \cdots, N$, one has

$$\begin{aligned}
\left\|\omega^{-1}E_q\right\|_{I_j} &\le C|\omega^{-1}|_{L^\infty(I_j)}\left\|\omega^2\eta_q - \pi^+(\omega^2\eta_q)\right\|_{I_j}\\
&\le C|\omega^{-1}|_{L^\infty(I_j)}\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} h_j^{k+1}\left\|D^\alpha\omega D^\beta\omega D^\gamma\eta_q\right\|_{I_j}\\
&\le C|\omega^{-1}|_{L^\infty(I_j)}\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} h_j^{k+1}\rho^{-(\alpha+\beta)}h_j^{-\gamma}\left\|\omega^2\eta_q\right\|_{I_j}\\
&\le C|\omega^{-1}|_{L^\infty(I_j)}\theta^{-1}\left\|\omega^2\eta_q\right\|_{I_j} \le C\theta^{-1}\left\|\omega\eta_q\right\|_{I_j},
\end{aligned}$$

and

$$\begin{aligned}
\left\|\omega^{-1}E_u\right\|_{I_j} &\le C|\omega^{-1}|_{L^\infty(I_j)}\left\|\omega^2\eta_u - \pi^-(\omega^2\eta_u)\right\|_{I_j}\\
&\le C|\omega^{-1}|_{L^\infty(I_j)}\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} h_j^{k+1}\left\|D^\alpha\omega D^\beta\omega D^\gamma\eta_u\right\|_{K_j}\\
&\le C|\omega^{-1}|_{L^\infty(I_j)}\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} h_j^{k+1}\rho^{-(\alpha+\beta)+\frac{1}{2}}h_j^{-\gamma}\left\|\omega(a\omega|\omega'|)^{\frac{1}{2}}\eta_u\right\|_{I_j}\\
&\le C|\omega^{-1}|_{L^\infty(I_j)}h_j^{\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|\omega(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_j}\\
&\le Ch_j^{\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|(a\omega|\omega|)^{\frac{1}{2}}\eta_u\right\|_{I_j}.
\end{aligned}$$

Another two inequalities can be verified in a similar way to (3.40a) and (3.40b). Note that $\pi^+v = v$ if v is a polynomial of degree at most k. By the Bramble-Hilbert Lemma, the properties of ω (3.22a)-(3.22e) and the inverse inequality,

$$\begin{aligned}
\left\|\omega^{-1}E_q\right\|_{I_j} &\le C|\omega^{-1}|_{L^\infty(K_j)}\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} h_j^{k}\left\|D^\alpha\omega D^\beta\omega D^\gamma\eta_q\right\|_{I_j}\\
&\le Ch_j^{-1}\theta^{-1}\left\|\omega\eta_q\right\|_{I_j},
\end{aligned}$$

and

$$\begin{aligned}
\left\|\omega^{-1}E_u\right\|_{I_j} &\le C|\omega^{-1}|_{L^\infty(K_j)}\sum_{|\alpha+\beta+\gamma|=k+1,\gamma\le k} h_j^{k}\left\|D^\alpha\omega D^\beta\omega D^\gamma\eta_u\right\|_{I_j}\\
&\le Ch_j^{-\frac{1}{2}}\theta^{-\frac{1}{2}}\left\|(a\omega|\omega'|)^{\frac{1}{2}}\eta_u\right\|_{I_j}.
\end{aligned}$$

Here constant C depends on $RO(I_j, \omega)$ and $RO(I_j, \omega')$. □

Step 5. Now we will prove Theorem 2.2.

Proof. (1) If $b-a'/2>0$, using a property of ω (3.22c), we have

$$\|\eta_u\|_{\mathcal{I}_0} \le C\left\|(b-a'/2)^{\frac{1}{2}}\omega\eta_u\right\|_{\mathcal{I}_0} \le C|(\eta_q,\eta_u)|_{\mathcal{W}}, \tag{3.41}$$

$$\epsilon^{-\frac{1}{2}}\|\eta_q\|_{\mathcal{I}_0} \le C\epsilon^{-\frac{1}{2}}\|\omega\eta_q\|_{\mathcal{I}_0} \le C|(\eta_q,\eta_u)|_{\mathcal{W}}. \tag{3.42}$$

The definition of ω implies that

$$\omega(x) \le w\left(A+s\ln(1/h)\rho\right) \le h^s, \qquad \forall\, x\in\mathcal{I}\setminus\mathcal{I}_s^+.$$

Set $s=m+k+1$. The above estimate, associated with Lemma 3.3, Lemma 3.4 and (3.24), gives rise to

$$\begin{aligned}
&|(\eta_q,\eta_u)|_{\mathcal{W}}\\
&\le C\left[\epsilon^{-\frac{1}{2}}\|\omega\xi_q\|_{\mathcal{I}_h}+\|\omega\xi_u\|_{\mathcal{I}_h}+\left|(w\xi_q)_N^-\right|\right]\\
&\le C\left[\epsilon^{-\frac{1}{2}}\|\xi_q\|_{\mathcal{I}_s^+}+\|\xi_u\|_{\mathcal{I}_s^+}\right]+Ch^s\left[\epsilon^{-\frac{1}{2}}\|\xi_q\|_{\mathcal{I}_h\setminus\mathcal{I}_s^+}+\|\xi_u\|_{\mathcal{I}_h\setminus\mathcal{I}_s^+}+\left|(\xi_q)_N^-\right|\right]\\
&\le Ch^{k+1}\|u\|_{H^{k+2}(\mathcal{I}^+_{k+m+1})}+Ch^s\epsilon^{-\frac{1}{2}}\\
&\le Ch^{k+1}.
\end{aligned} \tag{3.43}$$

On the other hand, the interpolation error estimate on $\mathcal{I}_0$ can be derived from Lemma 3.3, i.e.,

$$\epsilon^{-\frac{1}{2}}\|\xi_q\|_{\mathcal{I}_0}+\|\xi_u\|_{\mathcal{I}_0} \le Ch^{k+1}\|u\|_{H^{k+2}(\mathcal{I}_0)} \le Ch^{k+1}. \tag{3.44}$$

Then, (2.9) follows the combination of (3.41)-(3.44).

