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Superconvergence of Any Order Finite Volume Schemes for 1D General Elliptic Equations

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Abstract We present and analyze a finite volume scheme of arbitrary order for elliptic equations in the one-dimensional setting. In this scheme, the control volumes are constructed by using the Gauss points in subintervals of the underlying mesh. We provide a unified proof for the inf-sup condition, and show that our finite volume scheme has optimal convergence rate under the energy and L^2 norms of the approximate error. Furthermore, we prove that the derivative error is superconvergent at all Gauss points and in some special cases, the convergence rate can reach h^{r+2} and even h^{2r}, comparing with h^{r+1} rate of the counterpart finite element method. Here r is the polynomial degree of the trial space. All theoretical results are justified by numerical tests.

Keywords High order · Finite volume schemes · Superconvergence

1 Introduction

The *finite volume method* (FVM) attracted a lot of attentions during the past several decades, we refer to [4–7, 11, 17–19, 21–23, 28, 35] and the references cited therein for an incomplete list of references. Due to the local conservation of numerical fluxes, the capability to deal with the problems on the domains with complex geometries, and other advantages, FVM has a wide range of applications in scientific and engineering computations (see, e.g., [18, 21]).

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There have been many studies of the mathematical theory for FVM, see, e.g., [4,28] and the monographs [5,18,19]. However, much attention has been paid to linear FVM schemes (see, e.g., [4,6,17,23,24]), high order FVM schemes have not been investigated as much or as satisfactorily as linear FVM schemes. Moreover, the analysis of high order FVM schemes are often done case by case. For instances, earlier work on quadratic FVM schemes can be traced back to [16,20,25], high order FVMs for 1D elliptic equations were derived in [22], and high order FVMs on rectangular meshes were derived and analyzed in [7], the quadratic FVM schemes on triangular meshes have also been intensively studied by [10,19,28]. To the best of our knowledge, no analysis of FVM scheme of an arbitrary order has been published yet.

In this paper, we study a family of FVM schemes of any order in the one-dimensional setting. Instead of a case-by-case study as in the literature for quadratic and cubic FVM schemes, we adopt a unified approach to establish the inf-sup condition. Earlier work (see, e.g., [11,19,20,28]) utilized element-wise analysis to prove that the bilinear form resulting from FVM is positive definite, which is a stronger property than the inf-sup condition. Hence, some assumption is needed for the mesh, such as quasi-uniformity and shape-regularity (in 2D). The major difference here is that we prove only the inf-sup condition (instead of positive definiteness of the bilinear form) and there is no mesh condition attached. With help of the inf-sup condition, we obtain the optimal rate of convergence in both the H^1 and L^2 norms.

We further study the superconvergence property of our FVM schemes. Note that while the superconvergence theory of *finite element methods* (FEM) has reached its maturity ([3,9,26,33,34]), we know very little about superconvergence property of the FVM. Most studies have been focused on the linear schemes (see, e.g., [6,12,13,28]). In this study, we prove that for a 1D elliptic equation in general, the superconvergence behavior of the FVM is similar to that of the counterpart finite element method (see, e.g., [8,15]). For instances, the convergence rate at nodal points is h^{2r}, the rate at interior Lobatto points (defined in Sect. 4) is h^{r+2}, and the convergence rate of the derivative at Gauss points is h^{r+1}. While in some special cases, surprising superconvergence results have been found and proved: The convergence rate of the derivative error at all Gauss points can reach h^{r+2} for a general Sturm–Liouville system and h^{2r} for a special Sturm–Liouville system. The derivative convergence rate h^{2r} doubles the global optimal rate h^r, which is much better than the counterpart finite element method's h^{r+1} rate.

We organize the rest of the paper as follows. In Sect. 2 we present an arbitrary order FVM scheme for elliptic equations in one-dimensional setting. In particular, we use the Gauss points of order $r \geq 1$ to construct the control volumes and choose the trial space as the Lagrange finite element of rth order with the interpolation points being the Lobatto points of order r. In Sect. 3 we provide a unified proof for the inf-sup condition and establish the optimal convergence rate both under H^1 and L^2 norms. In Sect. 4, we study the superconvergence property at some special points of our FVM schemes of any order. In Sect. 5, a post-processing technique based on superconvergence results in the section 4 is proposed to recover the derivative of the solution. Numerical experiments supporting our theory are presented in Sect. 6. Some concluding remarks are provided in Sect. 7.

In the rest of this paper, "$A \lesssim B$" means that A can be bounded by B multiplied by a constant which is independent of the parameters which A and B may depend on. "$A \sim B$" means "$A \lesssim B$" and "$B \lesssim A$".

2 FVM Schemes of Any Order

In this section, we develop finite volume schemes for the following two-point boundary value problem on the interval $\Omega = (a, b)$:

$$\begin{aligned} -(\alpha u')'(x) + \beta(x)u'(x) + \gamma(x)u(x) = f(x), \quad \forall x \in \Omega, \\ u(a) = u(b) = 0, \end{aligned} \tag{2.1}$$

where $\alpha \geq \alpha_0 > 0$, $\gamma - \frac{1}{2}\beta' \geq \kappa > 0$, $\alpha, \beta, \gamma \in L^\infty(\bar{\Omega})$, f is a real-valued function defined on $\bar{\Omega}$.

We first introduce the primal partition and its corresponding trial space. For a positive integer N, let $\mathbb{Z}_N := \{1, \ldots, N\}$ and $a = x_0 < x_1 < \ldots < x_N = b$ be $N + 1$ distinct points on $\bar{\Omega}$. For all $i \in \mathbb{Z}_N$, we denote $\tau_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$, let $h = \max_{i \in \mathbb{Z}_N} h_i$ and

$$\mathcal{P} = \{\tau_i : i \in \mathbb{Z}_N\}$$

be a partition of Ω. The corresponding trial space is chosen as the Lagrange finite element of rth order, $r \geq 1$, defined by

$$U_{\mathcal{P}}^r = \{v \in C(\Omega) : v|_{\tau_j} \in \mathbb{P}_r, \forall \tau_j \in \mathcal{P}, v|_{\partial\Omega} = 0\},$$

where $\mathbb{P}_r$ is the set of all polynomials of degree no more than r. Obviously, $\dim U_{\mathcal{P}}^r = Nr - 1$.

