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Pointwise Error Estimates for the LDG Method Applied to 1-d Singularly Perturbed Reaction-Diffusion Problems

Huiqing Zhu · Zhimin Zhang

Abstract — The local discontinuous Galerkin method (LDG) is considered for solving one-dimensional singularly perturbed two-point boundary value problems of reaction-diffusion type. Pointwise error estimates for the LDG approximation to the solution and its derivative are established on a Shishkin-type mesh. Numerical experiments are presented. Moreover, a superconvergence of order $2k+1$ of the numerical traces is observed numerically.

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1. Introduction

In this work, we apply the local discontinuous Galerkin method (LDG) to one-dimensional singularly perturbed reaction-diffusion problems

$$\begin{cases} -\epsilon u'' + b(x)u = f(x) & \text{on } \mathcal{I} = (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

Here $\epsilon \ll 1$, b and f are sufficiently smooth with property $b(x) \geqslant \beta^2 > 0$. When ϵ is small, the analytical solution of problem (1.1) typically exhibits boundary layers at both ends of the boundary. Since the solution of the standard finite element method (FEM) exhibits large nonphysical oscillation, many stabilization techniques have been developed, including upwind scheme, streamline-diffusion FEM, discontinuous Galerkin methods, and least-squares FEM (cf. [8, 10, 12]). Another way for solving this problem is to construct a numerical scheme on layer-adapted meshes, such as the Shishkin mesh and the Bakhvalov mesh (cf. [4, 5, 8, 10, 12, 13, 15]). These layer-adapted meshes are designed a priori and are based on the knowledge of the boundary layers. In the literature, the following uniform error estimate was established for the central differencing or linear finite elements:

$$\max_{j=0,\dots,N} |u(x_j) - u_N(x_j)| \leqslant \begin{cases} N^{-2}, & \text{for a Bakhvalov-type mesh,} \\ (\ln N/N)^2, & \text{for a Shishkin-type mesh,} \end{cases}$$

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where u_N denotes the numerical solution; x_j denotes the nodal points of the mesh. A uniform nodal convergence of the linear FEM solution on equidistributing meshes has also been developed in one-dimensional setting [5]. For FEM with higher degree approximation space, some uniform nodal error estimates at the nodes of the Shishkin mesh inside the boundary layer region were derived as a corollary of the L^2 error estimates [13].

Recently, DG methods were intensively investigated for elliptic problems and convection-diffusion problems by many researchers (cf. [1, 2, 3, 12, 14, 15]). Many error estimates on the Shishkin mesh have been established in L^2 sense. The numerical results presented so far show that DG methods are more 'local' than the standard FEM [12] and don't exhibit any oscillation on uniform meshes. However, no theoretical results have been reported for uniform pointwise error estimates of DG methods till now. In this article, the local discontinuous Galerkin method (LDG) is analyzed for solving problem (1.1). The Shishkin-type mesh is employed. Based on the L^2 error estimates of the LDG method obtained in [15] for the reaction-diffusion problem (1.1), we derive pointwise error estimates for the LDG approximation to the solution and its derivative on different parts of the domain. The results are uniformly valid with respect to the singular perturbation parameter ϵ. Numerical experiments illustrate that the errors $u - U$ and $\sqrt{\epsilon}(u' - \epsilon^{-1}Q)$ converge at a rate of $O((\ln N/N)^{k+1})$, which indicates that our theoretical results are not sharp. Here U and Q denote the numerical solution and the numerical derivative respectively. We also observed a uniform superconvergence of order $O((\ln N/N)^{2k+1})$ of the numerical traces, which has not been reported before. A similar superconvergence phenomenon of the LDG method for singularly perturbed convection-diffusion problems was reported in [12].

The outline of the paper is as follows: the LDG formulation, the regularity results, the Shishkin mesh, and the L^2 error estimates (cf. [15]) are introduced in Section 2. Section 3 presents the pointwise error estimates for Q. The technique used in this section cannot be extended to the error estimate for U inside the boundary layer so that another stability property of the LDG formulation is needed. In Section 4, we present this stability property and prove that the numerical solution U converges uniformly in parameter ϵ. Results of numerical tests are presented in Section 5.

Notation. Throughout this paper C denotes a generic positive constant that is independent of ϵ and the mesh size. It may take different values in different places. For any $\mathcal{D} \subseteq \mathcal{I}_N$ the norm on $L^2(\mathcal{D})$ is defined as $\|v\|_{\mathcal{D}} := (\int_D v^2 dx)^{1/2}$.

2. The LDG Method

We begin by introducing an arbitrary mesh and an appropriate functional setting. Denote by $\mathcal{I}_N$ a decomposition of domain $\mathcal{I}$, and $\mathcal{I}_N = \{I_j, j = 1, 2, \ldots, N\}$. Here N denotes the total number of mesh elements; I_j indicates the j-th element of the mesh. The LDG formulation, the existence and the uniqueness of the numerical solution are ensured for an arbitrary mesh. We will use Shishkin meshes in the analysis after Section 2.1. Let x_j ($j = 0, 1, \ldots, N$) be mesh nodes with $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$. Define $v(x_j^{\pm}) = \lim_{\delta\to 0} v(x_j \pm \delta)$ as in [6]. Then we set $I_j = [x_{j-1}^+, x_j^-]$ (which includes two end points x_{j-1}^+ and x_j^-) and $h_j = x_j - x_{j-1}$ for $j = 1, 2, \ldots, N$. For each element $I_j \in \mathcal{I}_N$, we set its outward unit normal $n_{I_j}(x_j^-) = 1$ and $n_{I_j}(x_{j-1}^+) = -1$. For each mesh point x_j, we denote $v_j = v(x_j)$ if v is continuous at x_j, $v_j^{\pm} = v(x_j^{\pm})$, $[\![v]\!]_0 = -v_0^+$, $[\![v]\!]_N = v_N^-$, and $[\![v]\!]_j = v_j^- n_{I_j}(x_j^-) + v_j^+ n_{I_{j+1}}(x_j^+) = v_j^- - v_j^+$ for $j = 1, \ldots, N-1$.

We denote by $H^m(\mathcal{I}_N)$ the space of functions on $\mathcal{I}$ whose restriction to each element I belongs to the Sobolev space $H^m(I)$. Define

$$V_N := \{v \in L^2(\mathcal{I}) : v|_{I_j} \in \mathcal{P}^k(I_j),\ \forall j = 1, \ldots, N\},$$

where $\mathcal{P}^k$ is the polynomial space of degree at most k. We use the notations

$$(\varphi, \psi) = \sum_{I_j \in \mathcal{I}_N} (\varphi, \psi)_{I_j} = \sum_{I_j \in \mathcal{I}_N} \int_{I_j} \varphi(x)\psi(x)dx$$

and

$$\langle \varphi, \psi \rangle_{\partial \mathcal{I}_N} = \sum_{I_j \in \mathcal{I}_N} \langle \varphi, \psi \rangle_{\partial I_j} = \sum_{I_j \in \mathcal{I}_N} [\varphi_j^- \psi_j^- - \varphi_{j-1}^+ \psi_{j-1}^+].$$

For simplicity, the subscripts $\mathcal{I}_N$ and $\partial \mathcal{I}_N$ will always be dropped. For any $\mathcal{D} \subseteq \mathcal{I}_N$ the Sobolev seminorm on $H^s(\mathcal{D})$ is defined as

$$|v|_{s,\mathcal{D}} := (v^{(s)}, v^{(s)})_{\mathcal{D}}^{1/2}.$$

Accordingly, the Sobolev norm on $H^r(\mathcal{D})$ is defined as

$$\|v\|_{r,\mathcal{D}} := \left(\sum_{s=0}^{r} |v|_{s,D}^2 \right)^{1/2}.$$

We drop the first subscript whenever $r = 0$, and the second one if $\mathcal{D} = \mathcal{I}_N$.