(2) Consider the error estimate in the case of $a(x)=$ *constant* and $b(x)=0$. Instead of (3.41), we use

$$\|\eta_u\|_{\mathcal{I}_0} \le C\|\omega\eta_u\|_{\mathcal{I}_0} \le C\ln N\left\|(a\omega|\omega'|)^{\frac{1}{2}}\,\eta_u\right\|_{\mathcal{I}_0} \le C\ln N|(\eta_q,\eta_u)|_{\mathcal{W}}. \tag{3.45}$$

By Lemma 3.3, Lemma 3.4 and (3.25), we obtain the same error bound as (3.43)

$$|(\eta_q,\eta_u)|_{\mathcal{W}} \le C\left[\epsilon^{-1/2}\|\omega\xi_q\|_{\mathcal{I}_h}+\left|(w\xi_q)_N^-\right|\right] \le Ch^{k+1},$$

which, combined with (3.44) and (3.45), proves the inequality (2.10).

(3) If we let $s=2m+k+1$, the estimate (3.45) becomes

$$\begin{aligned}
|(\eta_q,\eta_u)|_{\mathcal{W}} &\le C\left[\epsilon^{-\frac{1}{2}}\|\omega\xi_q\|_{\mathcal{I}_h}+\left|(w\xi_q)_N^-\right|\right]\\
&\le C\sqrt{\epsilon}h^{k+1}\|u\|_{H^{k+2}(\mathcal{I}^+_{k+2m+2})}+Ch^s\epsilon^{-\frac{1}{2}}N\\
&\le C\sqrt{\epsilon}h^{k+1}.
\end{aligned}$$

Therefore, the combination of (3.42),(3.45) leads to

$$\|\eta_u\|_{\mathcal{I}_0}+\epsilon^{-1}\|\eta_q\|_{\mathcal{I}_0} \le Ch^{k+1},$$

which, combined with (3.10), completes the proof of (2.11). □

4. Numerical Results

In this section, numerical results are presented to verify theoretical results. We first consider the LDG method in Section 4.1. The numerical experiments for the SDFEM are presented in Section 4.2. Then we compare it with the numerical results of the SDFEM in Section 4.3. Uniform meshes with N elements are used for all examples. Similar numerical results are observed when quasi-uniform meshes are used.

4.1. The numerical experiments for the LDG method. We use uniform meshes and choose $\epsilon = 10^{-5}, 10^{-10}$. The stabilization parameter is chosen to be $\lambda = 1/h$. In order to observe the order of convergence of the error, at each refinement of the mesh, we compute the approximate order of convergence as follows. Denote by $\mathcal{I}_n$ the subdomain $\mathcal{I}_h \backslash \bigcup_{i=N-n+1}^{N} I_i$ for any integer $0 \le n \le N$. Define the local error of $(\epsilon^{-1}e_q, e_u) = (u' - \epsilon^{-1}Q, u - U)$ as

$$E_q^k(N,n) := \left\|\epsilon^{-1}e_q\right\|_{\mathcal{I}_n}, \qquad E_u^k(N,n) := \|e_u\|_{\mathcal{I}_n},$$

where the index $k = 1, 2, 3$ indicates the choice of the degree of approximation polynomial space. The corresponding rates of convergence are computed by using the formula

$$r_q^k(N,n) := \log_2\left(\frac{E_q^k(N,n)}{E_q^k(2N,n)}\right), \qquad r_u^k(N,n) := \log_2\left(\frac{E_u^k(N,n)}{E_u^k(2N,n)}\right),$$

for any $k = 1, 2, 3$.

Let $n_j = j\,[\![\ln N]\!]$, where $[\![s]\!]$ denotes the smallest integer that is greater than s. If $n = n_j$, the number of elements dropped depends on the total number of elements N (or $2N$) when calculating the errors and rates. Otherwise, if n is a fixed positive integer, it is independent of N. In the following analysis, Table 1, and Table 2, we will drop parameter 'N' in these notations and will use notations $E_q^k(n)$, $E_u^k(n)$, $r_q^k(n)$ and $r_u^k(n)$.

Example 4.1. We apply the LDG method to the test problem (1.1) with $a(x) = 1$, $b(x) = 0$, and $f(x) = \sin\pi x$ such that the exact solution $u(x)$ is

$$u(x) = \frac{1 + e^{-1/\epsilon} - 2e^{-(1-x)/\epsilon}}{\pi(1+\pi^2\epsilon^2)(1-e^{-1/\epsilon})} + \frac{\epsilon\pi\sin\pi x - \cos\pi x}{\pi(1+\pi^2\epsilon^2)}.$$

Table 1 displays the errors $\left(E_q^k(n), E_u^k(n)\right)$ and convergence rates $\left(r_q^k(n), r_u^k(n)\right)$ for $\epsilon = 10^{-5}$ and 10^{-10}, respectively. The main factor h^{k+1} of the convergence rate in (2.9) of Theorem 2.2 is observed. Table 1 illustrates that the L^2 error converges at an optimal rate when the last n_1 elements are dropped, which verifies our theoretical results. It also shows that when $\epsilon \ll h$ is extremely small (for example, $\epsilon = 10^{-10}$), the L^2 errors on a larger subdomain converge at the optimal rate. Actually, for the k-th degree polynomial approximation, we only cut off k elements from $\mathcal{I}$ near the outflow boundary to get $\mathcal{I}_0$. It means that the same order convergence rate can be obtained even when we discard much less elements than what we did in the theoretical error estimates. The reason is because the boundary layer is much more narrow when $\epsilon \ll h$ and the LDG method is locally and globally conservative, so that it can capture the solution and its derivative on the margin area of the boundary layer without producing a wide artificial layer. Once again, this confirms the observations of Figure 4.4 and Figure 4.6 in [18], i.e., DG methods is more 'local' than traditional finite element method.