We next present a dual partition and its corresponding test space. Let $G_1, \ldots, G_r$ be r Gauss points, i.e., zeros of the Legendre polynomial of rth degree, on the interval $[-1, 1]$. The Gauss points on each interval τ_i are defined as the affine transformations of G_j to τ_i, that is,

$$g_{i,j} = \frac{1}{2}(x_i + x_{i-1} + h_i G_j), \quad j \in \mathbb{Z}_r.$$

With these Gauss points, we construct a dual partition

$$\mathcal{P}' = \{\tau'_{1,0}, \tau'_{N,r}\} \cup \{\tau'_{i,j} : (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{r_i}\},$$

where

$$\tau'_{1,0} = [a, g_{1,1}], \tau'_{N,r} = [g_{N,r}, b], \tau'_{i,j} = [g_{i,j}, g_{i,j+1}],$$

here

$$r_i = \begin{cases} r & \text{if } i \in \mathbb{Z}_{N-1} \\ r-1 & \text{if } i = N \end{cases} \quad \text{and} \quad g_{i,r+1} = g_{i+1,1}, \forall i \in \mathbb{Z}_{N-1}.$$

The test space $V_{\mathcal{P}'}$ consists of the piecewise constant functions with respect to the partition $\mathcal{P}'$, which vanish on the intervals $\tau'_{1,0} \cup \tau'_{N,r}$. In other words,

$$V_{\mathcal{P}'} = \text{Span}\left\{\psi_{i,j} : (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{r_i}\right\},$$

where $\psi_{i,j} = \chi_{[g_{i,j}, g_{i,j+1}]}$ is the characteristic function on the interval $\tau'_{i,j}$. We find that $\dim V_{\mathcal{P}'} = Nr - 1 = \dim U_{\mathcal{P}}^r$.

We are ready to present our finite volume schemes. Integrating (2.1) on each control volume $[g_{i,j}, g_{i,j+1}]$, $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{r_i}$ yields

$$\int_{g_{i,j}}^{g_{i,j+1}} -(\alpha u')'(x) + \beta(x)u'(x) + \gamma(x)u(x)dx = \int_{g_{i,j}}^{g_{i,j+1}} f(x)dx.$$

In other words,

$$\alpha(g_{i,j})u'(g_{i,j}) - \alpha(g_{i,j+1})u'(g_{i,j+1}) + \int_{g_{i,j}}^{g_{i,j+1}} \big(\beta(x)u'(x) + \gamma(x)u(x)\big)dx$$
$$= \int_{g_{i,j}}^{g_{i,j+1}} f(x)dx. \tag{2.2}$$

Let $w_{\mathcal{P}'} \in V_{\mathcal{P}'}$, $w_{\mathcal{P}'}$ can be represented as

$$w_{\mathcal{P}'} = \sum_{i=1}^{N}\sum_{j=1}^{r_i} w_{i,j}\psi_{i,j},$$

where $w'_{i,j}s$ are constants. Multiplying (2.2) with $w_{i,j}$ and then summing up for all $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{r_i}$, we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{r_i} w_{i,j}\left(\alpha(g_{i,j})u'(g_{i,j}) - \alpha(g_{i,j+1})u'(g_{i,j+1}) + \int_{g_{i,j}}^{g_{i,j+1}} \big(\beta(x)u'(x) + \gamma(x)u(x)\big)dx\right)$$
$$= \int_a^b f(x)w_{\mathcal{P}'}(x)dx,$$

or equivalently,

$$\sum_{i=1}^{N}\sum_{j=1}^{r}[w_{i,j}]\alpha(g_{i,j})u'(g_{i,j}) + \sum_{i=1}^{N}\sum_{j=1}^{r_i} w_{i,j}\left(\int_{g_{i,j}}^{g_{i,j+1}} \big(\beta(x)u'(x) + \gamma(x)u(x)\big)dx\right)$$
$$= \int_a^b f(x)w_{\mathcal{P}'}(x)dx,$$

where $[w_{i,j}] = w_{i,j} - w_{i,j-1}$ is the jump of w at the point $g_{i,j}$, $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_r$ with $w_{1,0} = 0$, $w_{N,r} = 0$ and $w_{i,0} = w_{i-1,r}$, $2 \le i \le N$.

We define the FVM bilinear form for all $v \in H_0^1(\Omega)$, $w_{\mathcal{P}'} \in V_{\mathcal{P}'}$ by

$$a_{\mathcal{P}}(v, w_{\mathcal{P}'}) = \sum_{i=1}^{N}\sum_{j=1}^{r}[w_{i,j}]\alpha(g_{i,j})v'(g_{i,j})$$
$$+ \sum_{i=1}^{N}\sum_{j=1}^{r_i} w_{i,j}\left(\int_{g_{i,j}}^{g_{i,j+1}} \big(\beta(x)v'(x) + \gamma(x)v(x)\big)dx\right). \tag{2.3}$$

The finite volume method for solving Eq. (2.1) reads as : Find $u_{\mathcal{P}} \in U_{\mathcal{P}}^r$ such that

$$a_{\mathcal{P}}(u_{\mathcal{P}}, w_{\mathcal{P}'}) = (f, w_{\mathcal{P}'}), \quad \forall w_{\mathcal{P}'} \in V_{\mathcal{P}'}. \tag{2.4}$$

3 Convergence Analysis

In this section, we prove the inf-sup condition and use it to establish some convergence properties of the FVM solution.

3.1 Inf-sup Condition

We begin with some notations and definitions. First we introduce the *broken* Sobolev space

$$W_{\mathcal{P}}^{m,p}(\Omega) = \{v \in C(\Omega) : v|_{\tau_i} \in W^{m,p}, \forall \tau_i \in \mathcal{P}\},$$

where m is a given positive integer and $1 \leq p \leq \infty$. When $p = 2$, we denote $H_{\mathcal{P}}^m$ for simplicity. For all $j \geq 0$, we define a semi-norm by

$$|v|_{j,p,\mathcal{P}} = \left(\sum_{\tau_i \in \mathcal{P}} |v|_{j,p,\tau_i}^p \right)^{\frac{1}{p}}$$

and a norm by

$$\|v\|_{m,p,\mathcal{P}} = \left(\sum_{j=0}^{m} |v|_{j,p,\mathcal{P}}^p \right)^{\frac{1}{p}}.$$

We often use $|\cdot|_{m,\mathcal{P}}$ instead of $|\cdot|_{m,2,\mathcal{P}}$ and $\|\cdot\|_{m,\mathcal{P}}$ instead of $\|\cdot\|_{m,2,\mathcal{P}}$ for simplicity.

