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Some recent advances on vertex centered finite volume element methods for elliptic equations

Dedicated to Professor Shi Zhong-Ci on the Occasion of his 80th Birthday

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Abstract In this paper, we report our recent advances on vertex centered finite volume element methods (FVEMs) for second order partial differential equations (PDEs). We begin with a brief review on linear and quadratic finite volume schemes. Then we present our recent advances on finite volume schemes of arbitrary order. For each scheme, we first explain its construction and then perform its error analysis under both H^1 and L^2 norms along with study of superconvergence properties.

Keywords high order, finite volume method, inf-sup condition, superconvergence

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1 Introduction

The finite volume method (FVM) is one of the most commonly used numerical methods for solving PDEs in practice. Due to its local conservation of numerical fluxes (a property not shared by the finite element method (FEM)), the capability of handling domains with complex geometries (a property shared by FEM but not by the finite difference method (FDM)), and other advantages, the FVM enjoyed a great population in computational fluid mechanics, heat transfer and hyperbolic equations (cf. [9, 31, 36, 44, 48, 53, 69–71, 73, 83, 87] and the references cited therein).

Earlier work on FVMs may be traced back to MacNeal's difference schemes on irregular networks [59], Samarskii and Tichonov's discretization schemes for convection-diffusion equations [76, 77, 89], and the work of McDonald [62] and MacCormack and Paullay [58]. Since then, the FVM has attracted much attention. Roughly speaking, there are two categories of FVM schemes: *cell-centered FVM* with unknown variables associated with centers of computational cells [1–3, 8, 29, 37, 42, 46, 47, 60, 66, 91, 97], and *vertex-centered FVM* with unknown variables associated with vertices of computational cells. There is also hybrid finite volume method, which uses a simple control volume partitions for higher-order schemes and more flexible algorithm construction [14]. In this article we concentrate only on vertex centered finite

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volume methods, and do not touch hybrid finite volume methods and cell centered finite volume methods, which are also very important variations.

The vertex-centered FVM is a special Petrov-Galerkin method and was once called *finite volume element method* (FVE) or *box method* or *generalized difference method* (GDM) in the literature. In 1982, the GDM was introduced by Li and his colleagues for solving two-point boundary value problems [51] and two-dimensional elliptic equations [54]. Studies of the GDM were carried out further by Li, Chen and other Chinese researchers later (cf. [23, 25, 49, 50, 52, 88, 98]). The *box method* was first analyzed by Bank and Rose [7] in 1987 and further studied in [40, 79]. The name of FVE was first introduced in [57] and many associated studies have been done by Cai and other researchers (cf. [10, 12, 13, 90]). Other studies on the vertex-centered FVM have been carried out by Samarski [78] for Cartesian meshes, and by Henrich [41] for unstructured meshes; see also [68, 81, 85, 86] for quadrilateral meshes.

This survey contains two parts. The first part is a brief review of lower-order (mainly linear and quadratic) FVMs and their recent development. The second part is the main theme of this survey, where we report our recent advances (cf. [15–18, 55, 96, 106, 107, 110]) on a class of vertex-centered FVMs of arbitrary order.

2 Linear and quadrature schemes

This section is dedicated to a brief review on construction and analysis of linear and quadratic FVMs for the following elliptic boundary value problem,

$$-\bigtriangledown\cdot(\alpha\bigtriangledown u)=f\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega,\tag{2.1}$$

where $\Omega\subset\mathbb{R}^d$ is a simply connected polygon, $\alpha\in L^\infty(\Omega)$ and is bounded from below: there exists a constant $\alpha_0>0$ such that $\alpha(x)\geqslant\alpha_0$ for almost all $x\in\Omega$, and $f\in L^2(\Omega)$ is a real-valued function defined on Ω.

2.1 Linear schemes

In this subsection, we present first the construction and then the analysis of linear FV schemes.

2.1.1 *Construction*

Linear FV schemes are often constructed under the framework of Petrov-Galerkin method. The primal partition $\mathcal{T}$ is often chosen to be a conforming, shape regular, but not necessarily quasi-uniform partition. There are many choices in the construction of the dual mesh. We first consider the 2-dimensional case. When $\mathcal{T}$ is a triangular mesh, there are two popular methods for constructing the dual mesh: one is the so-called *barycenter dual mesh* (cf. [41], see Figure 1(a) for a control volume from barycenter dual mesh), the other is the so-called *circum-center dual mesh* (cf. [59], see Figure 1(b) for a control volume from circum-center dual mesh). For quadrilateral mesh, there are also two popular control volumes (cf. [79,86]), namely the so-called diagonal control volume (see Figure 2) and the center control volume (see Figure 3).

The construction of high spatial control volumes is somehow complicated. In the following, we present a method introduced in [19, 96] for constructing high space dimensional control volumes. Let $\tau\in\mathcal{T}$ be a simplex of d-dimension. For each n-dimensional ($n<d$) subsimplex K_n of τ, we choose Q_{K_n} as the barycenter of K_n. For example, if $n=1$, Q_{K_1} is selected to be the middle point of the segment K_1; if $n=2$, Q_{K_2} is selected to be the barycenter of the triangle K_2; if $n=3$, Q_{K_3} is selected to be the barycenter of the tetrahedron $K_3,\ldots$ We first choose an arbitrary point Q in the interior of the simplex τ, then we connect Q_τ with the barycenters $Q_{K_{d-1}}$ of those $(d-1)$-dimensional faces K_{d-1} by straight lines. In each K_{d-1}, we connect by straight lines $Q_{K_{d-1}}$ with the barycenters $Q_{K_{d-2}}$ of those $(d-2)$-dimensional faces of K_{d-1}. Continue this process, till all the barycenters of the triangles connect with the middle points of their edges. Then the contribution of τ to the control volume D_P of the vertex P of τ is the volume surrounding P by the above straight lines. And D_P is the union of all the contributions from those $\tau'\in\mathcal{T}$ which have P as a vertex. See Figure 4 for the construction of the control volume in 3D.