The errors of Table 1 are plotted in Figure 1 and Figure 2. To display the rapid change of the errors on the last element I_N of the uniform mesh when $\epsilon = 10^{-10}$, we combine the graph of the errors of $E^k(N,0)$ and some related graphs with the graph of $E^k(N,k)$. The first graph of Figure 2 shows that the L^2 error of the LDG linear approximation to the exact solution converges at the optimal rate even on uniform meshes.

TABLE 1. L^2 errors of LDG method and convergence rates of Example 1 for $\epsilon = 10^{-5}$ and $\epsilon = 10^{-10}$.

	$\epsilon = 10^{-5}$					
N	$E_u^1(n_1)$	$r_u^1(n_1)$	$E_u^2(n_1)$	$r_u^2(n_1)$	$E_u^3(n_1+1)$	$r_u^3(n_1+1)$
32	1.15e-04	-	1.03e-06	-	5.28e-09	-
64	3.04e-05	1.92	1.29e-07	2.99	3.54e-10	3.90
128	7.92e-06	1.94	1.62e-08	3.00	2.33e-11	3.93
256	2.01e-06	1.98	2.03e-09	3.00	1.49e-12	3.97
512	5.09e-07	1.99	2.54e-10	3.00	9.42e-14	3.98
N	$E_q^1(n_1)$	$r_q^1(n_1)$	$E_q^2(n_1)$	$r_q^2(n_1)$	$E_q^3(n_1+1)$	$r_q^3(n_1+1)$
32	2.52e-04	-	1.84e-06	-	1.29e-008	-
64	6.34e-05	1.99	2.42e-07	2.92	8.12e-10	3.99
128	1.59e-05	2.00	3.16e-08	2.94	5.09e-11	4.00
256	3.97e-06	2.00	4.02e-09	2.98	3.2187e-12	3.98
512	9.93e-07	2.00	5.08e-10	2.98	6.74e-13	2.26

	$\epsilon = 10^{-10}$					
N	$E_u^1(1)$	$r_u^1(1)$	$E_u^2(2)$	$r_u^2(2)$	$E_u^3(3)$	$r_u^3(3)$
32	1.28e-04	-	1.04e-06	-	5.67e-09	-
64	3.25e-05	1.98	1.299e-07	3.00	3.73e-10	3.93
128	8.19e-06	1.99	1.62e-08	3.00	2.39e-11	3.96
256	2.05e-06	1.99	2.03e-09	3.00	1.51e-12	3.98
512	5.15e-07	2.00	2.54e-10	3.00	9.53e-14	3.99
N	$E_q^1(1)$	$r_q^1(1)$	$E_q^2(2)$	$r_q^2(2)$	$E_q^3(3)$	$r_q^3(3)$
32	2.54e-04	-	1.97e-06	-	1.30e-08	-
64	6.35e-05	2.00	2.55e-07	2.95	8.14e-10	4.00
128	1.59e-05	2.00	3.24e-08	2.98	5.09e-11	4.00
256	4.10e-06	1.95	4.08e-09	2.99	3.30e-12	3.95
512	3.11e-06	0.40	5.13e-10	2.99	6.65e-13	2.31

4.2. The numerical experiments for the SDFEM. The SDFEM solves (1.1) for $u_h \in V_h$ by the following scheme

$$\epsilon(u_h', v') + (au_h', v) + (bu_h, v) + \delta(-\epsilon u_h'' + au_h' + bu_h, av') = (f, v) + \delta(f, av'),$$

where the parameter $\delta = 1/h$ and

$$V_h = \left\{ v \in H_0^1(\mathcal{I}) \;\mid\; v_{|I_j} \in \; P^k(I_j), \quad \forall\, I_j \in \mathcal{I}_h \right\}.$$

Remark 4.2. In the literature, local error estimates of the SDFEM for the solution of the singularly perturbed problem on two-dimensional domain Ω were obtained in [14]. Similar convergence rates were proved for discontinuous Galerkin (DG) methods in [10] and for Continuous Interior Penalty (CIP) Method in [2]. The L^2 error bound $O(\ln(1/h)h^{k+\frac{1}{2}})$ was established for these methods on a subdomain $\Omega_0 \subset \Omega$ where $\partial\Omega_0$ is $O(h\ln(1/h))$ distance away from the outflow boundary of Ω. Here h denotes the maximum size of the quasi-uniform mesh, u_h denotes the numerical solution.

Similar to the LDG method, we denote the error and the convergence rate by:

$$E_u^k(N,n) := \|u - u_h\|_{\mathcal{I}_n}, \qquad E_{u'}^k(N,n) := \sqrt{h}\,\|(u-u_h)'\|_{\mathcal{I}_n},$$