By introducing a new variable $q = \epsilon u'$, we rewrite the problem (1.1) as

$$\begin{cases} q = \epsilon u' & \text{in } \mathcal{I} = (0, 1), \\ -q' + bu = f & \text{in } \mathcal{I} = (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Then an approximate solution (U, Q) of the LDG method will be found in $V_N \times V_N$, which is a finite-dimensional subspace of $H^1(\mathcal{I}_N) \times H^1(\mathcal{I}_N)$. We consider the following LDG formulation: Find $(Q, U) \in V_N \times V_N$, such that

$$\begin{cases} (Q, w) = -\epsilon(U, w') + \langle \epsilon \hat{u}, w \rangle_{\partial \mathcal{I}_N}, \\ (Q, v') - \langle \hat{q}, v \rangle_{\partial \mathcal{I}_N} + (bU, v) = (f, v), \end{cases} \tag{2.2}$$

for all $(w, v) \in V_N \times V_N$.

We take the following numerical traces:

$$\begin{cases} \hat{q}(x_j) = Q_j^+ - \lambda_j [\![U]\!]_j, & \text{for } j = 0, 1, \ldots, N-1, \\ \hat{q}(1) = Q(1^-) - \lambda_N U(1^-), \\ \hat{u}(x_j) = U_j^-, & \text{for } j = 1, \ldots, N-1, \\ \hat{u}(0) = 0, \quad \hat{u}(1) = 0, \end{cases} \tag{2.3}$$

where $0 \leqslant \lambda_j \leqslant C$ $(j = 0, 1, \ldots, N)$ for some constant C. Optimal L^2 convergence of the numerical solution and its derivative was established in [15] when $\lambda_N \geqslant O(1)$ and $\lambda_j = 0$ for all $j = 0, 1, \ldots, N-1$.

Substituting the numerical traces (2.3) into (2.2) yields the following system:

$$\begin{cases} a(Q,w)+b_1(U,w)=0, \\ b_2(v,Q)+c(U,v)=f(v), \end{cases} \tag{2.4}$$

where

$$\begin{cases} a(Q,w)=\epsilon^{-1}(Q,w), \\ b_1(U,w)=(U,w')-\sum_{j=1}^{N-1}U_j^-[\![w]\!]_j, \\ b_2(Q,v)=(Q,v')-\sum_{j=0}^{N-1}Q_j^+[\![v]\!]_j-Q_N^-v_N^-, \\ c(U,v)=(bU,v)+\sum_{j=0}^{N}\lambda_j[\![U]\!]_j[\![v]\!]_j, \\ f(v)=(f,v). \end{cases} \tag{2.5}$$

Assume that $a_j^+=a_j^-=a_j$ for any $j=0,1,\dots,N$. Using the integration by parts we can verify that $b_1(v,w)=-b_2(w,v)$.

Lemma 2.1 (Existence and uniqueness, [15]). *If $\lambda_j\geqslant 0$ $(j=0,1,\dots,N)$, then the LDG solution determined by* (2.2) *and numerical traces* (2.3) *exists and is unique.*

To discuss the error estimates, we define

$$\mathcal{A}(\phi,\psi;w,v)=a(\phi,w)+b_1(\psi,w)+b_2(\phi,v)+c(\psi,v),\quad \text{for all } (\phi,\psi),\ (w,v)\in V_N\times V_N. \tag{2.6}$$

By the consistency of the numerical traces, it is straightforward to verify the orthogonality

$$\mathcal{A}(q-Q,u-U;w,v)=0,\quad \text{for all } (w,v)\in V_N\times V_N. \tag{2.7}$$

As a consequence, (2.6) introduces a norm for any $(w,v)\in V_N\times V_N$,

$$|(w,v)|_{\mathcal{A}}^2:=\mathcal{A}(w,v;w,v)=\epsilon^{-1}\|w\|^2+\|\sqrt{b}v\|^2+\sum_{j=0}^{N}\lambda_j[\![v]\!]_j^2. \tag{2.8}$$

2.1. Shishkin Mesh

The exact solution of (1.1) admits a decomposition of the form

$$u=\bar{u}+u_\epsilon,\quad u_\epsilon=u_{\epsilon,1}+u_{\epsilon,2},$$

where the component functions have the regularity [9, 13]

$$|\bar{u}^{(j)}(x)|\leqslant C,\quad |u_{\epsilon,1}^{(j)}(x)|\leqslant C\epsilon^{-j/2}e^{-\beta x/\sqrt{\epsilon}},\quad |u_{\epsilon,2}^{(j)}(x)|\leqslant C\epsilon^{-j/2}e^{-\beta(1-x)/\sqrt{\epsilon}} \tag{2.9}$$

for any x in $\mathcal{I}$ and $j=0,1,\dots,k+2$. Then we have $q=\bar{q}+q_\epsilon$ with $\bar{q}=\epsilon\bar{u}'$ and $q_\epsilon=q_{\epsilon,1}+q_{\epsilon,2}=\epsilon u_{\epsilon,1}'+\epsilon u_{\epsilon,2}'$. The components of q satisfy

$$|\bar{q}^{(j)}|\leqslant C\epsilon,\quad |q_{\epsilon,1}^{(j)}(x)|\leqslant C\epsilon^{-(j-1)/2}e^{-\beta x/\sqrt{\epsilon}},\quad |q_{\epsilon,2}^{(j)}(x)|\leqslant C\epsilon^{-(j-1)/2}e^{-\beta(1-x)/\sqrt{\epsilon}} \tag{2.10}$$

for any x in $\mathcal{I}$ and $j=0,1,\dots,k+1$.