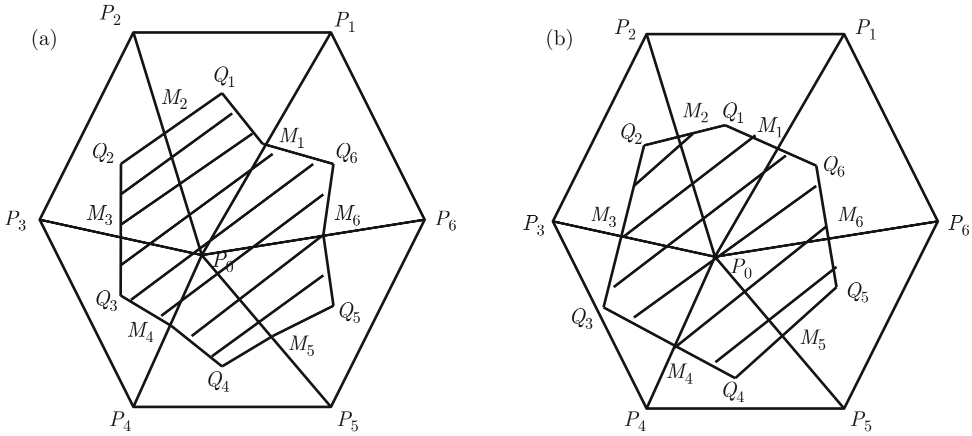

Figure 1 (a) Barycenter type control volume; (b) Circum-center type control volume. The control volume D_{P_0} is the polygon surrounded by the fold line $M_1Q_1M_2Q_2\cdots M_6Q_6M_1$, where M_i is the midpoint of the segments $P_0P_i, i=1,\dots,6$. For the barycenter type, Q_i is the barycenter, while for the circum-center type, Q_i is the circum-center of the triangle $P_0P_iP_{i+1}, i=1,\dots,6$

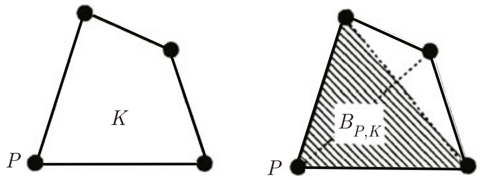

Figure 2 Contribution from one quadrilateral K to the diagonal control volume B_P

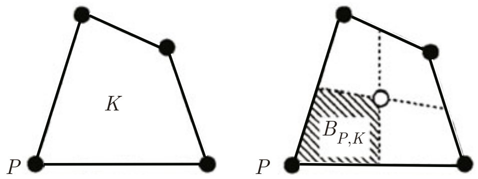

Figure 3 Contribution from one quadrilateral K to the center control volume B_P

Once $\mathcal{T}$ and $\mathcal{T}'$ are constructed, the trial and test spaces can be set up. Let the trial space be the standard conforming finite element space $U_{\mathcal{T}}$ defined as:

$$U_{\mathcal{T}} := \{v \in C(\overline{\Omega}) : v|_{\tau} \in P_1, \text{ for all } \tau \in \mathcal{T}, \ v|_{\partial\Omega} = 0\}. \tag{2.2}$$

If we denote by $\mathcal{N}_{\mathcal{T}}$ the set of all the interior vertices, then $U_{\mathcal{T}} = \text{span}\{\phi_P : P \in \mathcal{N}_{\mathcal{T}}\}$, where for all $P \in \mathcal{N}_{\mathcal{T}}$, ϕ_P is the standard first order nodal basis satisfying $\phi_P \in U_{\mathcal{T}}$, and $\phi_P(x_{P'}) = \delta_{P,P'}$, $\forall\, P' \in \mathcal{N}_{\mathcal{T}}$. Moreover, if we denote by ψ_P the characteristic function on the control volume D_P, then the test space $V_{\mathcal{T}'} = \text{span}\{\psi_P : P \in \mathcal{N}_{\mathcal{T}}\}$. A finite volume solution of (2.1) is a function $u_{\mathcal{T}} \in U_{\mathcal{T}}$ which satisfies the following local conservation law,

$$-\int_{\partial D_P} \alpha \frac{\partial U_{\mathcal{T}}}{\partial \mathbf{n}} ds = \int_{D_P} f dx \tag{2.3}$$

on each control volume $D_P, P \in \mathcal{N}_{\mathcal{T}}$.

Let the bilinear form

$$a_{\mathcal{T}}(u,v) = -\sum_{P \in \mathcal{N}_{\mathcal{T}}} \int_{\partial D_P} (a \triangledown u) \cdot \mathbf{n}\, v ds \tag{2.4}$$

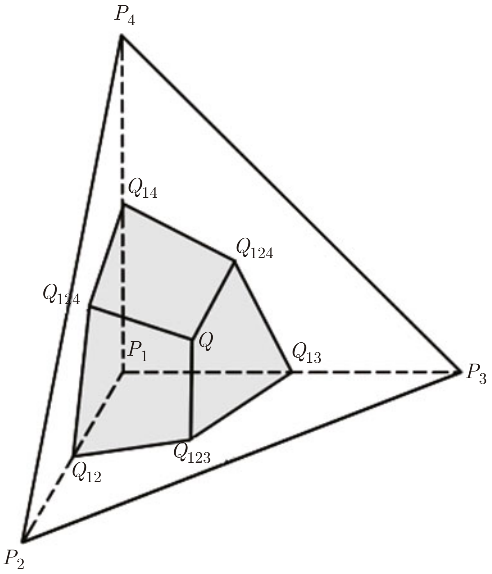

Figure 4 Contribution from one simplex to the control volume D_{P_1}. Q is an arbitrary point in τ, $Q_{123}, Q_{124}, Q_{134}$ are the barycenters of $\Delta P_1P_2P_3, \Delta P_1P_2P_4, \Delta P_1P_3P_4$, respectively, and Q_{12}, Q_{13}, Q_{14} are the middle points of the segments P_1P_2, P_1P_3 and P_1P_4, respectively.

for all $u \in H_0^1(\Omega)$ and $v \in V_{\mathcal{T}}$. The linear finite volume approximation $u_{\mathcal{T}} \in U_{\mathcal{T}}$ satisfies

$$a_{\mathcal{T}}(u_{\mathcal{T}}, v_{\mathcal{T}}) = \int_{\Omega} f v_{\mathcal{T}} dx, \quad \forall\, v_{\mathcal{T}} \in V_{\mathcal{T}'}. \tag{2.5}$$