Secondly, for all $w_{\mathcal{P}'} \in V_{\mathcal{P}'},\ w_{\mathcal{P}'} = \sum_{i=1}^{N} \sum_{j=1}^{r_i} w_{i,j} \psi_{i,j}$, let

$$|w_{\mathcal{P}'}|_{1,\mathcal{P}'}^2 = \sum_{i=1}^{N} \sum_{j=1}^{r} h_i^{-1} [w_{i,j}]^2, \quad \|w_{\mathcal{P}'}\|_{0,\mathcal{P}'}^2 = \sum_{i=1}^{N} \sum_{j=1}^{r_i} h_i w_{i,j}^2$$

and

$$\|w_{\mathcal{P}'}\|_{\mathcal{P}'}^2 = |w_{\mathcal{P}'}|_{1,\mathcal{P}'}^2 + \|w_{\mathcal{P}'}\|_{0,\mathcal{P}'}^2.$$

Noticing that $w_{1,0} = w_{N,r} = 0$, it is easy to obtain the following Poincaré type inequality

$$\|w_{\mathcal{P}'}\|_{0,\mathcal{P}'} \lesssim |w_{\mathcal{P}'}|_{1,\mathcal{P}'}, \quad \forall w_{\mathcal{P}'} \in V_{\mathcal{P}'}, \tag{3.1}$$

where the hidden constant depends only on Ω and r.

Thirdly, we denote by A_j, $j \in \mathbb{Z}_r$ the weights of the Gauss quadrature

$$Q_r(F) = \sum_{j=1}^{r} A_j F(G_j)$$

for computing the integral

$$I(F) = \int_{-1}^{1} F(x) dx.$$

It is well-known that $Q_r(F) = I(F)$ for all $F \in \mathbb{P}_{2r-1}(-1, 1)$. Naturally, the weights on interval τ_i, $i \in \mathbb{Z}_N$ are

$$A_{ij} = \frac{h_i}{2} A_j, \quad j \in \mathbb{Z}_r.$$

Before the presentation of the inf-sup condition, we define a linear mapping $\Pi_{\mathcal{P}} : U^r_{\mathcal{P}} \to V_{\mathcal{P}'}$ by

$$\Pi_{\mathcal{P}} v_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{r_i} v_{i,j} \psi_{i,j},$$

where the coefficients $v_{i,j}$ are determined by the constraints

$$[v_{i,j}] = A_{i,j} v'_{\mathcal{P}}(g_{i,j}), \quad (i,j) \in \mathbb{Z}_N \times \mathbb{Z}_{r_i}.$$

Note that $v_{\mathcal{P}} \in U^r_{\mathcal{P}}$, the derivative $v'_{\mathcal{P}} \in \mathbb{P}_{r-1}(\tau_i)$, $i \in \mathbb{Z}_N$, then

$$\sum_{i=1}^{N} \sum_{j=1}^{r} A_{i,j} v'_{\mathcal{P}}(g_{i,j}) = \int_a^b v'_{\mathcal{P}}(x) dx = v_{\mathcal{P}}(b) - v_{\mathcal{P}}(a) = 0.$$

Therefore,

$$\begin{aligned} v_{N,r-1} &= \sum_{i=1}^{N} \sum_{j=1}^{r_i} [v_{i,j}] = \sum_{i=1}^{N} \sum_{j=1}^{r} A_{i,j} v'_{\mathcal{P}}(g_{i,j}) - A_{N,r} v'_{\mathcal{P}}(g_{N,r}) \\ &= -A_{N,r} v'_{\mathcal{P}}(g_{N,r}). \end{aligned}$$

In other words, we also have

$$[v_{N,r}] = v_{N,r} - v_{N,r-1} = A_{N,r} v'_{\mathcal{P}}(g_{N,r}).$$

Consequently,

$$|\Pi_{\mathcal{P}} v_{\mathcal{P}}|^2_{1,\mathcal{P}'} = \sum_{i=1}^{N} \sum_{j=1}^{r} h_i^{-1} [v_{i,j}]^2 = \sum_{i=1}^{N} \sum_{j=1}^{r} h_i^{-1} \left(A_{i,j} v'_{\mathcal{P}}(g_{i,j})\right)^2 \sim \int_a^b \left(v'_{\mathcal{P}}(x)\right)^2 dx.$$

Namely, we have

$$|\Pi_{\mathcal{P}} v_{\mathcal{P}}|_{1,\mathcal{P}'} \sim |v_{\mathcal{P}}|_{1,\mathcal{P}}. \tag{3.2}$$

With all these preparations, we are now ready to present the inf-sup condition of $a_{\mathcal{P}}(\cdot,\cdot)$.

Theorem 3.1 *Assume that the mesh size h is sufficiently small, then*

$$\inf_{v_{\mathcal{P}} \in U^r_{\mathcal{P}}} \sup_{w_{\mathcal{P}'} \in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(v_{\mathcal{P}}, w_{\mathcal{P}'})}{\|v_{\mathcal{P}}\|_{1,\mathcal{P}} \|w_{\mathcal{P}'}\|_{\mathcal{P}'}} \ge c_0, \tag{3.3}$$

where $c_0 > 0$ is a constant depending only on r, α_0, κ and Ω.

Proof It follows from the bilinear form (2.3) that

$$a_{\mathcal{P}}(v_{\mathcal{P}}, \Pi_{\mathcal{P}} v_{\mathcal{P}}) = I_1 + I_2, \quad \forall v_{\mathcal{P}} \in U^r_{\mathcal{P}}$$

with

$$I_1 = \sum_{i=1}^{N} \sum_{j=1}^{r} [v_{i,j}] \alpha(g_{i,j}) v'_{\mathcal{P}}(g_{i,j}), \quad I_2 = \sum_{i=1}^{N} \sum_{j=1}^{r_i} v_{i,j} \int_{g_{i,j}}^{g_{i,j+1}} \left(\beta(x) v'_{\mathcal{P}}(x) + \gamma(x) v_{\mathcal{P}}(x)\right) dx.$$

Since $(v'_{\mathcal{P}})^2 \in \mathbb{P}_{2r-2}(\tau_i)$, $i \in \mathbb{Z}_N$, we have

$$\int_{x_{i-1}}^{x_i} (v'_{\mathcal{P}}(x))^2 dx = \sum_{j=1}^{r} A_{i,j}(v'_{\mathcal{P}}(g_{i,j}))^2.$$

Therefore,

$$I_1 \geq \alpha_0 \sum_{i=1}^{N} \sum_{j=1}^{r} A_{i,j}(v'_{\mathcal{P}}(g_{i,j}))^2 = \alpha_0 |v_{\mathcal{P}}|^2_{1,\mathcal{P}}.$$

We next estimate I_2. Let $V(x) = \int_a^x \big(\beta(s)v'_{\mathcal{P}}(s) + \gamma(s)v_{\mathcal{P}}(s)\big)\, ds$ and denote by

$$E_i = \int_{x_{i-1}}^{x_i} v'_{\mathcal{P}}(x)V(x)dx - \sum_{j=1}^{r} A_{i,j} v'_{\mathcal{P}}(g_{i,j})V(g_{i,j}),$$

the error of Gauss quadrature in the interval τ_i, $i \in \mathbb{Z}_N$. Then

$$I_2 = -\sum_{i=1}^{N}\sum_{j=1}^{r}[v_{i,j}]V(g_{i,j}) = -\int_a^b v'_{\mathcal{P}}(x)V(x)dx + \sum_{i=1}^{N} E_i.$$