We now define a *Shishkin mesh* $\mathcal{I}_N$ that is appropriate for the problem (1.1). Let $N \geqslant 4$ be a multiple of 4. Let $K_1 \geqslant k+1$. Define

$$\tau := \min\Big\{\frac{1}{4}, \frac{K_1}{\beta}\sqrt{\epsilon}\ln N\Big\} \tag{2.11}$$

and

$$H := 2\frac{(1-2\tau)}{N} \quad \text{and} \quad h := \frac{4\tau}{N}.$$

The *nodes* of the mesh $\mathcal{I}_N$ are defined recursively by setting $x_0 = 0$ and

$$x_j = \begin{cases} x_{j-1} + h, & \text{for } j = 1, 2, \dots, N/4, \\ x_{j-1} + H, & \text{for } j = N/4+1, \dots, 3N/4, \\ x_{j-1} + h, & \text{for } j = 3N/4+1, \dots, N. \end{cases}$$

Since the mesh $\mathcal{I}_N$ is piecewise uniform we define

$$\mathcal{I}_{\mathrm{R}} = \bigcup_{j=N/4+1}^{3N/4} I_j \quad \text{and} \quad \mathcal{I}_{\mathrm{BL}} = \mathcal{I} \setminus \mathcal{I}_{\mathrm{R}}.$$

Clearly, $\mathcal{I}_{\mathrm{R}}$ is a uniform discretization of the interval $(\tau, 1-\tau)$ of meshsize H, and $\mathcal{I}_{\mathrm{BL}}$ is that of the interval $(0,\tau)\cup(1-\tau,1)$ of meshsize h. Let us note that $N \leqslant 2/H$ and $H \leqslant 2/N$. In this article, we always assume that $\epsilon < \frac{1}{N}$.

2.2. L^2 Error Estimates

We use polynomial interpolation of degree $k \geqslant 1$. Let $I = (a^+, a^-)$ be an arbitrary interval. For $v \in C^0(\bar{I})$, we define the projection $\pi^{\pm}v \in \mathcal{P}^k(I)$ by the following two conditions:

$$\pi^{\pm}v(a^{\pm}) = v(a^{\pm}), \qquad \int_I [v(x) - \pi^{\pm}v(x)]p(x)dx = 0, \quad \text{for all } p \in \mathcal{P}^{k-1}(I).$$

Define the interpolation of u and q as $u_I = \pi^- u$ and $q_I = \pi^+ q$ on I_j for any $j = 1, 2, \dots, N$. Let $\xi_u := u - u_I$, $\eta_u := u_I - U$, $e_u := u - U$, $\xi_q := q - q_I$, $\eta_q := q_I - Q$, and $e_q := q - Q$. As a consequence, we get $e_u = \xi_u + \eta_u$, $e_q = \xi_q + \eta_q$.

To estimate the interpolation errors ξ_u and ξ_q, we need two preliminary lemmas. The first lemma was proven in [11].

Lemma 2.2 ([11, Lemma 3.7])**.** *For any $v \in C(\overline{I})$, the interpolation operators $\pi^{\pm}$ satisfy*

$$\|\pi^{\pm}v\|_I^2 \leqslant C(\|v\|_I^2 + |v(a^{\pm})|^2) \tag{2.12}$$

on the reference element $I = (a^+, a^-)$.

The second lemma gives elementwise error bounds of the interpolation.

Lemma 2.3. *If $\mathcal{I}_N$ is an arbitrary decomposition of the domain $\mathcal{I}$, then there exists a constant C such that*

$$\|\xi_u'\|_{I_j} \leqslant Ch_j^k \|u^{(k+1)}\|_{I_j}, \tag{2.13a}$$

$$\|\xi_{\bar{q}}'\|_{I_j} \leqslant Ch_j^k \|\bar{q}^{(k+1)}\|_{I_j}, \tag{2.13b}$$

$$\|\xi_{q_\epsilon}'\|_{I_j} \leqslant Ch_j^k \|q_\epsilon^{(k+1)}\|_{I_j}, \tag{2.13c}$$

for any $I_j \in \mathcal{I}_N$ $(j = 1, 2, \dots, N)$.

Proof. Notice that the mapping $\Phi : H^{k+1}(K) \to H^1(K)$ defined by $\Phi(u) = u - u_I$ is linear and continuous. Meanwhile, we have $\Phi(v) = 0$ for all $v \in \mathcal{P}^k(K)$. Using the scaling arguments and the Bramble–Hilbert Lemma we obtain (2.13a). Since we assumed that $q \in H^{k+1}(K)$ in (2.10), we can prove (2.13b) and (2.13c) in a similar way. □

The following error estimate is cited from [15], and is valid for $K_1 \geqslant k+1$.

Lemma 2.4 ([15, Theorem 3.5]). *If $\lambda_j = 0$ for all $j = 0, 1, \ldots, N-1$ and $\lambda_N \geqslant O(1)$, then there exists a constant C independent of ϵ and N such that*

$$\epsilon^{-\frac{1}{2}}\|\eta_q\| + \|\eta_u\| \leqslant C\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1} + N^{-(k+1)}\Big], \tag{2.14}$$

Moreover, if $\bar{u} \in V_N$, then we have

$$\epsilon^{-\frac{1}{2}}\|e_q\| + \|e_u\| \leqslant C\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1} + N^{-(k+\frac{3}{2})}\Big]. \tag{2.15}$$

The following lemma can be derived directly from the proof of [15, Lemmas 4.4 and 4.6]. It is useful when estimating e_u inside the boundary layer region.

Lemma 2.5. (1) *Suppose $\lambda_j \sim O(\frac{h}{\epsilon})$ for all $j = 0, 1, \ldots, N$. There exists a constant C such that for any $(w, v) \in V_N \times V_N$*

$$|A(\xi_q, \xi_u; w, v)| \leqslant C\Big(\frac{\ln N}{N}\Big)^{k+1}|(w, v)|_{\mathcal{A}}. \tag{2.16}$$

(2) *Suppose $\lambda_j \sim O(1)$ for all $j = 0, 1, \ldots, N$. There exists a constant C such that for any $(w, v) \in V_N \times V_N$*

$$|A(\xi_q, \xi_u; w, v)| \leqslant C\Big(\frac{\ln N}{N}\Big)^{k+1/2}|(w, v)|_{\mathcal{A}}. \tag{2.17}$$

(3) *Suppose $\lambda_j \sim O(\frac{\epsilon}{h})$ for all $j = 0, 1, \ldots, N$. There exists a constant C such that for any $(w, v) \in V_N \times V_N$*

$$|A(\xi_q, \xi_u; w, v)| \leqslant C\Big(\frac{\ln N}{N}\Big)^{k}|(w, v)|_{\mathcal{A}}. \tag{2.18}$$

Proof. Recall the definition of the bilinear form $\mathcal{A}$

$$\begin{aligned}|\mathcal{A}(\xi_q, \xi_u; w, v)| &\leqslant |a(\xi_q, w)| + |b_1(\xi_u, w)| + |b_2(\xi_q, v)| + |c(\xi_u, v)| \\ &= S_1 + S_2 + S_3 + S_4.\end{aligned}$$