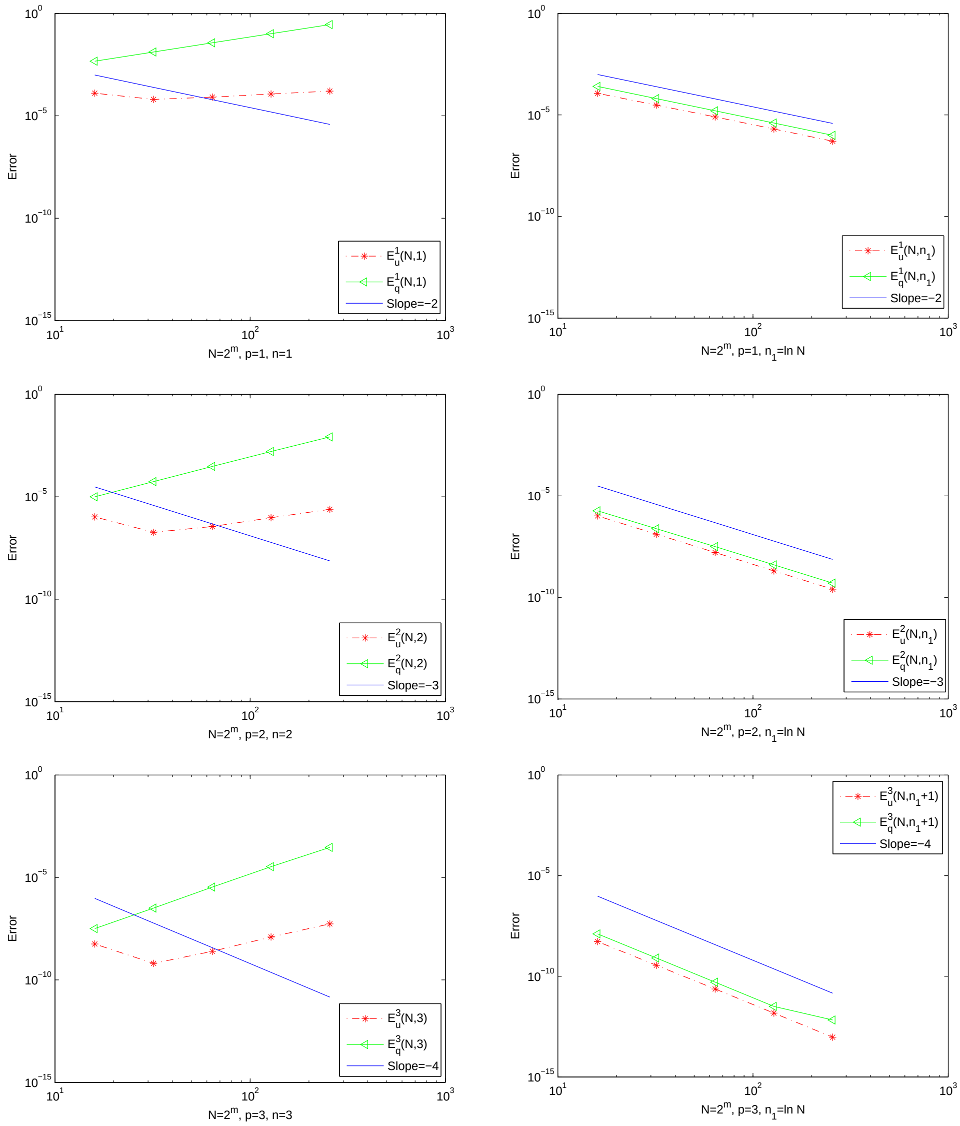

FIGURE 1. The convergence rates of the LDG method for $k = 1, 2, 3$, $\epsilon = 10^{-5}$.

$$r_u^k(N,n) := \log_2 \frac{E_u^k(N,n)}{E_u^k(2N,n)}, \qquad r_{u'}^k(N,n) := \log_2 \frac{E_{u'}^k(N,n)}{E_{u'}^k(2N,n)},$$

for $k = 1, 2, 3$. This definition has slight difference from the one defined in Section 4.1.

Example 4.3. To compare with the LDG method, we apply the SDFEM to the test problem (1.1) with the same exact solution as Example 4.1.

Table 2 displays the L^2 errors $\left(E_q^k(n), E_u^k(n)\right)$ and convergence rates $\left(r_q^k(n), r_u^k(n)\right)$ on subdomain $\mathcal{I}_0$ for $\epsilon = 10^{-5}$ and 10^{-10}, respectively. The main factor $h^{k+1/2}$ of