Using the fact that $v_{\mathcal{P}}(a) = v_{\mathcal{P}}(b) = 0$ and

$$\int_a^b \beta(x)v'_{\mathcal{P}}(x)v_{\mathcal{P}}(x)dx = -\frac{1}{2}\int_a^b \beta'(x)v^2_{\mathcal{P}}(x)dx,$$

we obtain

$$-\int_a^b v'_{\mathcal{P}}(x)V(x)dx = \int_a^b \left(\gamma(x) - \frac{\beta'(x)}{2}\right) v^2_{\mathcal{P}}(x)dx \geq \kappa \|v_{\mathcal{P}}\|_0^2. \tag{3.4}$$

On the other hand, by [14, p. 98, (2.7.12)], for all $i \in \mathbb{Z}_N$

$$E_i = \frac{h_i^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}(v'_{\mathcal{P}}V)^{(2r)}(\xi_i),$$

where $\xi_i \in \tau_i$. By the Leibnitz formula of derivatives, we have

$$\left|(v'_{\mathcal{P}}V)^{(2r)}(\xi_i)\right| \leq \sum_{k=r+1}^{2r}\binom{2r}{k}\left|(\beta v'_{\mathcal{P}} + \gamma v_{\mathcal{P}})^{(k-1)}(v'_{\mathcal{P}})^{(2r-k)}(\xi_i)\right| \leq c_1 \|v_{\mathcal{P}}\|^2_{r,\infty,\tau_i}$$

with

$$c_1 = \max\left\{\|\beta\|^2_{2r-1,\infty,\tau_i}, \|\gamma\|^2_{2r-1,\infty,\tau_i}\right\}\sum_{k=r+1}^{2r}\binom{2r}{k}.$$

Therefore, by the inverse inequality that

$$\|v_{\mathcal{P}}\|_{r,\infty,\tau_i} \lesssim h_i^{-(r-\frac{1}{2})}|v_{\mathcal{P}}|_{1,\tau_i},$$

we have

$$|E_i| \leq \frac{c_1(r!)^4}{(2r+1)[(2r)!]^3} h_i^2 |v_{\mathcal{P}}|^2_{1,\tau_i}.$$

Combining the above estimates, we obtain

$$I_2 \geq \kappa \|v_{\mathcal{P}}\|^2_{0,\mathcal{P}} - \frac{c_1(r!)^4}{(2r+1)[(2r)!]^3} h^2 \, |v_{\mathcal{P}}|^2_{1,\mathcal{P}} .$$

Then for sufficiently small h, we have

$$a_{\mathcal{P}}(v_{\mathcal{P}}, \Pi_{\mathcal{P}} v_{\mathcal{P}}) \geq \frac{\alpha_0}{2} |v_{\mathcal{P}}|^2_{1,\mathcal{P}} + \frac{\kappa}{2} \|v_{\mathcal{P}}\|^2_{0,\mathcal{P}} \geq \frac{1}{2} \min\{\alpha_0, \kappa\} \|v_{\mathcal{P}}\|^2_{1,\mathcal{P}}.$$

Noticing (3.1) and (3.2), we obtain

$$\|\Pi_{\mathcal{P}} v_{\mathcal{P}}\|_{\mathcal{P}'} \lesssim \|v_{\mathcal{P}}\|_{1,\mathcal{P}}.$$

Therefore, for any $v_{\mathcal{P}} \in U^r_{\mathcal{P}}$,

$$\sup_{w_{\mathcal{P}'} \in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(v_{\mathcal{P}}, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}'}} \geq \frac{a_{\mathcal{P}}(v_{\mathcal{P}}, \Pi_{\mathcal{P}} v_{\mathcal{P}})}{\|\Pi_{\mathcal{P}} v_{\mathcal{P}}\|_{\mathcal{P}'}} \geq c_0 \|v_{\mathcal{P}}\|_{1,\mathcal{P}},$$

where c_0 is a constant depending only on r, α_0, κ and Ω. The inf-sup condition (3.3) then follows. □

Remark 3.2 In the above proof, the partition $\mathcal{P}$ does not need to satisfy any quasi-uniform or shape-regularity conditions. Moreover, (3.3) holds even the order of polynomials in each sub-interval τ_i is different.

3.2 Energy Norm Error Estimate

Following [28], we use the inf-sup condition (3.3) and the framework of Petrov–Galerkin method to present and analyze the finite volume method (2.4).

We first introduce a semi-norm and a norm of the broken space $H^2_{\mathcal{P}}(\Omega) = W^{2,2}_{\mathcal{P}}(\Omega)$ which are

$$|v|^2_{\mathcal{P}} = \sum_{\tau_i \in \mathcal{P}} |v|^2_{1,\tau_i} + h_i^2 |v|^2_{2,\tau_i}, \quad \|v\|^2_{\mathcal{P}} = \|v\|^2_{0,\mathcal{P}} + |v|^2_{\mathcal{P}}.$$

It is straightforward to show that,

$$|v_{\mathcal{P}}|_{\mathcal{P}} \sim |v_{\mathcal{P}}|_{1,\mathcal{P}}, \qquad \|v_{\mathcal{P}}\|_{\mathcal{P}} \sim \|v_{\mathcal{P}}\|_{1,\mathcal{P}}, \qquad \forall v_{\mathcal{P}} \in U^r_{\mathcal{P}}.$$

With these equivalences, the inf-sup condition (3.3) can be written as

$$\inf_{v_{\mathcal{P}} \in U^r_{\mathcal{P}}} \sup_{w_{\mathcal{P}'} \in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(v_{\mathcal{P}}, w_{\mathcal{P}'})}{\|v_{\mathcal{P}}\|_{\mathcal{P}} \|w_{\mathcal{P}'}\|_{\mathcal{P}'}} \geq c_2, \tag{3.5}$$

where $c_2 > 0$ also depends only on r, α_0, κ and Ω. Moreover, we define a discrete semi-norm $\big|\cdot\big|_{G,1}$ for all $v \in H^1_0(\Omega)$ by

$$\big|v\big|_{G,1} = \left(\sum_{i=1}^{N} \sum_{j=1}^{r} A_{i,j} (v'(g_{i,j}))^2 \right)^{\frac{1}{2}}.$$

We next discuss the relationship between $|\cdot|_{\mathcal{P}}$ and $|\cdot|_{G,1}$. On one hand,

$$|v_{\mathcal{P}}|_{G,1} = |v_{\mathcal{P}}|_{1,\mathcal{P}} \sim |v_{\mathcal{P}}|_{\mathcal{P}}, \quad \forall v_{\mathcal{P}} \in U^r_{\mathcal{P}}.$$