The error bounds of S_1, S_2, S_3, S_4 have been derived in the proof of [15, Lemmas 4.4 and 4.6], i.e.,

$$S_1 \leqslant \epsilon^{-1}\|\xi_q\|_{\mathcal{I}_N}\|w\|_{\mathcal{I}_N} \leqslant C\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1} + N^{-(k+1)}\Big]|(w,v)|_{\mathcal{A}},$$

$$S_2 = (\xi_u, w') + \sum_{j=1}^{N-1}(\xi_u)_j^- [\![(w)_j]\!] = 0,$$

$$S_3 = -\xi_q(1^-)v(1^-) \leqslant C|\xi_q(1^-)||(w,v)|_{\mathcal{A}} \leqslant C\sqrt{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1}|(w,v)|_{\mathcal{A}},$$

$$S_4 \leqslant C\Bigg(\|\xi_u\|_{\mathcal{I}_N} + \sqrt{\sum_{j=0}^{N}\lambda_j[\![(\xi_u)]\!]_j^2}\Bigg)|(w,v)|_{\mathcal{A}}$$

$$\leqslant \begin{cases} C(\frac{\ln N}{N})^{k+1}|(w,v)|_{\mathcal{A}}, & \text{if } \lambda_j \sim O(\frac{h}{\epsilon}),\ j=0,1,\dots,N, \\ C(\frac{\ln N}{N})^{k+\frac12}|(w,v)|_{\mathcal{A}}, & \text{if } \lambda_j \sim O(1),\ j=0,1,\dots,N, \\ C(\frac{\ln N}{N})^{k}|(w,v)|_{\mathcal{A}}, & \text{if } \lambda_j \sim O(\frac{\epsilon}{h}),\ j=0,1,\dots,N. \end{cases}$$

Collecting the estimates of S_1, S_2, S_3, S_4 establishes the estimates (2.16)–(2.18). □

3. Pointwise Error Estimates for e_q

In this section we shall discuss the pointwise error estimates for e_q. First, we introduce the inverse inequality and the trace inequality, which will be frequently used in the analysis. Let $K = [a,b]$ and $h_K = b-a$. There exists a constant C which depends solely on k such that

$$\|v'\|_K \leqslant Ch_K^{-1}\|v\|_K \quad \text{for all } v \in \mathcal{P}^k(K). \tag{3.1}$$

The trace inequality is similar to [7, Lemma A.1]. There exists a constant C independent of ϵ and N such that for any $x \in K$

$$|v(x)| \leqslant C[h_K^{-\frac12}\|v\|_K + h_K^{\frac12}\|v'\|_K] \quad \text{for all } v \in C^1(K), \tag{3.2}$$

and

$$|v(x)| \leqslant Ch_K^{-\frac12}\|v\|_K \quad \text{for all } v \in \mathcal{P}^k(K). \tag{3.3}$$

Theorem 3.1. *If the assumption of Lemma 2.4 holds true, then there exists a constant C such that for all $x \in I_j$, $j = N/4+1, \dots, 3N/4$*

$$\epsilon^{-\frac12}|e_q(x)| \leqslant C\Big[\sqrt[4]{\epsilon(\ln N)^2}\Big(\frac{\ln N}{N}\Big)^{k+1/2} + N^{-(k+1/2)}\Big]. \tag{3.4}$$

Proof. Using (3.3) and (2.12) we have for any $x \in I_j$

$$\begin{aligned} |e_q(x)| &\leqslant |\xi_{\bar q}(x)| + |q_\epsilon(x)| + |\pi^+ q_\epsilon(x)| + |\eta_q(x)|, \\ &\leqslant |\xi_{\bar q}(x)| + |q_\epsilon(x)| + CH^{-\frac12}(\|\pi^+ q_\epsilon\|_{I_j} + \|\eta_q\|_{I_j}) \\ &\leqslant |\xi_{\bar q}(x)| + |q_\epsilon(x)| + CH^{-\frac12}(\|q_\epsilon\|_{I_j} + \sqrt{H}|q_\epsilon(x_{j-1}^+)| + \|\eta_q\|_{I_j}) \\ &\leqslant |\xi_{\bar q}(x)| + |q_\epsilon(x)| + |q_\epsilon(x_{j-1}^+)| + CH^{-\frac12}(\|q_\epsilon\|_{I_j} + \|\eta_q\|_{I_j}) \end{aligned} \tag{3.5}$$

The error bound of $\|\eta_q\|_{I_j}$ is given in (2.14), which is

$$H^{-\frac12}\|\eta_q\|_{I_j} \leqslant \sqrt{\frac{\epsilon}{H}}(\epsilon^{-\frac12}\|\eta_q\|_{I_j}) \leqslant C\sqrt{\epsilon N}\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1} + N^{-(k+1)}\Big]. \tag{3.6}$$

To estimate $|\xi_{\bar{q}}(x)|$, we use (3.2) and (2.13b) to get

$$|\xi_{\bar{q}}(x)| \leqslant C(H^{-\frac{1}{2}}\|\xi_{\bar{q}}\|_{I_j} + \sqrt{H}\|\xi'_{\bar{q}}\|_{I_j}) \leqslant C\epsilon H^{k+\frac{1}{2}} \leqslant C\epsilon^{\frac{1}{2}}N^{-(k+1)}. \tag{3.7}$$

For $\|q_\epsilon\|_{I_j}$, $|q_\epsilon(x)|$, and $|q_\epsilon(x_{j-1}^+)|$, we use (2.13b), (2.10) to get

$$\begin{aligned} \|q_\epsilon\|_{I_j} &\leqslant C\Big[\int_{I_j} \epsilon e^{-2\beta(1-x)/\sqrt{\epsilon}}dx + \int_{I_j} \epsilon e^{-2\beta x/\sqrt{\epsilon}}dx\Big]^{\frac{1}{2}} \\ &\leqslant C\epsilon^{\frac{3}{4}}e^{-\beta\tau/\sqrt{\epsilon}} \leqslant C\epsilon^{\frac{3}{4}}N^{-(k+1)}, \\ \max\{|q_\epsilon(x_{j-1}^+)|, |q_\epsilon(x)|\} &\leqslant |q_{\epsilon,1}(x_{\frac{N}{4}})| + |q_{\epsilon,2}(x_{\frac{3N}{4}})| \\ &\leqslant C\epsilon^{\frac{1}{2}}e^{-\beta\tau/\sqrt{\epsilon}} \leqslant C\epsilon^{\frac{1}{2}}N^{-(k+1)}. \end{aligned} \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) into (3.5) yields (3.4). (Here we used the assumption $\epsilon < N^{-1}$.) □

Theorem 3.2. *Under the assumption of Lemma 2.4, there exists a constant C such that for all $x \in I_j$, $j = 1, \ldots, N/4, 3N/4+1, \ldots, N$*

$$|e_q(x)| \leqslant C\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1/2} + N^{-(k+1/2)}\Big]. \tag{3.9}$$

Proof. Consider e_q at $x \in I_j$ for any $j = 1, \ldots, N/4, 3N/4+1, \ldots, N$. Using (3.2) and (2.14), one has

$$\begin{aligned} |e_q(x)| &\leqslant C\Big(\sqrt{h}\|e'_q\|_{I_j} + \sqrt{\frac{1}{h}}\|e_q\|_{I_j}\Big) \\ &\leqslant C\Big(\sqrt{h}\|\xi'_q\|_{I_j} + \sqrt{h}\|\eta'_q\|_{I_j} + \sqrt{\frac{1}{h}}\|\xi_q\|_{I_j} + \sqrt{\frac{1}{h}}\|\eta_q\|_{I_j}\Big) \\ &\leqslant C\Big(\sqrt{h}\|\xi'_{\bar{q}}\|_{I_j} + \sqrt{h}\|\xi'_{q_\epsilon}\|_{I_j} + \sqrt{\frac{N}{\ln N}}\epsilon^{-\frac{1}{2}}\|e_q\|_{I_j}\Big) \\ &\leqslant C\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1/2} + N^{-(k+1/2)}\Big], \end{aligned} \tag{3.10}$$

where we have used Lemma 2.13 and (2.10) to derive

$$\begin{aligned} \sqrt{h}\|\xi'_{\bar{q}}\|_{I_j} &\leqslant C\epsilon h^{k+1/2}, \\ \sqrt{h}\|\xi'_{q_\epsilon}\|_{I_j} &\leqslant Ch^{k+1/2}\|q_\epsilon^{(k+1)}\|_{I_j} \leqslant C\sqrt{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1/2}. \end{aligned}$$