2.1.2 *Analysis*

Linear FV schemes have been much analyzed in the literature. In 1987, Bank and Rose [7] discovered that for the Poisson equation on a polygonal domain in two dimensions, the stiffness matrices of the finite volume method are identical to that of the finite element method for very general grids and based on this, they proved the following optimal order convergence property: if the solution $u \in H^2(\Omega)$, then

$$\|u - u_{\mathcal{T}}\|_1 \lesssim h_{\mathcal{T}} \|u\|_2, \tag{2.6}$$

where $h_{\mathcal{T}}$ is the largest mesh size of the elements in $\mathcal{T}$. In 1989, Hackbusch [40] studied the linear FVM for general elliptic equations on a polygonal domain in two dimensions and proved that the error between the FEM solution and the FVM solution is of first order in general case and is of second order for barycenter type FVM schemes. In 2002, Ewing et al. [35] showed that when mesh size is sufficiently small, the stiffness matrix of the linear FVM is a small perturbation of that of the linear FEM for general elliptic equations. In 2009, Xu and Zou [96] extended (2.6) to general elliptic equations on polygonal domains in arbitrary spatial dimensions by establishing the inf-sup condition of the bilinear form (2.4) with a simple and transparent proof.

Optimal L_2 error estimate for linear FVM cannot be obtained directly from Aubin-Nitsche's duality argument since the FV bilinear form $a_{\mathcal{T}}$ is defined on $H_0^1(\Omega) \times V_{\mathcal{T}'}$ instead of $H_0^1(\Omega) \times H_0^1(\Omega)$. In 1989, optimal L^2 second-order error estimate for the Poisson equation is obtained by Hackbusch in [40] with the assumption $u \in H^2$ and $f \in H^1$. In 1994, Chen [24] proved the following estimate,

$$\|u - u_{\mathcal{T}}\|_0 \lesssim h_{\mathcal{T}}^2 \|u\|_{3,p} \tag{2.7}$$

by assuming $u \in W^{3,p}(\Omega)$ for arbitrary $p > 1$. Later, Huang and Xi [43] and Ewing et al. [35] constructed counter-examples of 1D and 2D to demonstrate that optimal second order L_2 error estimate cannot be

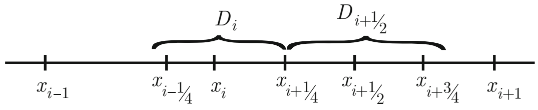

Figure 5 Control volumes D_i and $D_{i+\frac{1}{2}}$

obtained by only assuming $u \in H^2$ and $f \in L_2$. In 2002, Chen et al. [26] improved (2.7) to the following result: $\|u - u_{\mathcal{T}}\|_0 \lesssim h_{\mathcal{T}}^2 |\ln h_{\mathcal{T}}|^{1/2} \|f\|_{1,1}$ and $\|u - u_{\mathcal{T}}\|_0 \lesssim h_{\mathcal{T}}^2 \|f\|_{1,p}$, $p > 1$. Optimal order L_2 error estimate for general self-adjoint elliptic problems has been obtained also by Chatzipantelidis in [20] for a nonconforming finite volume method by assuming $u \in H^2$ and $f \in H^1$. In [35], Ewing et al. showed that if $f \in H^\alpha$ and $u \in H^{1+\alpha}$ for $\alpha \in (0, 1]$, then the order of the L_2 convergence is 2α.

The super-convergence of the linear FVM on a domain of two dimensions has been studied by Cai [10] directly from the FVM scheme and by Wu and Li [94] through studying the relationship of the FVM scheme and the FEM scheme. In [96], for general elliptic equations on arbitrary space dimensional domain, Xu and Zou proved the following supercloseness result,

$$|u_{e,\mathcal{T}} - u_{\mathcal{T}}|_{1,\Omega} \lesssim h^2 |f|_{1,\Omega}, \tag{2.8}$$

where $u_{e,\mathcal{T}}$ is the corresponding linear finite element solution. We recall that Hackbusch proved (2.8) for 2D barycenter type FVM scheme in [40].

Then based on the supercloseness property of FEM $|u_I - u_{e,\mathcal{T}}|_{1,\Omega} \lesssim h^2 |f|_{1,\Omega}$, where u_I is the linear interpolation of u in the space $U_{\mathcal{T}}$, a supercloseness result $|u_I - u_{\mathcal{T}}|_{1,\Omega} \lesssim h^2 |f|_{1,\Omega}$ can be established for FVM.

2.2 Quadratic schemes

In this subsection, we will follow [53, 96] to present the construction and analysis of quadratic FV schemes. Note that other quadratic FV schemes have also been constructed and analyzed by other researchers such as Emonot [34], Liebau [56], Plexousakis and Zouraris [72], and Cai et al. [11].

2.2.1 *Construction*

To design quadratic finite volume schemes, one should construct a control volume not only for nodes of the partition, but also for mid-points/barycenters of the edges/sides. In Figure 5, we show a simple method to construct quadratic control volumes in 1D: for a node x_i, its corresponding control volume is $D_i := [x_{i-1/4}, x_{i+1/4}]$, for a middle point $x_{i+1/2}$, its control volume $D_{i+1/2} := [x_{i+1/4}, x_{i+3/4}]$.

In Figures 6(a) and 6(b), we explain how to construct a class of control volumes for quadratic FV schemes on triangular meshes introduced in [96]. The quadratic simplicial FVM schemes proposed in [34, 56] and [53] belong to this class. Let $\mathcal{N}_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}}$ be the set of interior vertices and the set of interior middle points, respectively. The dual partition consists of polygons K_{P_0} surrounding $P_0 \in \mathcal{N}_{\mathcal{T}}$ and K_{M_0} surrounding $M_0 \in \mathcal{M}_{\mathcal{T}}$. In the control volume D_{P_0}, $|P_0P_{0i}| = \alpha |P_0P_i|$, $|P_0M_{0i}| = \frac{3}{2}\beta |P_0Q_i|$, $1 \leqslant i \leqslant 7$, where $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{2}{3}$ are two given parameters. In the control volume D_{M_0}, $|P_0M_{0i}| = \frac{3}{2}\beta |P_0Q_i|$, $i = 1, 2$ and $|P_2M_{1i}| = \frac{3}{2}\beta |P_2Q_i|$, $i = 1, 2$. Note that different choices of α and β lead to different FVM schemes. The case $\alpha = \beta$ produces a very simple partition: $D_P \cap \tau$ is a triangle homothetic to τ, and $D_M \cap \tau$ is a pentagon. Li et al. considered in [53] an FVM scheme corresponding to $\alpha = \beta = 1/3$. For the case $\alpha \neq \beta$, we refer to [56] in which $\alpha = 1/4, \beta = 1/3$ and [34] in which $\alpha = 1/6, \beta = 1/4$.