On the other hand, for all $v \in H^1_0(\Omega) \cap H^2_{\mathcal{P}}(\Omega)$,

$$(v'(g_{i,j}))^2 \lesssim h_i^{-1}\|v'\|^2_{L^2(\tau_i)} + h_i\|v''\|^2_{L^2(\tau_i)}, \quad (i,j) \in \mathbb{Z}_N \times \mathbb{Z}_r,$$

where the hidden constant depends only on r. Thus by the fact $A_{ij} \le h_i$, we have

$$|v|^2_{G,1} \lesssim \sum_{i=1}^N h_i \left(h_i^{-1}\|v'\|^2_{L^2(\tau_i)} + h_i\|v''\|^2_{L^2(\tau_i)}\right) = |v|^2_{\mathcal{P}}.$$

Namely,

$$|v|_{G,1} \lesssim |v|_{\mathcal{P}}, \quad \forall v \in H^1_0(\Omega) \cap H^2_{\mathcal{P}}(\Omega),$$

where the hidden constant depends only on r.

We are in a perfect position to show our main result.

Theorem 3.3 *Assume that u is the solution of* (2.1)*, $u_{\mathcal{P}}$ is the solution of* (2.4)*. Then the finite volume bilinear form $a_{\mathcal{P}}(\cdot,\cdot)$ is variationally exact:*

$$a_{\mathcal{P}}(u, w_{\mathcal{P}'}) = (f, w_{\mathcal{P}'}), \quad \forall\, w_{\mathcal{P}'} \in V_{\mathcal{P}'}, \tag{3.6}$$

and bounded : for all $v \in H^1_0(\Omega) \cap H^2_{\mathcal{P}}(\Omega)$, $w_{\mathcal{P}'} \in V_{\mathcal{P}'}$,

$$|a_{\mathcal{P}}(v, w_{\mathcal{P}'})| \le M\|v\|_{\mathcal{P}}\|w_{\mathcal{P}'}\|_{\mathcal{P}'}, \tag{3.7}$$

where the constant $M > 0$ depends only on r, α_0, κ and Ω. Consequently,

$$\|u - u_{\mathcal{P}}\|_{\mathcal{P}} \le \frac{M}{c_2} \inf_{v_{\mathcal{P}} \in U^r_{\mathcal{P}}} \|u - v_{\mathcal{P}}\|_{\mathcal{P}}, \tag{3.8}$$

where c_2 is the same as in (3.5)*.*

Proof First, the formula (3.6) follows by multiplying (2.1) by an arbitrary function $w_{\mathcal{P}'} \in V_{\mathcal{P}'}$ and then using Newton–Leibniz formula on each control volume $[g_{i,j}, g_{i,j+1}]$, $(i,j) \in \mathbb{Z}_N \times \mathbb{Z}_{r_i}$.

Secondly, we show (3.7). By the Cauchy-Schwartz inequality, from (2.3) there holds for all $v \in H^1_0(\Omega) \cap H^2_{\mathcal{P}}(\Omega)$, $w_{\mathcal{P}'} \in V_{\mathcal{P}'}$ that

$$\begin{aligned} a_{\mathcal{P}}(v, w_{\mathcal{P}'}) &\le |v|_{G,1}\left(\sum_{i=1}^N\sum_{j=1}^r \frac{\alpha^2(g_{i,j})}{A_{ij}}([w_{i,j}])^2\right)^{\frac12} \\ &\quad + \max(|\beta|,|\gamma|)\left(\sum_{i=1}^N\sum_{j=1}^{r_i} h_i w_{i,j}^2\right)^{\frac12}\left(\sum_{i=1}^N(|v|^2_{1,\tau_i} + \|v\|^2_{0,\tau_i})\right)^{\frac12} \\ &\le M\|v\|_{\mathcal{P}}\|w_{\mathcal{P}'}\|_{\mathcal{P}'}, \end{aligned}$$

where the constant M depends only on r, α_0, κ and Ω.

Finally, combining the inf-sup condition (3.5), (3.6) and (3.7), we derive (3.8) using similar arguments as in Babuska and Aziz [2], or Xu and Zikatanov [27]. □

Corollary 3.4 *Assume that* $u \in H_0^1(\Omega) \cap H_{\mathcal{P}}^{r+1}(\Omega)$ *is the solution of* (2.1)*, and* $u_{\mathcal{P}}$ *is the solution of FVM scheme* (2.4)*, then*

$$\|u - u_{\mathcal{P}}\|_1 \lesssim h^r |u|_{r+1,\mathcal{P}}, \tag{3.9}$$

where the hidden constant is independent of $\mathcal{P}$.

Proof It follows from the definition of $\|\cdot\|_{\mathcal{P}}$ and (3.8) that

$$\|u - u_{\mathcal{P}}\|_1 \le \|u - u_{\mathcal{P}}\|_{\mathcal{P}} \lesssim \inf_{v_{\mathcal{P}} \in U_{\mathcal{P}}^r} \|u - v_{\mathcal{P}}\|_{\mathcal{P}}.$$

Notice that

$$\inf_{v_{\mathcal{P}} \in U_{\mathcal{P}}^r} \|u - v_{\mathcal{P}}\|_{\mathcal{P}} \le \|u - u_I\|_1 + \left(\sum_{i=1}^N h_i^2 |u - u_I|_{2,\tau_i}^2 \right)^{\frac{1}{2}},$$

where u_I is the Lagrange interpolation of u at the Lobatto points (defined in the next section) in the trial space $U_{\mathcal{P}}^r$. By the standard approximation theory, we obtain the estimate (3.9). □

4 Superconvergence

In this section, we will present the superconvergence properties of the FVM solution. To this end, we need to use an interpolation of a function on the so-called Lobatto points. This kind of interpolation has been used in the literature for superconvergence analysis, see, e.g., [31,32]. We denote $L_1, L_2, \dots, L_{r-1}$ the zeros of $P_r'(x)$, where $P_r(x)$ is the Legendre polynomial of order r. Moreover, we denote $L_0 = -1$, $L_r = 1$ and $\mathbb{N}_r = \{0, 1, \dots, r\}$ for $r \ge 1$. The family of points L_j, $j \in \mathbb{N}_r$ are called Lobatto points of degree r. The Lobatto points on τ_i are defined by using the affine transformations from $[-1, 1]$ to τ_i, i.e,

$$l_{i.j} = \frac{1}{2}(x_i + x_{i-1} + h_i L_j), \quad j \in \mathbb{N}_r.$$

Let $u_I \in U_{\mathcal{P}}^r$ be the interpolation of u such that

$$u_I(l_{i,j}) = u(l_{i,j}), \quad (i, j) \in \mathbb{Z}_N \times \mathbb{N}_r,$$

then by [33, p. 146 (1.2)]

$$|(u - u_I)'(g_{i,j})| \lesssim h^r |u|_{r+2,1,\omega_{i,j}'}, \tag{4.1}$$

where $\omega_{i,j}' = (g_{i,j-1}, g_{i,j+1})$.