□

Remark 3.3. If an additional assumption $\bar{u} \in V_N$ is added to the regularity of the exact solution $u(x, y)$, it follows that $\bar{q} \in V_N$ and $\xi_{\bar{q}} = 0$. The analysis of $\xi_{\bar{q}}$ could be removed from the proof of the nodal error estimate of e_q. Therefore, from (2.15) the above error bounds (3.4) and (3.9) turn out to be

$$\epsilon^{-\frac{1}{2}}|e_q(x)| \leqslant C\Big[\sqrt[4]{\epsilon N^2}\Big(\frac{\ln N}{N}\Big)^{k+1} + N^{-(k+1)}\Big],$$

for any $x \in \mathcal{I}_{\mathrm{R}}$, and

$$|e_q(x)| \leqslant C\Big[\sqrt[4]{\epsilon}\Big(\frac{\ln N}{N}\Big)^{k+1/2} + N^{-(k+1)}\Big],$$

for any $x \in \mathcal{I}_{\mathrm{BL}}$.

4. Pointwise Error Estimates for e_u

The pointwise error estimate outside the boundary layer can be done by a similar technique as in Section 3.

Theorem 4.1. *If the assumption of Lemma 2.4 holds true, then there exists a constant C such that for any $x \in I_j$, $j = N/4+1, \dots, 3N/4$*

$$|e_u(x)| \leqslant C\Big[\sqrt[4]{\epsilon(\ln N)^2}\Big(\frac{\ln N}{N}\Big)^{k+\frac{1}{2}} + N^{-(k+\frac{1}{2})}\Big]. \tag{4.1}$$

Proof. Applying (3.3) and (2.12), we have

$$\begin{aligned}|e_u(x)| &\leqslant |\xi_{\bar u}(x)| + |u_\epsilon(x)| + |\pi^- u_\epsilon(x)| + |\eta_u(x)| \\ &\leqslant |\xi_{\bar u}(x)| + |u_\epsilon(x)| + CH^{-\frac{1}{2}}(\|\pi^- u_\epsilon\|_{I_j} + \|\eta_u\|_{I_j}) \\ &\leqslant |\xi_{\bar u}(x)| + |u_\epsilon(x)| + CH^{-\frac{1}{2}}(\|u_\epsilon\|_{I_j} + \sqrt{H}|u_\epsilon(x_j^-)| + \|\eta_u\|_{I_j}) \\ &\leqslant |\xi_{\bar u}(x)| + |u_\epsilon(x)| + |u_\epsilon(x_j^-)| + CH^{-\frac{1}{2}}(\|u_\epsilon\|_{I_j} + \|\eta_u\|_{I_j}).\end{aligned} \tag{4.2}$$

To estimate $|\xi_{\bar u}(x)|$, we use (3.2) and (2.13b) to get

$$|\xi_{\bar u}(x)| \leqslant C(H^{-\frac{1}{2}}\|\xi_{\bar u}\|_{I_j} + \sqrt{H}\|\xi'_{\bar u}\|_{I_j}) \leqslant CN^{-(k+1/2)}. \tag{4.3}$$

The estimation of $\|u_\epsilon\|_{I_j}$, $|u_\epsilon(x)|$, and $|u_\epsilon(x_j^-)|$ follows from (2.9):

$$\begin{aligned}\|u_\epsilon\|_{I_j} &\leqslant C\Big(\int_{I_j} e^{-2\beta(1-x)/\sqrt{\epsilon}}dx + \int_{I_j} e^{-2\beta x/\sqrt{\epsilon}}dx\Big)^{\frac{1}{2}} \\ &\leqslant Ce^{-\beta\tau/\sqrt{\epsilon}} \leqslant CN^{-(k+1)}, \\ \max\{|u_\epsilon(x)|, |u_\epsilon(x_j^-)|\} &\leqslant Ce^{-\alpha\tau/\epsilon} \leqslant CN^{-(k+1)}.\end{aligned} \tag{4.4}$$

The error bound of $\|\eta_u\|_{I_j}$ has been provided by (2.14), which is

$$H^{-\frac{1}{2}}\|\eta_u\|_{I_j} \leqslant C\Big[\sqrt[4]{\epsilon(\ln N)^2}\Big(\frac{\ln N}{N}\Big)^{k+\frac{1}{2}} + N^{-(k+\frac{1}{2})}\Big]. \tag{4.5}$$

Substituting (4.3), (4.4) and (4.5) into (4.2) yields (4.1). □

Remark 4.2. If we similarly assume $\bar u \in V_N$ as in Remark 3.3, it follows that $\xi_{\bar u} = 0$ and the analysis of $\xi_{\bar u}$ could be removed from the proof of the nodal error estimate of e_u. By (2.15) the above error bound (4.1) becomes

$$|e_u(x)| \leqslant C\Big[\sqrt[4]{\epsilon N^2}\Big(\frac{\ln N}{N}\Big)^{k+1} + N^{-(k+1)}\Big],$$

for any $x \in \mathcal{I}_{\mathrm{R}}$.

Remark 4.3. Applying the same technique used in Theorem 4.1 to e_u on boundary layer region $\mathcal{I}_{\mathrm{BL}}$, we can obtain an error bound

$$|e_u(x)| \leqslant Ch^{-\frac{1}{2}}|(e_q, e_u)|_{\mathcal{A}}, \quad \text{for all } x \in \mathcal{I}_{\mathrm{BL}},$$

in which a negative power of the parameter ϵ appears. Therefore, this error bound of e_u is not uniformly valid with respect to ϵ.

To obtain a uniform pointwise error estimate on $\mathcal{I}_{\rm BL}$, we will use the results of Lemma 2.5 and a stability property of the compact form. To simplify the proof, we only estimate e_u with the choice of the parameter $\lambda_j \sim O(1)$. Convergence rates in L^∞ sense when using other choices of the parameter (e.g., $\lambda_j \sim O(\frac{h}{\epsilon})$ or $\lambda_j \sim O(\frac{\epsilon}{h})$) can also be derived in a similar way.

Define another norm $\|(\cdot,\cdot)\|_{\mathcal{A}}$ as follows:

$$\|(w,v)\|_{\mathcal{A}}^2 = \epsilon^{-1}\|w\|^2 + C_D h\|v'\|^2 + \|\sqrt{b}v\|^2 + \sum_{j=0}^{N} \lambda_j [\![v]\!]_j^2.$$

Apparently, this norm is stronger than (2.8), i.e., $|(w,v)|_{\mathcal{A}} \leqslant \|(w,v)\|_{\mathcal{A}}$ for any $(w,v) \in V_N \times V_N$.