2.2.2 *Analysis*

A crucial step in FVM error analysis is to establish the inf-sup condition. A commonly used technique is the so-called *element stiffness matrix analysis*, in which all eigenvalues of an element stiffness matrix are calculated and validity of the inf-sup condition is determined according to positivity of those eigenvalues.

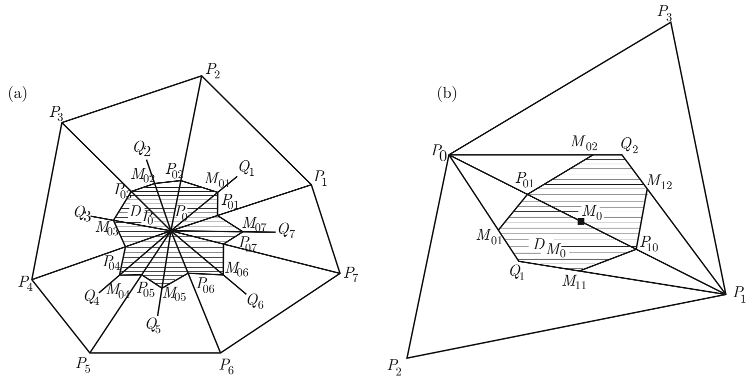

Figure 6 (a) Control volume D_{P_0}; (b) Control volume D_{M_0}

This approach depends heavily on shape regularity of the partition. In the monograph [53], the inf-sup condition for quadratic FV schemes in 2D is proved under some strong shape conditions of the triangulation. The authors of [96] established the inf-sup condition of some quadratic FVM schemes for elliptic equations in 2D under much weaker conditions. They showed that the inf-sup condition holds if the minimal angle $\theta_0 \geqslant 7.11^\circ$ for an FVM scheme proposed in [34], if $\theta_0 \geqslant 9.98^\circ$ for an FVM scheme proposed in [56], and if $\theta_0 \geqslant 20.95^\circ$ for an FVM scheme proposed in [53]. Moreover, it is observed that the inf-sup condition may hold with very small θ_0 for some specific quadratic FVM schemes, for example, θ_0 can be less than 3° for a *near optimal* FVM scheme. Once the inf-sup condition is established, the optimal convergence rate of the H^1 error can be obtained by a standard argument for Petrov-Galerkin method.

The element stiffness matrix analysis has been applied to discussion of other FV schemes, see, e.g., [27, 53].

3 A class of FV schemes of arbitrary order

3.1 Schemes for one-dimensional elliptic equations

In this subsection, we report our recent work [16] on arbitrary order FV schemes for the following two-point boundary value problem on (a, b):

$$-(\alpha u')' + \beta u' + \gamma u = f, \quad u(a) = u(b) = 0, \tag{3.1}$$

where coefficient functions $\alpha, \beta, \gamma \in L^\infty([a, b])$ satisfy $\alpha \geqslant \alpha_0 > 0$, $\gamma - \frac{1}{2}\beta' \geqslant \kappa > 0$, and $f \in L^2([a, b])$.

3.1.1 *Construction*

Let $a = x_0 < x_1 < \cdots < x_N = b$ be $N + 1$ distinct points in $[a, b]$ and let $\mathcal{T}_h = \{\tau_i = [x_{i-1}, x_i] : i = 1, \ldots, N\}$ be the corresponding partition of $[a, b]$ with mesh size $h = \max_{1 \leqslant i \leqslant N}(x_i - x_{i-1})$. We choose the trial space as the Lagrange finite element of r-th order, $r \geqslant 1$, defined by

$$\mathcal{U}_h^r = \{v \in C([a, b]) : v|_\tau \in \mathbb{P}_r, \tau \in \mathcal{T}_h, v(a) = v(b) = 0\},$$

where $\mathbb{P}_r$ is the set of all polynomials of degree no more than r. For all $\tau \in \mathcal{T}_h$, let $g_{\tau,1}, \ldots, g_{\tau,r}$ be r Gauss points in the interval τ. A dual partition $\mathcal{T}_h'$ is obtained by dividing $[a, b]$ with Gauss points $g_{\tau,j}, \tau \in \mathcal{T}_h, j = 1, \ldots, r$ and each element of $\mathcal{T}_h'$ is a control volume. The test space $\mathcal{V}_h$ consists of piecewise constant functions with respect to the dual partition $\mathcal{T}_h'$, which vanish on the two extremal points a and b. In other words, $\mathcal{V}_h = \mathrm{Span}\,\{\psi_{i,j} : i = 1, \ldots, N, j = 1, \ldots, r_i\}$, where $r_i = r$ if $i \leqslant N$ and $r_N = r - 1$, and $\psi_{i,j} = \chi_{\tau'_{i,j}}$ is the characteristic function on the control volume $\tau'_{i,j} = [g_{\tau_i,j}, g_{\tau_i,j+1}]$.

The FV solution is a function $u_h \in \mathcal{U}_h^r$ satisfying the following conservation law,

$$\int_{\tau'_{i,j}} (-(\alpha u_h')'(x) + \beta u_h'(x) + \gamma u_h(x))dx = \int_{\tau'_{i,j}} f(x)dx \tag{3.2}$$

on each control volume $\tau'_{i,j}, i = 1, \dots, N, j = 1, \dots, r_i$.