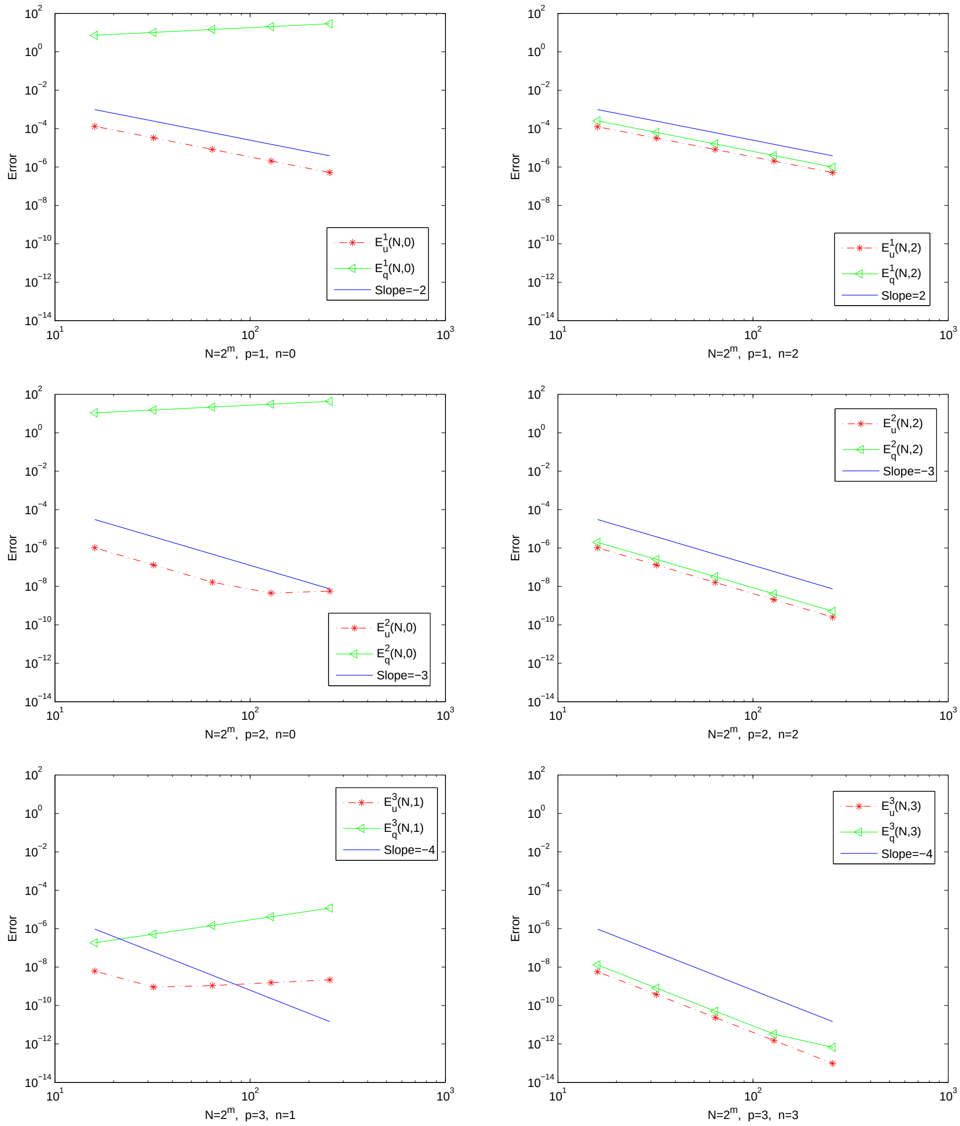

FIGURE 2. The convergence rates of the LDG method for $k = 1, 2, 3$, $\epsilon = 10^{-10}$.

the convergence rate for the approximation of $\sqrt{h}u'$ is observed, while the approximation of u converges at the optimal rate $O(h^{k+1})$. Figure 3 and Figure 4 illustrate that the above rates of convergence can be seen when collecting the errors on a subdomain $\mathcal{I}_0$ which is $O(h \ln(1/h))$ away from the outflow boundary. This subdomain is much smaller than the subdomain used in the LDG method. Actually, for the polynomial approximation of degree k, about $(k+1)\ln N$ elements need to be cut off to maintain the rates of convergence for different values of ϵ, which is observed from Figure 3 and Figure 4.

4.3. The comparison of the LDG method and the SDFEM. The comparison of the data from Table 1 and Table 2 shows that L^2 errors of the LDG method

TABLE 2. L^2 errors of SDFEM and convergence rates of Example 2 for $\epsilon = 10^{-5}$ and $\epsilon = 10^{-10}$.

	$\epsilon = 10^{-5}$					
N	$E_u^1(n_2)$	$r_u^1(n_2)$	$E_u^2(n_3)$	$r_u^2(n_3)$	$E_u^3(n_4)$	$r_u^3(n_4)$
32	2.41e-04	-	1.80e-06	-	6.64e-08	-
64	6.28e-05	1.94	2.02e-07	3.16	2.35e-09	4.82
128	1.60e-05	1.97	2.30e-08	3.13	8.47e-11	4.79
256	4.00e-06	2.00	2.69e-09	3.10	3.24e-12	4.71
512	9.80e-07	2.03	3.21e-10	3.06	2.93e-13	3.47
N	$E_{u'}^1(n_2)$	$r_{u'}^1(n_2)$	$E_{u'}^2(n_3)$	$r_{u'}^2(n_3)$	$E_{u'}^3(n_4)$	$r_{u'}^3(n_4)$
32	2.81e-03	-	4.10e-05	-	4.58e-07	-
64	1.08e-03	1.38	7.87e-06	2.38	3.12e-08	3.88
128	4.08e-04	1.40	1.41e-06	2.48	2.63e-09	3.57
256	1.49e-04	1.45	2.50e-07	2.50	2.38e-10	3.46
512	5.39e-05	1.48	4.41e-08	2.50	2.25e-11	3.41
	$\epsilon = 10^{-10}$					
N	$E_u^1(n_2)$	$r_u^1(n_2)$	$E_u^2(n_3)$	$r_u^2(n_3)$	$E_u^3(n_4)$	$r_u^3(n_4)$
32	2.42e-04	-	1.80e-06	-	6.64e-08	-
64	6.32e-05	1.94	2.02e-07	3.16	2.36e-09	4.82
128	1.62e-05	1.96	2.30e-08	3.13	8.52e-11	4.79
256	4.11e-06	1.98	2.69e-09	3.10	3.28e-12	4.70
512	1.04e-06	1.99	3.22e-10	3.06	2.94e-13	3.48
N	$E_{u'}^1(n_2)$	$r_{u'}^1(n_2)$	$E_{u'}^2(n_3)$	$r_{u'}^2(n_3)$	$E_{u'}^3(n_4)$	$r_{u'}^3(n_4)$
32	2.81e-03	-	4.10e-05	-	4.58e-07	-
64	1.08e-03	1.39	7.87e-06	2.38	3.12e-08	3.88
128	4.08e-04	1.40	1.41e-06	2.48	2.63e-09	3.57
256	1.49e-04	1.45	2.51e-07	2.50	2.39e-10	3.46
512	5.39e-05	1.47	4.43e-08	2.50	2.26e-11	3.40

are smaller than that of SDFEM, although the errors are collected from more mesh elements. The error curves in Figure 1 - Figure 4 confirms the above conclusion. Moreover, the LDG approximation of the derivative converges at a optimal rate $O(h^{k+1})$, which is the same as LDG approximation of u.