Lemma 4.4. *If $\lambda_j \sim O(1)$ for all $j = 0,1,\ldots,N$, then there exists a positive constant C such that the estimate*

$$\sup_{(w,v)\in V_N\times V_N} \frac{\mathcal{A}(\phi,\psi;w,v)}{\|(w,v)\|_{\mathcal{A}}} \geqslant C_D \|(\phi,\psi)\|_{\mathcal{A}} \tag{4.6}$$

holds for any $(\phi,\psi) \in V_N \times V_N$. The constant C_D is specified at the end of the proof.

Proof. (1) We begin by testing the bilinear form $\mathcal{A}$ defined in (2.6) with $(w,v) = (\phi,\psi)$. Using the symmetry of bilinear $b_i(\cdot,\cdot)$ we obtain

$$\mathcal{A}(\phi,\psi;\phi,\psi) = \frac{1}{\epsilon}\|\phi\|^2 + c(\psi,\psi). \tag{4.7}$$

(2) Next, we choose $(w,v) = (-\gamma h\psi', \gamma\frac{h}{\epsilon}\psi)$ for some positive constant γ. It follows from the definition of the compact form (2.6) that

$$\mathcal{A}(\phi,\psi;w,v) = a(\phi,-\gamma h\psi') - b_2(-\gamma h\psi',\psi) + b_2(\phi,\gamma h\epsilon^{-1}\psi) + c(\psi,\gamma h\epsilon^{-1}\psi), \tag{4.8}$$

where

$$\begin{aligned}
a(\phi,-\gamma h\psi') &= \epsilon^{-1}(\phi,-\gamma h\psi') = -(\phi,\gamma h\epsilon^{-1}\psi'),\\
-b_2(-\gamma h\psi',\psi) &= \gamma h\|\psi'\|^2 - \sum_{j=0}^{N-1}(\gamma h\psi')_j^+[\![\psi]\!]_j - (\gamma h\psi')_N^-\psi_N^-,\\
b_2(\phi,\gamma h\epsilon^{-1}\psi) &= (\phi,\gamma h\epsilon^{-1}\psi') - \sum_{j=0}^{N-1}\gamma h\epsilon^{-1}\phi_j^+[\![\psi]\!]_j - \gamma h\epsilon^{-1}\phi_N^-\psi_N^-,\\
c(\psi,\gamma h\epsilon^{-1}\psi) &= \gamma h\epsilon^{-1}c(\psi,\psi).
\end{aligned}$$

Applying the Schwarz inequality and (3.2) to $b_2(-\gamma h\psi',\psi)$ yields

$$\begin{aligned}
b_2(-\gamma h\psi',\psi) &= \gamma h\|\psi'\|^2 - \sum_{j=0}^{N-1}(\gamma h\psi')_j^+[\![\psi]\!]_j - (\gamma h\psi')_N^-\psi_N^-\\
&\geqslant \gamma h\|\psi'\|^2 - C\gamma^2 h\|\psi'\|^2 - \frac{1}{2}\sum_{j=0}^{N}\lambda_j[\![\psi]\!]_j^2\\
&\geqslant \gamma h(1-C\gamma)\|\psi'\|^2 - \frac{1}{2}\sum_{j=0}^{N}\lambda_j[\![\psi]\!]_j^2.
\end{aligned} \tag{4.9}$$

Consider $b_2(\phi,\gamma h\epsilon^{-1}\psi)$ in a similar way. Since the first term of $b_2(\phi,\gamma h\epsilon^{-1}\psi)$ and $a(\phi,-\gamma h\psi')$ canceled out, we were left with boundary terms, which can be estimated as follows:

$$\begin{aligned} a(\phi,-\gamma h\psi')+b_2(\phi,\gamma h\epsilon^{-1}\psi) &= -\sum_{j=0}^{N-1}\gamma h\epsilon^{-1}\phi_j^+[\![\psi]\!]_j-\gamma h\epsilon^{-1}\phi_N^-\psi_N^- \\ &\geqslant -\frac{1}{2\epsilon}\|\phi\|^2-C\frac{\gamma^2h^2}{\epsilon}\sum_{j=0}^{N}\lambda_j[\![\psi]\!]_j^2. \end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8) yields

$$\begin{aligned} \mathcal{A}(\phi,\psi;-\gamma h\psi',\gamma h\epsilon^{-1}\psi) &\geqslant \gamma h(1-C\gamma)\|\psi'\|^2-\frac{1}{2\epsilon}\|\phi\|^2 \\ &\quad -\Big[\frac{1}{2}-\gamma h\epsilon^{-1}(1-C\gamma h)\Big]\sum_{j=0}^{N}\lambda_j[\![\psi]\!]_j^2. \end{aligned}$$

Here the constant C depends on $\min\lambda_j$. Choosing a sufficiently small constant γ such that $0<C_d\leqslant\gamma(1-C\gamma)\leqslant C$ and $(1-C\gamma h)>0$, we derive

$$\mathcal{A}(\phi,\psi;-\gamma h\psi',\gamma h\epsilon^{-1}\psi)\geqslant C_dh\|\psi'\|^2-\frac{1}{2\epsilon}\|\phi\|^2-\frac{1}{2}c(\psi,\psi). \tag{4.11}$$

Let $(w,v)=(\phi-\gamma h\psi',\psi+\gamma h\epsilon^{-1}\psi)$. Combining (4.7) and (4.11) we obtain

$$\begin{aligned} \mathcal{A}(\phi,\psi;w,v) &\geqslant \frac{1}{2\epsilon}\|\phi\|^2+C_dh\|\psi'\|^2+\frac{1}{2}c(\psi,\psi) \\ &\geqslant \min\Big\{C_d,\frac{1}{2}\Big\}\|(\phi,\psi)\|_{\mathcal{A}}^2 \\ &\geqslant C\min\Big\{C_d,\frac{1}{2}\Big\}\|(\phi,\psi)\|_{\mathcal{A}}\|(w,v)\|_{\mathcal{A}}. \end{aligned} \tag{4.12}$$

When proving (4.12), we used (3.2), (3.1) and the fact that $\|(w,v)\|_{\mathcal{A}}$ is bounded by $\|(\phi,\psi)\|_{\mathcal{A}}$ when the constant γ is sufficiently small. Let $C_D=C\min\{C_d,\frac{1}{2}\}$. The result (4.6) now directly follows from (4.12). □