Our FVM bilinear form for all $v \in H_0^1(\Omega), w_h = \sum_{i=1}^N \sum_{j=1}^{r_i} w_{i,j}\psi_{i,j} \in \mathcal{V}_h$ is defined by

$$a_h(v, w_h) = \sum_{i=1}^N \sum_{j=1}^r [w_{i,j}]\alpha(g_{i,j})v'(g_{i,j}) + \sum_{i=1}^N \sum_{j=1}^{r_i} w_{i,j} \int_{\tau'_{i,j}} (\beta v'(x) + \gamma v(x))dx \tag{3.3}$$

with $[w_{i,j}] = w_{i,j} - w_{i,j-1}$. Under this definition, the local conservation law (3.2) is equivalent to the following variational equation,

$$a_h(u_h, w_h) = (f, w_h), \quad \forall\, w_h \in \mathcal{V}_h. \tag{3.4}$$

3.1.2 *Analysis*

By (3.4), the analysis can be done under the framework of Petrov-Galerkin method. So we start with a discussion of the inf-sup condition.

Theorem 3.1. *For h sufficiently small,*

$$\inf_{v_h \in \mathcal{U}_h^r} \sup_{w_h \in \mathcal{V}_h} \frac{a_h(v_h, w_h)}{\|v_h\|_1 |w_h|'} \gtrsim 1, \tag{3.5}$$

where the hidden constant is independent of h and the semi-norm in the test space is defined by

$$|w_h|' = \left(\sum_{i=1}^N \sum_{j=1}^r h_i^{-1}[w_{i,j}]^2 \right)^{\frac{1}{2}}, \quad \forall\, w_h \in \mathcal{V}_h,$$

with $h_i = x_i - x_{i-1}$.

Proof. We prove (3.5) only for $\beta = \gamma = 0$. A proof for general situation can be found in [16]. Given $v_h \in \mathcal{U}_h^r$, we choose $w_h \in \mathcal{V}_h$ such that $[w_{i,j}] = A_{i,j}v_h'(g_{\tau_i,j})$, where $A_{i,j}, i = 1, \dots, N, j = 1, \dots, r$ are weights of r-point Gauss quadrature on τ_i. Therefore,

$$a_h(v_h, w_h) = \sum_{i=1}^N \sum_{j=1}^r A_{i,j}\alpha(g_{\tau_i,j})[v_h'(g_{\tau_i,j})]^2 \geqslant \alpha_0 \sum_{i=1}^N \sum_{j=1}^r A_{i,j}[v_h'(g_{\tau_i,j})]^2.$$

Since $v_h' \in \mathbb{P}_{r-1}$ in each τ_i, we have $\sum_{j=1}^r A_{i,j}[v_h'(g_{\tau_i,j})]^2 = \int_{\tau_i}[v_h'(x)]^2dx$. Then $a_h(v_h, w_h) \geqslant \alpha_0|v_h|_1^2$. The desired result (3.5) follows directly from the fact $|w_h|' \sim |v_h|_1$. □

With this inf-sup property, the following optimal H^1 and L^2 error estimates are proved in [16].

Note that for this special case, there is no need for the assumption "sufficiently small h".

Theorem 3.2. *Assume that $u \in H_0^1([a, b])$ is the solution to* (3.1) *and u_h the solution to FV scheme* (3.4). *Then we have the H^1-error estimate*

$$\|u - u_h\|_1 \lesssim h^r|u|_{r+1}, \tag{3.6}$$

and the L^2 error estimate

$$\|u - u_h\|_0 \lesssim h^{r+1}\|u\|_{r+2}. \tag{3.7}$$

We observe that for the L^2 error estimate, we require a slightly stronger regularity assumption $u \in H^{r+2}$. Moreover, we have the following superconvergence property.

Theorem 3.3. *Let u be the solution to* (3.1) *and u_h, the solution ot FV scheme* (3.4). *Then we have*

$$\|u_I - u_h\|_1 \lesssim h^{r+1}|u|_{r+2}, \tag{3.8}$$

where $u_I \in \mathcal{U}_h^r$ interpolates u at all Lobatto points of $\mathcal{T}_h$. Furthermore, 1) *if P is a node of $\mathcal{T}_h$, then*

$$|(u - u_h)(P)| \lesssim h^{2r}\|u\|_{2r+1,\infty}; \tag{3.9}$$

2) *if P is an interior Lobatto point, then*

$$|(u - u_h)(P)| \lesssim h^{r+2}|u|_{r+2,\infty}; \tag{3.10}$$

3) *if P is a Gauss point, then*

$$|(u - u_h)'(P)| \lesssim h^{r+1}|u|_{r+2,\infty}, \tag{3.11}$$

$$|(u - u_h)'(P)| \lesssim h^{\min\{r+2,2r\}}\|u\|_{2r+1,\infty}, \quad \textit{when } \beta = 0, \tag{3.12}$$

$$|(u - u_h)'(P)| \lesssim h^{2r}\|u\|_{2r+1,\infty}, \quad \textit{when } \beta = \gamma = 0. \tag{3.13}$$

We see that at Gauss points, when $\beta = \gamma = 0$, the derivative convergence rate h^{2r} doubles the global optimal rate h^r, which is much better than the counterpart finite element method's h^{r+1} rate, when $\beta = 0$, the derivative convergence rate h^{r+2} is one order higher than the counterpart finite element method's h^{r+1}; at the nodal points, the convergence rate h^{2r} almost doubles the global optimal rate h^{r+1} and equals to the counterpart finite element method's h^{2r} rate; and at the Lobatto points, the convergence rate h^{r+2} is one order higher than the optimal global rate h^{r+1}, which is the same as the counterpart finite element method. We refer to [4, 33, 45, 75, 92, 108, 109] for superconvergence results of the finite element method.

3.2 *p*-Version FV schemes for one-dimensional elliptic equations

Theoretical studies of FVM focus often on the *h-version* methods, i.e., convergence properties are discussed when the mesh size approaches zero while the polynomial degree of the trial space is fixed. Contrary to the fact that many papers on p-version FEMs have been published (cf. [5, 39, 63, 64, 80, 103, 105]), no *p-version* FVM has been published in the literature yet.

In [17], the convergence of FV schemes (3.4) is discussed when the polynomial order r tends to infinity while the partition $\mathcal{T}_h$ is fixed. To this end, we assume the exact solution $u \in H^1$ satisfies the following regularity assumption,

$$\|u\|_{k,\infty} \lesssim M^k, \quad \forall\, k \geqslant 0 \tag{3.14}$$

for some constant $M > 0$. Note that similar regularity assumption has been made in the discussion of p-version FEM schemes, e.g., [104].