Numerical results shows that a small artificial layer does exist for small ϵ if the mesh size h is not very large. Furthermore, if we choose $\epsilon \ll h$ is extremely small, it is observed that the LDG solution U has no oscillation on uniform mesh. The reason is that the huge error caused by the boundary layer, which is inside one element, will not pollute local errors on other elements. This property does not hold true for the SDFEM. As stated in Section 4.2, the artificial layer of the SDFEM contains certain amount of mesh elements for different values of ϵ. To illustrate this point visually, we plot the exact solution (solid blue curve), nodal values of the LDG solution (red star) the SDFEM solution(red star) of the Example 4.1 together in Figure 5 when $\epsilon = 10^{-10}$, $k = 1$ and $N = 32$. The graphs for the case $\epsilon = 10^{-5}$ are very similar to Figure 5.

The first column of Figure 5 includes the graph of the numerical solution and numerical derivative of the LDG method. The second column is about the SDFEM, with penalty parameters $\delta = 1/h$. It can be seen that the SDFEM solution exhibits a numerical layer with width $O(h \ln(1/h))$ when $\delta = 1/h$. Because of the large

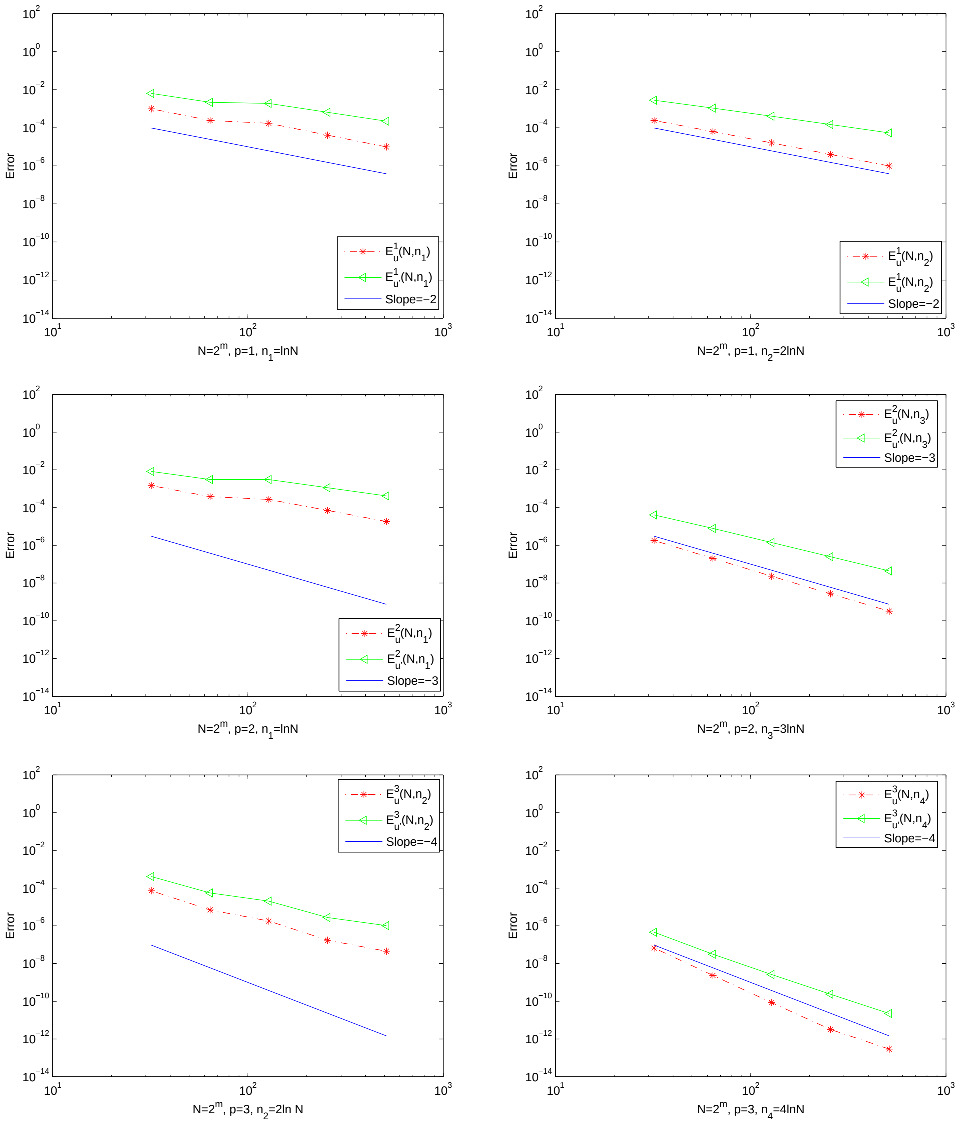

FIGURE 3. The convergence rates of SDFEM for $k = 1, 2, 3$, $\epsilon = 10^{-5}$.

derivative on the last mesh element I_N, the difference of the numerical derivatives in the second row of Figure 5 is not obvious. To see more details, we drop the last element I_N and plot the graph of the numerical derivatives on a subdomain $\mathcal{I} \setminus I_N$ in the third row of Figure 5. It can be seen that the SDFEM approximation for the derivative also has a artificial layer, which is wider than LDG.

All these numerical observations seems to support the conclusion that the LDG method capture the boundary layer of Problem (1.1) more locally than FEM and SDFEM.