Theorem 4.5. *Under the assumption of Lemma 4.4, there exists a constant C such that for any $x\in I_j$, $j=1,\dots,N/4,3N/4+1,\dots,N$*

$$|e_u(x)|\leqslant C\Big(\frac{\ln N}{N}\Big)^k. \tag{4.13}$$

Proof. Firstly, we consider $u(x_j^-)$ for $j=1,\dots,N/4$,

$$\begin{aligned} |e_u(x_j^-)| &= |\eta_u(x_j^-)| \\ &= \Big|\sum_{i=1}^{j-1}\int_{I_i}\eta_u'dx+\int_{x_{j-1}}^{x}\eta_u'dx+\sum_{i=0}^{j-1}[\![\eta_u]\!]_j\Big| \\ &\leqslant \sqrt{\frac{\tau}{h}}\Big(\sum_{i=1}^{j}h\|\eta_u'\|_{I_i}^2\Big)^{\frac{1}{2}}+\sqrt{N}\Big(\sum_{i=0}^{j-1}[\![\eta_u]\!]_j^2\Big)^{\frac{1}{2}} \\ &\leqslant C\sqrt{N}\|(\eta_q,\eta_u)\|_{\mathcal{A}}. \end{aligned} \tag{4.14}$$

Secondly, we estimate $u(x_j^-)$ for $j = 3N/4+1,\dots,N$,

$$\begin{aligned}|e_u(x_j^-)| &= |\eta_u(x_j^-)|\\ &= \Big|\sum_{i=3N/4+1}^{j-1}\int_{I_i}\eta_u' dx + \int_{x_{j-1}}^{x}\eta_u' dx + \sum_{i=3N/4}^{j-1}[\![\eta_u]\!]_j - \eta_u(x^-_{\frac{3N}{4}})\Big|\\ &\leqslant C\sqrt{\frac{\tau}{h}}\Big(\sum_{i=3N/4+1}^{j} h\|\eta_u'\|^2_{I_i}\Big)^{\frac12} + C\sqrt{N}\Big(\sum_{i=0}^{j-1}\lambda_j[\![\eta_u]\!]_j^2\Big)^{\frac12} + |\eta_u(x^-_{\frac{3N}{4}})| \\ &\leqslant C\sqrt{N}\|(\eta_q,\eta_u)\|_{\mathcal{A}} + |\eta_u(x^-_{\frac{3N}{4}})|\\ &\leqslant C\sqrt{N}\|(\eta_q,\eta_u)\|_{\mathcal{A}} + C\Big(\frac{\ln N}{N}\Big)^{k+\frac12}.\end{aligned} \tag{4.15}$$

Here, in the last inequality, we used a practical assumption $\epsilon(\ln N)^2 < 1$ and (4.1). Next, we consider $e_u(x)$ for any $x \in I_j$ for $j = 1,\dots,N/4, 3N/4+1,\dots,N$.

$$\begin{aligned}|e_u(x)| &= \Big|-\int_{x_{j-1}}^{x} e_u' dx + e_u(x_j^-)\Big|\\ &\leqslant \sqrt{h}\|e_u'\|_{I_j} + |\eta_u(x_j^-)|\\ &\leqslant C\big(\sqrt{h}\|\xi_u'\|_{I_j} + \sqrt{h}\|\eta_u'\|_{I_j} + |\eta_u(x_j^-)|\big)\\ &\leqslant C\big((\ln N/N)^{k+\frac12} + \|(\eta_q,\eta_u)\|_{\mathcal{A}} + |e_u(x_j^-)|\big),\end{aligned} \tag{4.16}$$

where we used Lemma 2.13 and (2.9) to derive

$$\sqrt{h}\|\xi_u'\|_{I_j} \leqslant Ch^{k+1/2}\|\xi_u^{(k+1)}\|_{I_j} \leqslant C(\ln N/N)^{k+\frac12}.$$

Notice that the estimate of $\|(\eta_q,\eta_u)\|_{\mathcal{A}}$ can be derived from Lemma 4.4 and (2) of Lemma 2.5

$$\begin{aligned}\|(\eta_q,\eta_u)\|_{\mathcal{A}} &\leqslant \sup_{(w,v)\in V_N\times V_N}\frac{|\mathcal{A}(\eta_q,\eta_u;w,v)|}{\|(w,v)\|_{\mathcal{A}}}\\ &\leqslant \sup_{(w,v)\in V_N\times V_N}\frac{|\mathcal{A}(\xi_q,\xi_u;w,v)|}{\|(w,v)\|_{\mathcal{A}}}\\ &\leqslant C\Big(\frac{\ln N}{N}\Big)^{k+1/2}.\end{aligned} \tag{4.17}$$

Combining (4.14)–(4.17) yields the error estimates (4.13). □

Remark 4.6. If other choices of the parameter (e.g., $\lambda_j \sim O(\frac{h}{\epsilon})$ or $\lambda_j \sim O(\frac{\epsilon}{h})$) are used, the same convergence rates of L^∞ error as in (4.13) can be established in a similar way.

The choices of the parameter $\lambda_N \geqslant O(1)$ and $\lambda_j = 0$ for all $j = 0,1,\dots,N-1$ (i.e., the assumption of Lemma 2.4) fail to guarantee the stability (4.6). Therefore, ϵ-uniform convergence cannot be theoretically proved when $\lambda_j = 0$ for all $j = 0,1,\dots,N-1$ by the technique used in this article.

5. Numerical Experiments

We choose $\epsilon = 10^{-4}, 10^{-8}$ and $N = 2^m$ for some positive integer $m \geqslant 2$. The L^∞ error and its convergence rate are denoted by

$$E_k(N) := \frac{1}{\sqrt{\epsilon}}\|e_q\|_{\infty,\mathcal{I}_N} + \|e_u\|_{\infty,\mathcal{I}_N},$$

with $k = 1, 2, 3$. To calculate the convergence rate, we define

$$r_k := \log_p\Big(\frac{E_k(N)}{E_k(2N)}\Big), \quad k = 1, 2, 3,$$

with base $p = 2(\ln N/\ln 2N)$. The notation r_k indicates the convergence rate with respect to $\ln N/N$. Similarly, we define the maximum-error for numerical traces as

$$E'_k(N) := \max_{j=0,\dots,N}\Big(\frac{1}{\sqrt{\epsilon}}|e_q(x_j)| + |e_u(x_j)|\Big), \quad r'_k := \log_p\Big(\frac{E'_k(N)}{E'_k(2N)}\Big)$$

for $k = 1, 2, 3$.

Example 1. We consider test problem (1.1) with $b(x) = 1$ and the other data such that the exact solution is

$$u(x) = \cos(2\pi x) - \frac{e^{-x/\sqrt{\epsilon}} + e^{(x-1)/\sqrt{\epsilon}}}{1 + e^{-1/\sqrt{\epsilon}}}.$$

Set the value of the stabilization parameter $\lambda_j = 0$ for all $j = 0, 1, \dots, N-1$, and $\lambda_N = k/h$. The transition number we used is given in (2.11) with $K_1 = k+1$. Table 1 displays the error E_k and the convergence rates r_k ($k = 1, 2, 3$), which are of order $(\ln N/N)^{k+1}$ and imply that our theoretical results are not sharp.

We also list maximum-norm errors of the numerical traces E'_k for the LDG approximation in Table 2 to show that there exists a similar superconvergence phenomenon as in solving convection-diffusion problems [12]. Table 2 clearly indicates that the errors of numerical traces converge at a superconvergence rate $(\ln N/N)^{2k+1}$. The transition number we used in the case of $\epsilon = 10^{-4}$ is the same as (2.11) with $K_1 = k + 1$. The same superconvergence behavior can be observed when $K_1 \geqslant k + 1$.