Theorem 3.4. *The inf-sup condition* (3.5) *holds with the hidden constant independent of the polynomial degree r. Let u and u_h be the solutions to* (3.1) *and* (3.4), *respectively. If u satisfies the regularity condition* (3.14), *then*

$$|u - u_h|_1 \lesssim r^{-\frac{1}{2}}\mathrm{e}^{-\sigma r}, \quad \|u - u_h\|_0 \lesssim r^{-\frac{1}{2}}\mathrm{e}^{-\sigma(r+1)} \tag{3.15}$$

and

$$\|u - u_h\|_\infty \lesssim \mathrm{e}^{-\sigma(r+1)}, \tag{3.16}$$

where $\sigma = -\ln\frac{heM}{4r}$ and the hidden constants in the above inequalities are independent of r.

Superconvergence properties of the p-version FV schemes are discussed in [17] as well.

3.3 Schemes for one-dimensional convection-diffusion equations

Due to the effect of *boundary layers*, numerical solutions to convection-diffusion equations attracted much attention in FEM community (see, e.g., [6,65,67,74,82,84,95,103,104]). The FVM for convection-diffusion equations has been discussed by Eymard et al. [36]. In this subsection, we present finite volume schemes of arbitrary order introduced in [15] for the following convection-diffusion problem on $(0,1)$:

$$-\epsilon u'' + pu' + qu = f, \quad u(0) = u(1) = 0, \tag{3.17}$$

where $0 < \epsilon \ll 1$ is a small parameter and

$$p(x) \geqslant p_0 > 0, \quad q(x) \geqslant q_0 > 0, \quad \forall\, x \in [0,1].$$

3.3.1 *Construction*

The problem (3.17) exhibits a boundary layer near $x = 1$. Let the primary partition $\mathcal{T}_h$ be a Shishkin type mesh obtained by introducing $\lambda = \min(\frac{1}{2}, \frac{\epsilon}{\beta}(r+2)\ln(N+1))$, and then dividing the intervals $(0, 1-\lambda)$ and $(1-\lambda, 1)$ into N equal-size subintervals, the mesh size $h = (1-\lambda)/N$. We denote by $\mathcal{N}_h$ the set of nodes, by $\mathcal{N}_L$ the set of Lobatto points, by $\mathcal{N}_G$ the set of Gauss points of $\mathcal{T}_h$, respectively. The corresponding trial space is chosen as the Lagrange finite element of r-th order, which is defined by

$$\mathcal{U}_h^r = \{v \in C(0,1) : v|_\tau \in \mathbb{P}_r, \forall\, \tau \in \mathcal{T}_h, v(0) = v(1) = 0\}.$$

Let the dual partition $\mathcal{T}_h'$ be obtained by dividing $[0,1]$ with Gauss points in each element of $\mathcal{T}_h$ and the test space $\mathcal{V}_h$ consist of piecewise constant functions with respect to the dual partition $\mathcal{T}_h'$, which vanish on the two extremal points 0 and 1. In other words, $\mathcal{V}_h = \operatorname{Span}\{\psi_{\tau'} : \tau' \in \mathcal{T}_{h0}'\}$, where $\mathcal{T}_{h0}' = \{\tau' \in \mathcal{T}_h' : 0 \notin \tau', 1 \notin \tau'\}$. The FV solution is a function $u_h \in \mathcal{U}_h^r$ satisfying the following conservation law,

$$\int_{\tau'} (-\epsilon u_h'' + pu_h' + qu_h)dx = \int_{\tau'} f dx \tag{3.18}$$

on each control volume $\tau' \in \mathcal{T}_{h0}'$. The FVM bilinear form for all $v \in H_0^1(\Omega), w_h = \sum_{\tau' \in \mathcal{T}_h'} w_{\tau'}\psi_{\tau'} \in \mathcal{V}_h$ is defined by

$$a_h(v, w_h) = \sum_{\tau' \in \mathcal{T}_{h0}'} w_{\tau'}(-\epsilon u'' + pu' + qu)dx. \tag{3.19}$$

Then the local conservation law (3.2) is equivalent to the variational equation

$$a_h(u_h, w_h) = (f, w_h), \quad \forall\, w_h \in \mathcal{V}_h. \tag{3.20}$$

3.3.2 *Analysis*

The analysis is also performed under the framework of Petrov-Galerkin method. By [65], the solution u can be decomposed into $u = \bar{u} + u_\epsilon$, where the regular part $\bar{u}$ and the singular part u_ϵ satisfy

$$\|\bar{u}^{(k)}\|_{L^\infty} \lesssim 1, \quad |u_\epsilon^{(k)}(x)| \lesssim \epsilon^{-k}\mathrm{e}^{-\beta(1-x)/\epsilon}, \quad \forall\, x \in (0,1), \quad k \geqslant 0. \tag{3.21}$$

Let the energy norm

$$\|v\|_\epsilon^2 = \epsilon(v', v') + (v, v), \quad v \in H_0^1(\Omega) \tag{3.22}$$

and the test space norm

$$\|w_h\|_\epsilon^2 = \epsilon \sum_{P \in \mathcal{N}_G} h^{-1}[w_P]^2 + (v, v). \tag{3.23}$$

We obtain the following result in [15].

Theorem 3.5. *For sufficiently small h, we have*

$$\inf_{v_h\in\mathcal{U}_h^r}\sup_{w_h\in\mathcal{V}_h}\frac{a_h(v_h,w_h)}{\|v_h\|_\epsilon\|w_h\|_\epsilon}\gtrsim 1. \tag{3.24}$$

Moreover, if u satisfies the regularity (3.21)*, we have the modified continuity*

$$a_h(u-I_\epsilon u,w_h)\lesssim\left(\left(\frac{\ln(N+1)}{N}\right)^{r+1}+\frac{1}{N^r}\right)\|w_h\|_\epsilon. \tag{3.25}$$

Consequently,

$$\|u-u_h\|_\epsilon\lesssim\left(\frac{\ln(N+1)}{N}\right)^{r}. \tag{3.26}$$

All hidden constants are independent of ϵ and h.