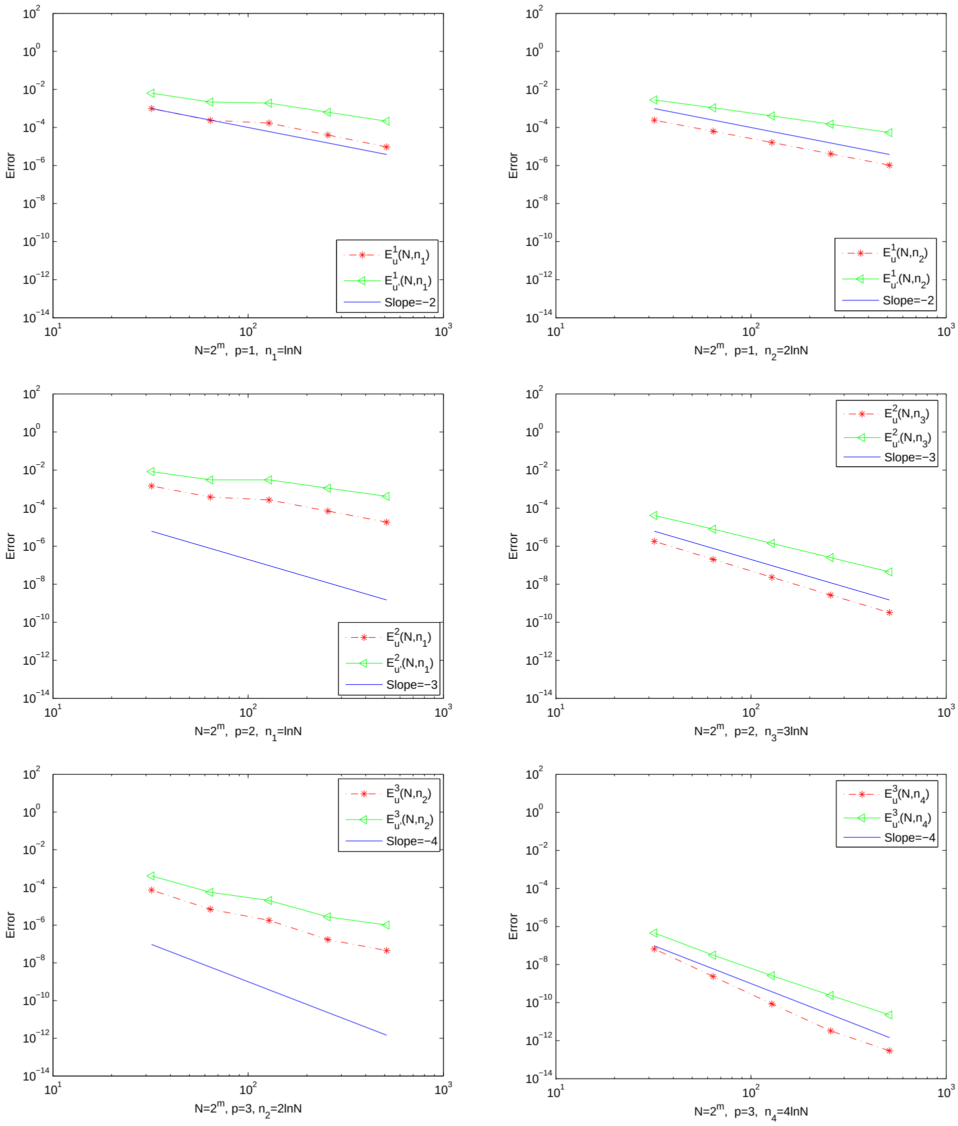

FIGURE 4. The convergence rates of SDFEM for $k = 1, 2, 3$, $\epsilon = 10^{-10}$.

5. Concluding remarks

In this article, we investigate the LDG discretization on uniform and quasi-uniform mesh for solving singularly perturbed problems in one dimensional setting. Based on a mixed discretization for the elliptic part, we establish an optimal rate of convergence on a subdomain, which is $O(h(\ln 1/h))$ distance away from the outflow boundary. Numerical experiments indicate that the our theoretical results are sharp. Numerical experiments also show that the LDG method can capture the solution and its derivative on the margin area of the boundary layer without producing a wide artificial layer. A comparison of the LDG method and SDFEM are provided.