However, when $\epsilon = 10^{-8}$, the superconvergence rate degrades considerably if $K_1 = k+1$ is used. So, we set $K_1 = 3k+1$ when $k = 1$ and $K_1 = 2k+1$ when $k = 2, 3$. The rates shown in Table 2 are of $O((\ln N/N)^{2k+1})$. It seems that K_1 must be sufficiently large to guarantee the rate $(\ln N/N)^{2k+1}$.

If we take $\lambda_j = 0$ for all $j = 0, 1, \dots, N-1$, and $\lambda_N \sim O(1)$, optimal convergence of the L^∞ error E_k is observed on $\mathcal{I}_N$ except on the last element I_N. Actually, on the last element I_N, $\|e_u\|_{\infty,\mathcal{I}_N}$ converges at optimal rate, while the rate of $\|e_q\|_{\infty,\mathcal{I}_N}$ has a slight degradation. Moreover, the superconvergence rate of numerical traces also has a slight degradation when $\lambda_N \sim O(1)$. Although it is not indicated by the theoretical proof, it seems that the parameter I_N should be sufficiently large when $\lambda_j = 0$ ($j = 0, 1, \dots, N-1$) are set to be zero.

Example 2. Consider problem (1.1) with $b(x) = 2 - x$, and a properly chosen f such that the exact solution is

$$u(x) = (1 - e^{-x/\sqrt{\epsilon}})(1 - e^{(x-1)/\sqrt{\epsilon}}).$$

Table 3 displays the error E_k and the convergence rates r_k ($k = 1, 2, 3$). The maximum-norm error of the numerical traces E'_k in Table 4 shows the same superconvergence rates of order $(\ln N/N)^{2k+1}$ as in Example 1. The values of λ_j are chosen the same as in Example 1.

			$\epsilon = 10^{-4}$			
N	E_1	r_1	E_2	r_2	E_3	r_3
32	1.05e-1	—	2.21e-2	—	4.50e-3	—
64	4.03e-2	1.88	5.53e-3	2.71	7.41e-4	3.53
128	1.40e-2	1.96	1.19e-3	2.85	1.02e-4	3.67
256	4.58e-3	1.99	2.39e-4	2.86	1.21e-5	3.81
512	1.48e-3	1.96	4.46e-5	2.92	1.29e-6	3.89
			$\epsilon = 10^{-8}$			
N	E_1	r_1	E_2	r_2	E_3	r_3
32	8.72e-2	—	1.90e-2	—	4.00e-3	—
64	3.54e-2	1.76	5.01e-3	2.61	6.85e-4	3.46
128	1.30e-2	1.86	1.13e-3	2.77	9.48e-5	3.67
256	4.43e-3	1.92	2.27e-4	2.86	1.13e-5	3.81
512	1.44e-3	1.96	4.23e-5	2.92	1.20e-6	3.89

Table 1. L^∞ errors and convergence rates of Example 1.

			$\epsilon = 10^{-4}$			
N	E'_1	r'_1	E'_2	r'_2	E'_3	r'_3
32	8.94e-3	—	4.27e-4	—	2.30e-5	—
64	1.90e-3	3.03	3.71e-5	4.76	8.08e-7	6.56
128	3.85e-4	2.96	2.64e-6	4.91	2.04e-8	6.83
256	7.23e-5	2.99	1.63e-7	4.97	4.19e-10	6.94
512	1.29e-5	3.00	9.26e-9	4.99	7.53e-12	6.98
			$\epsilon = 10^{-8}$			
N	E'_1	r'_1	E'_2	r'_2	E'_3	r'_3
32	8.85e-2	—	4.01e-3	—	6.46e-4	—
64	2.53e-2	2.45	4.22e-4	4.41	3.14e-5	5.92
128	5.67e-3	2.78	3.24e-5	4.76	9.29e-7	6.53
256	1.11e-3	2.92	2.07e-6	4.92	2.04e-8	6.83
512	2.00e-4	2.97	1.31e-7	4.80	3.75e-10	6.94

Table 2. The error of numerical traces (Example 1).

6. Summary and Conclusion Remarks

In this paper, the LDG method was analyzed for the one-dimensional singularly perturbed two point boundary value problems of reaction-diffusion type. Uniform pointwise convergence rates for the LDG approximation to the solution and its derivative were established. We saw that our theoretical results are inferior to the rates of convergence observed in the numerical experiments, which are of order $(\ln N/N)^{k+1}$. Other techniques such as Green's function or discrete Green's function may be useful to prove optimal pointwise convergence, which is an on-going work.

The techniques used in this article can be extended to two-dimensional singularly perturbed reaction-diffusion problems if corresponding L^2 error estimates are available. On the other hand, it has been observed that numerical traces converge at a superconvergence rate

			$\epsilon = 10^{-4}$			
N	E_1	r_1	E_2	r_2	E_3	r_3
32	1.05e-01	—	2.21e-02	—	4.50e-03	—
64	4.03e-02	1.88	5.53e-03	2.71	7.38e-04	3.54
128	1.40e-02	1.96	1.19e-03	2.86	1.02e-04	3.67
256	4.58e-03	1.99	2.39e-04	2.86	1.21e-05	3.81
512	1.46e-03	1.99	4.46e-05	2.92	1.29e-06	3.89
			$\epsilon = 10^{-8}$			
N	E_1	r_1	E_2	r_2	E_3	r_3
32	1.05e-01	—	2.21e-02	—	4.50e-03	—
64	4.03e-02	1.88	5.53e-03	2.71	7.38e-04	3.54
128	1.40e-02	1.96	1.19e-03	2.86	1.02e-04	3.67
256	4.58e-03	1.99	2.39e-04	2.86	1.21e-05	3.81
512	1.46e-03	1.99	4.46e-05	2.92	1.29e-06	3.89

Table 3. L^∞ errors and convergence rates of Example 2.

			$\epsilon = 10^{-4}$			
N	E_1'	r_1'	E_2'	r_2'	E_3'	r_3'
32	1.52e-02	—	1.05e-03	—	7.67e-05	—
64	3.62e-03	2.80	9.91e-05	4.62	2.98e-06	6.36
128	7.50e-04	2.92	7.27e-06	4.85	7.90e-08	6.74
256	1.42e-04	2.97	4.57e-07	4.94	1.66e-09	6.90
512	2.55e-05	2.99	2.61e-08	4.98	3.02e-11	6.96
			$\epsilon = 10^{-8}$			
N	E_1'	r_1'	E_2'	r_2'	E_3'	r_3'
32	1.51e-02	—	1.05e-03	—	7.63e-05	—
64	3.62e-03	2.80	9.88e-05	4.62	2.96e-06	6.36
128	7.49e-04	2.92	7.25e-06	4.85	7.84e-08	6.74
256	1.42e-04	2.97	4.56e-07	4.94	1.65e-09	6.90
512	2.55e-05	2.99	2.60e-08	4.98	3.10e-11	6.90

Table 4. The error of numerical traces (Example 2).

$O((\ln N/N)^{2k+1})$. The theoretical proof of this uniform superconvergence at nodal points is a future work.

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