The superconvergence at Gauss points is discussed by introducing the discrete norm

$$\|v\|_{\epsilon,G}^2=\epsilon\sum_{\tau\in\mathcal{T}_h}\sum_{P\in\mathcal{N}_G\cap\tau}A_{\tau,P}v'(P)^2+(v,v), \tag{3.27}$$

where $A_{\tau,P}$s are weights for the r-point Gaussian quadrature on the interval τ.

Theorem 3.6. *If u satisfies the regularity* (3.21)*, then*

$$\|u-u_h\|_{\epsilon,G}\lesssim\left(\frac{\ln(N+1)}{N}\right)^{r+1}+\frac{1}{N^r}, \tag{3.28}$$

in particular when the regular part $\bar{u}\in U_h^r$,

$$\|u-u_h\|_{\epsilon,G}\lesssim\left(\frac{\ln(N+1)}{N}\right)^{r+1}. \tag{3.29}$$

Moreover, we have the following highest order superconvergence property at all mesh nodes.

Theorem 3.7. *If u satisfies the regularity* (3.21) *and its regular part $\bar{u}\in U_h^r$, then*

$$|(u-u_h)(P)|\lesssim\left(\frac{\ln(N+1)}{N}\right)^{2r},\quad \forall\, P\in\mathcal{N}_h, \tag{3.30}$$

where the hidden constant is independent of ϵ and N.

3.4 Schemes on quadrilateral meshes for elliptic equations

We consider FV schemes of arbitrary order for (2.1) on a simply connected polygon $\Omega\subset\mathbb{R}^2$.

3.4.1 *Construction*

We partition Ω into the union of a finite number of convex quadrilaterals and denote this quadrilateral mesh by $\mathcal{T}_h$, where h is the largest diameter of all quadrilaterals. We denote by $\mathcal{N}_h$ the set of nodes, by $\mathcal{N}_L$ the set of all Lobatto points, by $\mathcal{N}_G$ the set of all Gauss points of $\mathcal{T}_h$, respectively. We choose the associated trial space as the standard FEM space of degree $r\geqslant 1$ defined by

$$\mathcal{U}_h^r=\{v\in C(\Omega)\mid \hat{v}_\tau=v\circ F_\tau\in\mathbb{Q}_r(\tau_0),\ \forall\,\tau\in\mathcal{T}_h,\text{ and } v|_{\partial\Omega}=0\},$$

where $\mathbb{Q}_r$ is the set of all polynomials of degree no more than r in each variable, $\tau_0=[-1,1]^2$ is the reference element, and the bilinear mapping F_τ maps τ_0 onto τ.

The dual mesh $\mathcal{T}_h'$ is constructed by dividing Ω with segments which connect Gauss points on an edge E with Gauss points on the opposite edge F of the same positions. In this way, each control volume in $\mathcal{T}_h'$ is a polygon V_h^P surrounding a Lobatto point $P\in\mathcal{N}_L$. See Figure 7 for control volumes associated

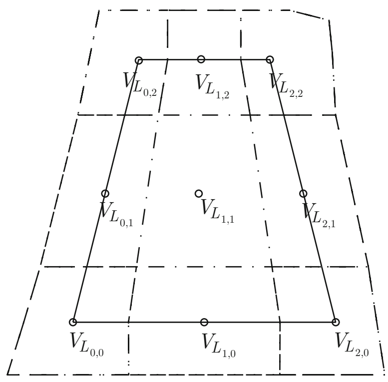

Figure 7 Control volumes associated with a quadrilateral τ ($r = 2$)

Lobatto points in a quadrilateral for $r = 2$. Note that this idea of control volumes construction was used on constructing quadratic FV schemes on rectangular meshes, see, e.g., [61].

The dual mesh $\mathcal{T}_h'$ consists of all control volumes V_h^P for $P \in \mathcal{N}_L$, i.e., $\mathcal{T}_h' = \{V_h^P \mid P \in \mathcal{N}_L\}$. The corresponding test space is defined as $\mathcal{V}_h = \text{Span}\{\psi_{V_h^P} \mid P \in \mathcal{N}_L^\circ\}$, where $\mathcal{N}_L^\circ = \mathcal{N} \setminus \partial\Omega$ and ψ_A is the characteristic function of some set $A \subset \Omega$.

The finite volume solution to (2.1) is a function $u_h \in \mathcal{U}_h^r$ satisfying the following conservation law,

$$-\int_{\partial V_h^P} \alpha \frac{\partial u_h}{\partial \mathbf{n}} ds = \int_{V_h^P} f dxdy \tag{3.31}$$

on each control volume V_h^P for $P \in \mathcal{N}_L^\circ$, where $\mathbf{n}$ is the unit outward normal on the boundary curve ∂V_h^P. Defining an FVM bilinear form for all $v \in H_0^1(\Omega), w_h = \sum_{P\in\mathcal{N}_L^\circ} w_h \psi_{V_h^P} \in \mathcal{V}_h$ as

$$a_h(v, w_h) = -\sum_{P\in\mathcal{N}_L^\circ} w_h \int_{\partial V_h^P} \alpha \frac{\partial v}{\partial \mathbf{n}} ds, \tag{3.32}$$

the finite volume method for solving equations (2.1) reads as: Find $u_h \in \mathcal{U}_h^r$ such that

$$a_h(u_h, w_h) = (f, w_h), \quad \forall\, w_h \in \mathcal{V}_h. \tag{3.33}$$

3.4.2 *Analysis*

It is known that stability proof is a challenging task in error analysis of higher order FV schemes. Earlier approaches in the literature (see, e.g., [27, 53, 56, 96]) often adopt the so-called *element stiffness matrix analysis* by calculating eigenvalues of the stiffness matrix. The stability is established if all eigenvalues of the stiffness matrix are positive. This technique has been successful for linear, quadratic, and cubic elements under various triangular mesh conditions. However, generalization to higher-order elements must be done case-by-case under more and more restrictive mesh conditions. On the other hand, those mesh conditions required for stability are usually sufficient but might not be necessary.