Theorem 4.1 *Let* $u \in H_0^1(\Omega) \cap H_{\mathcal{P}}^{r+2}(\Omega)$ *be the solution of* (2.1)*, and* $u_{\mathcal{P}}$ *the solution of FVM scheme* (2.4)*. Then*

$$|a_{\mathcal{P}}(u - u_I, w_{\mathcal{P}'})| \lesssim h^{r+1} \left(|u|_{r+2,\mathcal{P}} + |u|_{r+1,\infty,\mathcal{P}} \right) \|w_{\mathcal{P}'}\|_{\mathcal{P}'}, \quad \forall w_{\mathcal{P}'} \in V_{\mathcal{P}'}. \tag{4.2}$$

Consequently,

$$\|u_I - u_{\mathcal{P}}\|_1 \lesssim h^{r+1} \left(|u|_{r+2,\mathcal{P}} + |u|_{r+1,\infty,\mathcal{P}} \right). \tag{4.3}$$

Proof Recalling the definition of bilinear form (2.3) and using integral by parts, we obtain

$$a_{\mathcal{P}}(u - u_I, w_{\mathcal{P}'}) = I_1 + I_2$$

where

$$I_1 = \sum_{i=1}^{N}\sum_{j=1}^{r}[w_{i,j}]\left(\alpha(g_{i,j})(u-u_I)'(g_{i,j}) - \beta(g_{i,j})(u-u_I)(g_{i,j})\right),$$

$$I_2 = \sum_{i=1}^{N}\sum_{j=1}^{r} w_{i,j}\int_{g_{i,j}}^{g_{i,j+1}}(\gamma-\beta')(u-u_I).$$

Note that for all $i \in \mathbb{Z}_N$,

$$(u-u_I)(g_{i,j}) \lesssim h_i^{r+1}|u|_{r+1,\infty,\tau_i}, \quad |u|_{r+2,1,\tau_i} \lesssim h_i^{\frac{1}{2}}|u|_{r+2,\tau_i}.$$

By the Cauchy–Schwartz inequality, (4.1) and the standard approximation theory,

$$\begin{aligned} a_{\mathcal{P}}(u-u_I, w_{\mathcal{P}'}) &\lesssim \|w_{\mathcal{P}'}\|_{\mathcal{P}'}\left(\sum_{i=1}^{N}\sum_{j=1}^{r}\left(h_i^{2(r+1)}|u|^2_{r+2,\tau_i} + h_i^{2r+3}|u|^2_{r+1,\infty,\tau_i} + \|u-u_I\|^2_{0,\tau_i}\right)\right)^{\frac{1}{2}} \\ &\lesssim h^{r+1}\left(|u|_{r+2,\mathcal{P}} + |u|_{r+1,\infty,\mathcal{P}}\right)\|w_{\mathcal{P}'}\|_{\mathcal{P}'}. \end{aligned}$$

The desired result (4.2) is proved.

We next show (4.3). By the inf-sup condition (3.3),

$$c_0\|u_I - u_{\mathcal{P}}\|_1 \le \sup_{w_{\mathcal{P}'}\in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(u_{\mathcal{P}} - u_I, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}'}} = \sup_{w_{\mathcal{P}'}\in V_{\mathcal{P}'}} \frac{a_{\mathcal{P}}(u - u_I, w_{\mathcal{P}'})}{\|w_{\mathcal{P}'}\|_{\mathcal{P}'}}.$$

Combining the above inequality with (4.2), we derive (4.3). □

As a direct consequence of (4.3), we have the following L^2 error estimate.

Corollary 4.2 *Let $u \in H_0^1(\Omega)\cap H_{\mathcal{P}}^{r+2}(\Omega)$ be the solution of* (2.1)*, and $u_{\mathcal{P}}$ the solution of FVM scheme* (2.4)*, then*

$$\|u - u_{\mathcal{P}}\|_0 \lesssim h^{r+1}\|u\|_{r+2,\mathcal{P}}, \tag{4.4}$$

where the hidden positive constant is independent of $\mathcal{P}$.

Proof By the triangle inequality,

$$\|u-u_{\mathcal{P}}\|_0 \le \|u-u_I\|_0 + \|u_I - u_{\mathcal{P}}\|_0.$$

By (4.3), we have

$$\|u_I - u_{\mathcal{P}}\|_0 \lesssim \|u_I - u_{\mathcal{P}}\|_1 \lesssim h^{r+1}\|u\|_{r+2,\mathcal{P}}.$$

Moreover, by the standard approximation theory,

$$\|u-u_I\|_0 \lesssim h^{r+1}\|u\|_{r+1} \lesssim h^{r+1}\|u\|_{r+2,\mathcal{P}}.$$

The above facts imply (4.4). □

Remark 4.3 In the above L^2 error estimate, we do not use the so-called Aubin-Nitsche technique. However, we need slightly stronger regularity assumption on the exact solution u.

We first study the superconvergence property at the nodes x_i, $i \in \mathbb{Z}_{N-1}$.

Theorem 4.4 *Let u be the solution of* (2.1), *and $u_{\mathcal{P}}$ the solution of FVM scheme* (2.4). *If the coefficient $\alpha \in C^{2r}(\Omega)$ and the solution $u \in W_{\mathcal{P}}^{2r+1,\infty}(\Omega)$, then*

$$|(u - u_{\mathcal{P}})(x_i)| \lesssim h^{2r} \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\mathcal{P}}, \quad \forall i \in \mathbb{Z}_{N-1}. \tag{4.5}$$

Proof Let $e = u - u_{\mathcal{P}}$ and

$$\epsilon(x) = \int_a^x \big(\beta(y)e'(y) + \gamma(y)e(y)\big)dy, \quad \forall x \in [a, b].$$

By the construction of the FVM scheme, both u and $u_{\mathcal{P}}$ satisfy (2.2), then for all $(i, j) \in \mathbb{Z}_{N-1} \times \mathbb{Z}_{r_i}$,

$$-\big(\alpha(g_{i,j+1})e'(g_{i,j+1}) - \alpha(g_{i,j})e'(g_{i,j})\big) + \epsilon(g_{i,j+1}) - \epsilon(g_{i,j}) = 0.$$

Namely,

$$\alpha(g_{i,j})e'(g_{i,j}) - \epsilon(g_{i,j}) = C_0, \tag{4.6}$$

where C_0 is a constant independent of i, j.