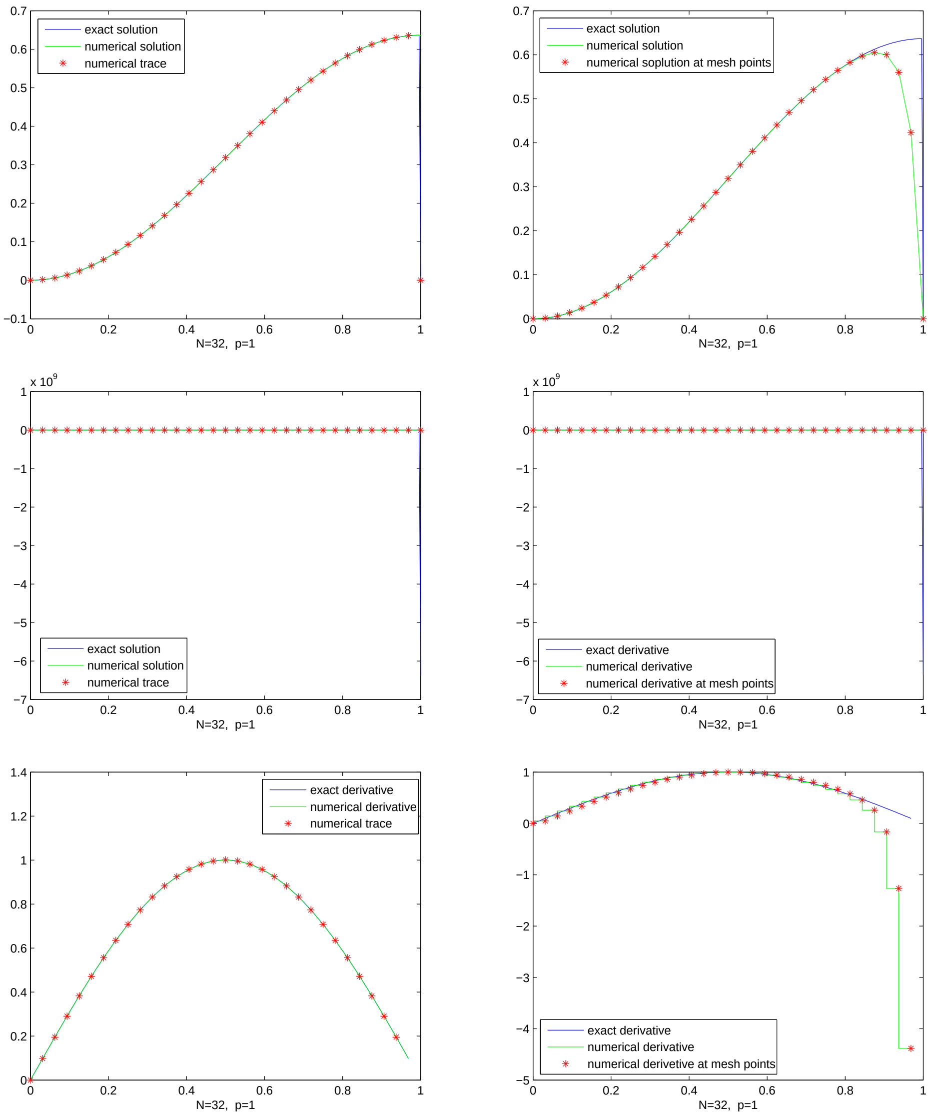

FIGURE 5. The LDG solution and the exact solution (column 1); the SDFEM solution and the exact solution (column 2). ($k = 1$, $\epsilon = 10^{-10}$)

References

[1] S. Adjerid, M. Aiffa, and J. E. Flaherty, *High-order finite element methods for singularly-perturbed elliptic and parabolic problems*, SIAM J. Appl. Math. **55** 1995 no:2, 520–543.

[2] E. Burman, J. Guzmán and D. Leykekhman, *Weighted error estimates of the continuous interior penalty method for singularly perturbed problems*, IMA Journal of Numerical Analysis, **29** (2009) no: 2, 284–314.

[3] P. Castillo, B. Cockburn, D. Schötzau and C. Schwab. *Optimal a priori error estimates for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems*, Math. Comp. **71** (2002), 455–478.

[4] F. Celiker and B. Cockburn, *Superconvergence of the numerical traces of discontinuous Galerkin and hybridized methods for convection-diffusion problems in one space dimension*, Math. Comp. **76** (2007), no. 257, 67–96.

[5] Fatih Celiker, Zhimin Zhang, and Huiqing Zhu, *Nodal superconvergence of SDFEM for singularly perturbed problems*, J. Sci. Comput., Online first, 27 April 2011.

[6] L. Chen and J. Xu, *An optimal streamline diffusion finite element method for a singularly perturbed problem*, Recent advances in adaptive computation, Contemp. Math., Amer. Math. Soc., Providence, RI, **383** (2005), 191–201.

[7] Y. Cheng and C.W. Shu, *Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension*, SIAM J. Numer. Anal. **47** (2010), 4044-4072.

[8] B. Cockburn and C.W. Shu, *The local discontinuous Galerkin finite element method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal. **35** (1998), 2440–2463.

[9] B. Cockburn, G.E. Karniadakis and C.-W. Shu, The development of discontinuous Galerkin methods, in Discontinuous Galerkin Methods, Theory, Computation and Applications, Lecture Notes in Pure and Applied Mathematics, Vol.196, Marcel Dekker, Inc., New York, 1998.

[10] J. Guzmán, *Local analysis of discontinuous Galerkin methods applied to singularly perturbed problems*, J. Numer. Math., **14** (2006) no: 1, 41–56.

[11] T.J.R. Hughes, *A simple scheme for developing upwind finite elements*, Int. J. Numer. Meth. Engng., **12** (1978), 1359-1365.

[12] T.J.R. Hughes and A.N. Brooks, *A multidimensional upwind scheme with no crosswind diffusion*, In: Analytical and Numerical Approaches to Asymptotics problem in Analysis (Eds. O. Axelsson, L. S. Frank, and A. van der Sluis), North Holland, Amsterdam, **49** (1981), 99–116.

[13] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*. Math. Comp. **46** (1986), 1–26.

[14] C. Johnson, A.H. Schatz, and L.B. Wahlbin, *Crosswind smear and pointwise error estimates in streamline diffusion finite element methods*, Math. Comp., **49** (1987) no: 179, 25–38.

[15] J.J. Miller, E. O'Riordan and G. I. Shishkin, *Fitted numerical methods for singularly perturbed reaction-diffusion problems in two and one dimensions*, World Scientific, Singapore, 1996.

[16] H.G. Roos, M. Stynes, and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, volume 24 of Springer Series in Computational Mathematics, Springer-Verlag, 2008.

[17] D. Schötzau and C. Schwab, *Time discretization of parabolic problems by the hp-version of the discontinuous Galerkin finite element method*, SIAM J. Numer. Anal. **38** (2000), 837–875.

[18] Z. Xie and Z. Zhang, *Superconvergence of DG method for one-dimensional singularly perturbed problems*, J. Comput. Math. **25** (2007), no. 2, 185–200.

[19] ——, *Uniform superconvergence analysis of the discontinuous Galerkin method for a singularly perturbed problem in 1-D*, Math. Comp. **79** (2010), no. 269, 35–45.

[20] Z. Zhang, *Finite element superconvergence approximation for one-dimensional singularly perturbed problems*, Numer. Methods Partial Differential Equations. **18** (2002), no. 3, 374–395.

[21] Huiqing Zhu, Haiyan Tian, and Zhimin Zhang *Convergence analysis of the LDG method for singularly perturbed two-point boundary value problems*, Comm. Math. Sci., **9** (2011), 1013–1032.

Department of Mathematics, The University of Southern Mississippi, Hattiesburg, MS 39406, USA

E-mail: Huiqing.Zhu@usm.edu

URL: http://www.math.usm.edu/zhu/

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

Beijing Computational Science Research Center, No. 3 Heqing Road, Haidian District, Beijing 100084, China.

E-mail: zzhang@math.wayne.edu

URL: http://www.math.wayne.edu/ zzhang/