In [106, 107], we provided a unified approach for FV schemes of any order over quadrilateral meshes which is completely different from the classical *element stiffness matrix analysis*. The key in our analysis is the construction of the following mapping Π which maps $v_h \in \mathcal{U}_h^r$ to $w_h \in \mathcal{V}_h$ satisfying

$$\lfloor w_h \rfloor_{i,j}^\tau = A_i A_j \frac{\partial^2 \hat{v}_\tau}{\partial\xi\partial\eta}(g_i, g_j), \quad \forall\, \tau \in \mathcal{T}_h, \quad i, j = 1, \ldots, r, \tag{3.34}$$

where $l_{i,j}^\tau$ are Lobatto points in τ and the jump $\lfloor w_h \rfloor_{i,j}^\tau = w_h(l_{i,j}^\tau) + w_h(l_{i-1,j-1}^\tau) - w_h(l_{i-1,j}^\tau) - w_h(l_{i,j-1}^\tau)$, $\hat{v}_\tau = v_h \circ F_\tau \in \mathcal{Q}_r$, and A_j are the weights of the Gauss quadrature $\sum_{j=1}^r A_j v(g_j)$ for computing the integral $\int_{-1}^1 v(x)dx$. Since the total number of the Gauss points is often greater than the dimension of the test space, it is a challenging task to show the well-definiteness of the above mapping over an arbitrary unstructured quadrilateral mesh. We complete this task in [106, 107] by first determining those redundant constraints and then discussing the relation between the number of Gauss points and the dimension of the test space.

With this uniquely determined mapping, the proof of the inf-sup property of $a_h(\cdot,\cdot)$ defined on trial-test spaces is transformed to show the coercivity of the bilinear $a_h(\cdot,\Pi\cdot)$ defined on trial space only. Moreover, the transferred bilinear form $a_h(\cdot,\Pi\cdot)$ can be expressed on each element as a Gauss quadrature of the corresponding FEM bilinear form. In this way, we proved the inf-sup property of $a_h(\cdot,\cdot)$ in [106, 107]. Consequently, the optimal convergence rate can be obtained with standard arguments.

Theorem 3.8. *Let $\mathcal{T}_h$ be a shape regular $h^{1+\gamma}$-distortion* ($\gamma > 0$) *quadrilateral mesh and suppose the coefficient α is piecewisely continuous with respect to $\mathcal{T}_h$. Then the inf-sup property* (3.5) *holds when the meshsize h is sufficiently small. Consequently, if $u \in H^{r+1}(\Omega)$, then*

$$|u - u_h|_1 \lesssim h^r |u|_{r+1}. \tag{3.35}$$

Note that this theorem provides an analysis for an arbitrary r without any artificial calculation. Moreover, our mesh-condition is more relaxed in two aspects: 1) We only require that the minimum interior angle of any quadrilateral is bounded below; and 2) the mesh distortion parameter $\gamma > 0$ can be arbitrarily small. Note that $\gamma = 0$ implies an arbitrary quadrilateral without any structure. Therefore, our mesh condition here is very much similar to the most relaxed mesh condition in finite element methods.

3.4.3 *Superconvergence of schemes over rectangular meshes*

When the underlying meshes are rectangular, our FV schemes have the following superconvergence properties.

First, it is shown in [106] that the FV solution u_h supercloses to the interpolation $u_I \in \mathcal{U}_h^r$ which satisfies $u_I(P) = u(P), \forall P \in \mathcal{N}_L$.

Theorem 3.9. *Assume that $u \in H_0^1(\Omega) \cap H^{r+2}(\Omega)$ is the solution to* (2.1)*, and u_h is the solution to the FV scheme* (3.33)*. Then*

$$|u_I - u_h|_1 \lesssim h^{r+1} \|u\|_{r+2}. \tag{3.36}$$

Consequently, we have the L^2 estimates

$$\|u - u_h\|_0 \lesssim h^{r+1} \|u\|_{r+2}. \tag{3.37}$$

Note that usually, the L^2 estimate is obtained by using the so-called *Aubin-Nitsche* technique. Here, our L^2 error estimate is obtained with the help of the superconvergence property (3.36). For this reason, we require a little stronger regularity condition $u \in H^{r+2}$ instead of the usually used $u \in H^{r+1}$ in the FEM community.

In [18], we also investigated the superconvergence properties of our FV schemes at some special points. To this end, we construct a correction function $w_h \in \mathcal{U}_h^r$ such that $w_h = 0$ at all nodes and

$$|a_h(u - u_I - w_h, \Pi v)| \lesssim h^{2r} \|u\|_{2r+1,\infty} \|v\|_1, \quad \forall\, v \in \mathcal{U}_h^r. \tag{3.38}$$

Moreover,

$$\|w_h\|_\infty \lesssim h^{r+2} \|u\|_{2r+1,\infty}, \quad |w_h(P)| \lesssim h^{r+3} \|u\|_{2r+1,\infty}, \quad \forall\, P \in \mathcal{N}_L. \tag{3.39}$$

With this correction function, we obtain the following superconvergence results.

Theorem 3.10. *Let u be the solution to* (3.1)*, and u_h the solution to* (3.33)*. Then, for a nodal point $P \in \mathcal{N}_h$,*

$$|(u - u_h)(P)| \lesssim h^{2r} |\ln h|^{\frac{1}{2}} \|u\|_{2r+1,\infty}. \tag{3.40}$$

Moreover, if $P \in \mathcal{N}_L$ is a Lobatto point, then

$$|(u - u_h)(P)| \lesssim h^{r+2} |\ln h|^{\frac{\lambda}{2}} \|u\|_{r+3,\infty}, \tag{3.41}$$

and if $P \in \mathcal{N}_G$ is a Gauss point, then

$$|\nabla(u-u_h)(P)| \lesssim h^{r+1}|\ln h|^{\frac{\lambda}{2}}\|u\|_{r+3,\infty}, \tag{3.42}$$

where $\lambda = \delta_{2,r}$ with δ the Kronecker symbol.

We find from (3.40) that the error at nodes of our FV schemes has the highest order superconvergence property which is similar to the counterpart FEM proved recently in [21].

For some related convergence/superconvergence results, we refer the readers to [22, 28, 30, 32, 38, 93, 99–102].

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