On the other hand, let $G(\cdot, \cdot)$ be the Green function for the problem (2.1). Then for all $v \in H_0^1(\Omega)$,

$$v(x) = A(v, G(x, \cdot)), \quad \forall x \in \Omega,$$

where the Galerkin bilinear form $A(\cdot, \cdot)$ is defined for all $v, w \in H_0^1(\Omega)$ by

$$A(v, w) = \int_a^b \alpha(y)v'(y)w'(y)dy + \int_a^b \big(\beta(y)v'(y) + \gamma(y)v(y)\big)w(y)dy.$$

In particular, for all $i \in \mathbb{Z}_{N-1}$,

$$e(x_i) = A(e, G(x_i, \cdot)).$$

Noting that $G(x_i, a) = G(x_i, b) = 0$, by(4.6)

$$\begin{aligned} e(x_i) &= \int_a^b \big(\alpha(y)e'(y) - \epsilon(y)\big)\frac{\partial G}{\partial y}(x_i, y)dy \\ &= \sum_{k=1}^{N}\sum_{j=1}^{r} A_{k,j}\big(\alpha(g_{k,j})e'(g_{k,j}) - \epsilon(g_{k,j})\big)\frac{\partial G}{\partial y}(x_i, g_{k,j}) + E_1 \\ &= C_0 \int_a^b \frac{\partial G}{\partial y}(x_i, y)dy + E_1 + E_2 = E_1 + E_2, \end{aligned}$$

where the residuals of the Gauss quadratures

$$E_1 = \int_a^b \big(\alpha(y)e'(y) - \epsilon(y)\big)\frac{\partial G}{\partial y}(x_i, y)dy - \sum_{k=1}^{N}\sum_{j=1}^{r} A_{k,j}\big(\alpha(g_{k,j})e'(g_{k,j}) - \epsilon(g_{k,j})\big)\frac{\partial G}{\partial y}(x_i, g_{k,j}),$$

and

$$E_2 = -C_0\left(\int_a^b \frac{\partial G}{\partial y}(x_i, y)dy - \sum_{k=1}^{N}\sum_{j=1}^{r} A_{k,j}\frac{\partial G}{\partial y}(x_i, g_{k,j})\right).$$

By [14, p. 98 (2.7.12)], there exist $\xi_k, \eta_k \in \tau_k$ such that

$$E_1 = \sum_{k=1}^{N} \frac{h_k^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}\left[\big((\alpha(y)e'(y) - \epsilon(y)\big)\frac{\partial G}{\partial y}(x_i, y)\right]^{(2r)}\Bigg|_{y=\xi_k},$$

$$E_2 = -\sum_{k=1}^{N} \frac{h_k^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}\left[\frac{\partial G}{\partial y}(x_i, y)\right]^{(2r)}\Bigg|_{y=\eta_k}.$$

We next estimate E_1 and E_2 separately. Note that $e^{(j)} = u^{(j)}$ for $j > r$ and the Green function $G(x_i, \cdot)$ has bounded derivatives of any order on each τ_k, $k \in \mathbb{Z}_N$, then

$$\begin{aligned}|E_1| &\lesssim \sum_{k=1}^{N} h_k^{2r+1}\left(\sum_{j=0}^{r}|e|_{j,\infty,\tau_k} + \sum_{j=r+1}^{2r+1}|u|_{j,\infty,\tau_k}\right)\\ &\lesssim \sum_{k=1}^{N} h_k^{2r+1}\left(\sum_{j=0}^{r} h^{-j}\|e\|_{0,\infty,\tau_k} + \sum_{j=r+1}^{2r+1}|u|_{j,\infty,\tau_k}\right)\\ &\lesssim h^r\|e\|_{0,\infty,\mathcal{P}} + h^{2r}\sum_{j=r+1}^{2r}|u|_{j,\infty,\mathcal{P}},\end{aligned}$$

where in the second inequality we have used the fact that (cf., [1])

$$|e|_{j,\infty,\tau_k} \lesssim h_k^{-j}\|e\|_{0,\infty,\tau_k} + h_k^{r+1-j}|e|_{r+1,\infty,\tau_k}, \quad \forall j \in \mathbb{Z}_r. \tag{4.7}$$

We next consider the term $\|e\|_{0,\infty,\mathcal{P}}$. By the triangular inequality

$$\begin{aligned}\|u_I - u_{\mathcal{P}}\|_{0,\infty,\mathcal{P}} &\lesssim h^{\frac{1}{2}}|u_I - u_{\mathcal{P}}|_1 \lesssim h^{\frac{1}{2}}(|u - u_I|_1 + |u - u_{\mathcal{P}}|_1)\\ &\lesssim h^{r+\frac{1}{2}}|u|_{r+1} \lesssim h^{r+\frac{1}{2}}|u|_{r+1,\infty,\mathcal{P}},\end{aligned}$$

then

$$\|e\|_{0,\infty,\mathcal{P}} \le \|u - u_I\|_{0,\infty,\mathcal{P}} + \|u_I - u_{\mathcal{P}}\|_{0,\infty,\mathcal{P}} \lesssim h^{r+\frac{1}{2}}|u|_{r+1,\infty,\mathcal{P}}.$$

In summary,

$$|E_1| \lesssim h^{2r}\sum_{j=r+1}^{2r+1}|u|_{j,\infty,\mathcal{P}}.$$

As for E_2, a direct calculation implies that

$$|E_2| \lesssim h^{2r} \|G\|_{2r,\infty,\mathcal{P}}.$$

Combining E_1 with E_2, we obtain (4.5). □

As a direct consequence of (4.5), we have

$$E_{node} = \left(\frac{1}{N} \sum_{i=1}^{N} [(u - u_{\mathcal{P}})(x_i)]^2 \right)^{\frac{1}{2}} \lesssim h^{2r}. \tag{4.8}$$

In the following we present the superconvergence property of $u'_{\mathcal{P}}$ at Gauss points, and $u_{\mathcal{P}}$ at interior Lobatto points. We start with a projector which maps a function $\hat{v}(t) \in H^1([-1, 1])$ to a polynomial $\hat{v}_r \in \mathbb{P}_r([-1, 1])$ defined by

$$\hat{v}_r(t) = \sum_{j=0}^{r} b_j M_j(t)$$

where M_j is the Lobatto polynomial of degree j and

$$b_0 = \frac{\hat{v}(1) + \hat{v}(-1)}{2}, \quad b_1 = \frac{\hat{v}(1) - \hat{v}(-1)}{2},$$

$$b_j = (j - \frac{1}{2}) \int_{-1}^{1} \hat{v}'(t) M'_j(t) dt, \quad j = 2, \dots, r.$$

For any $x \in \tau_i, i \in \mathbb{Z}_N$, we denote

$$v(x) = \hat{v}(\frac{x_i + x_{i-1} + h_i t}{2}), \quad v_r(x) = \hat{v}_r(\frac{x_i + x_{i-1} + h_i t}{2}).$$

Then (see [9, p. 27])

$$|(v - v_r)(l_{i,j})| \lesssim h^{r+2} \|v\|_{r+2,\infty,\mathcal{P}}, \quad j \in \mathbb{Z}_{r-1}, \tag{4.9}$$

and

$$|(v - v_r)'(g_{i,j})| \lesssim h^{r+1} \|v\|_{r+2,\infty,\mathcal{P}}, \quad j \in \mathbb{Z}_r. \tag{4.10}$$

For a point $x \in [a, b]$, let $G_{\mathcal{P}} \in U^r_{\mathcal{P}}$ be the Galerkin approximation of $G(x, \cdot)$, that is

$$A(v_{\mathcal{P}}, G_{\mathcal{P}}) = A(v_{\mathcal{P}}, G(x, \cdot)), \quad \forall v_{\mathcal{P}} \in U^r_{\mathcal{P}}.$$

There hold (see[9, p. 33])

$$\|G_{\mathcal{P}}\|_{2,1,\mathcal{P}} = \sum_{i=1}^{N} \|G_{\mathcal{P}}\|_{2,1,\tau_i} \lesssim 1 \tag{4.11}$$

and

$$A(u - u_r, G_{\mathcal{P}}) \lesssim h^{r+2} \|u\|_{r+2,\infty,\mathcal{P}} \|G_{\mathcal{P}}\|_{2,1,\mathcal{P}} \lesssim h^{r+2} \|u\|_{r+2,\infty,\mathcal{P}}, \tag{4.12}$$

where $u_r \in U^r_{\mathcal{P}}$ is the projector of u and the hidden constants are independent of the partition $\mathcal{P}$.

With all the preparations, we are ready to present superconvergence properties at Gauss and interior Lobatto points.

Theorem 4.5 *If the conditions of Theorem 4.4 hold, then*

$$|(u-u_{\mathcal{P}})'(g_{i,j})| \lesssim h^{r+1}\|u\|_{2r+1,\infty,\mathcal{P}}, \quad (i,j)\in \mathbb{Z}_N\times\mathbb{Z}_r, \tag{4.13}$$

and

$$|(u-u_{\mathcal{P}})(l_{i,j})| \lesssim h^{r+2}\|u\|_{2r+1,\infty,\mathcal{P}}, \quad (i,j)\in \mathbb{Z}_N\times\mathbb{Z}_{r-1}. \tag{4.14}$$

Proof For any $x\in\Omega$, by the definition of $G(\cdot,\cdot)$ and $G_{\mathcal{P}}$, we have

$$u_r(x)-u_{\mathcal{P}}(x) = A(u_r-u_{\mathcal{P}}, G(x,\cdot)) = A(u-u_{\mathcal{P}}, G_{\mathcal{P}}) + A(u_r-u, G_{\mathcal{P}}).$$

We next estimate $A(u-u_{\mathcal{P}}, G_{\mathcal{P}})$. We have

$$A(u-u_{\mathcal{P}}, G_{\mathcal{P}}) = \int_a^b \big(\alpha(y)e'(y)-\epsilon(y)\big)G'_{\mathcal{P}}(y)dy,$$

where $e(y)$ and $\epsilon(y)$ are the same as in Theorem 4.4.

Let

$$E_3 = \int_a^b \big(\alpha(y)e'(y)-\epsilon(y)\big)G'_{\mathcal{P}}(y)dy - \sum_{k=1}^N\sum_{j=1}^r A_{k,j}\big(\alpha(g_{k,j})e'(g_{k,j})-\epsilon(g_{k,j})\big)G'_{\mathcal{P}}(g_{k,j}).$$

Then

$$\begin{aligned} A(u-u_{\mathcal{P}}, G_{\mathcal{P}}) &= \sum_{k=1}^N\sum_{j=1}^r A_{k,j}\big(\alpha(g_{k,j})e'(g_{k,j})-\epsilon(g_{k,j})\big)G'_{\mathcal{P}}(g_{k,j}) + E_3 \\ &= a_{\mathcal{P}}(e, \Pi_{\mathcal{P}}G_{\mathcal{P}}) + E_3 = E_3. \end{aligned}$$

Since E_3 is the residual between the exact integral and the Gauss quadrature, there exists a $\xi_k\in\tau_k$ such that

$$E_3 = \sum_{k=1}^N \frac{h_k^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}\Big[\big(\alpha(y)e'(y)-\epsilon(y)\big)G'_{\mathcal{P}}(y)\Big]^{(2r)}\bigg|_{y=\xi_k}.$$

Note that $G_{\mathcal{P}}\in U^r_{\mathcal{P}}$, we have $G^{(j)}_{\mathcal{P}}(\xi_k)=0$ if $j\ge r+1$. Then

$$\begin{aligned} &\Big|\Big[\big(\alpha(y)e'(y)-\epsilon(y)\big)G'_{\mathcal{P}}(y)\Big]^{(2r)}(\xi_k)\Big| \\ &\lesssim \sum_{j=r+1}^{2r}\binom{2r}{j}\|\alpha e'-\epsilon\|_{j,\infty,\tau_k}\left|G'_{\mathcal{P}}\right|_{2r-j,\infty,\tau_k} \\ &\lesssim \left(\sum_{j=1}^{2r+1}|e|_{j,\infty,\tau_k}\right)\sum_{j=r+1}^{2r}\binom{2r}{j}|G_{\mathcal{P}}|_{2r+1-j,\infty,\tau_k}. \end{aligned}$$

By the same arguments in the proof of Theorem 4.4, we have

$$\sum_{j=1}^{2r}|e|_{j,\infty,\tau_k} \lesssim \sum_{j=r+1}^{2r+1}|u|_{j,\infty,\mathcal{P}}.$$

On the other hand, the fact that $G_{\mathcal{P}} \in U^r_{\mathcal{P}}$ yields the inverse inequalities

$$|G_{\mathcal{P}}|_{j,\infty,\tau_k} \lesssim h_k^{1-j} |G_{\mathcal{P}}|_{2,1,\tau_k}, \ \forall k \in \mathbb{Z}_N, \ j \in \mathbb{Z}_r,$$

and thus

$$\sum_{j=r+1}^{2r} \binom{2r}{j} |G_{\mathcal{P}}|_{2r+1-j,\infty,\tau_k} \lesssim h_k^{1-r} |G_{\mathcal{P}}|_{2,1,\tau_k}.$$

Then by (4.7) and (4.11), we have

$$|E_3| \lesssim \sum_{k=1}^{N} |G_{\mathcal{P}}|_{2,1,\tau_k} h_k^{r+2} \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\mathcal{P}} \lesssim h^{r+2} \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\mathcal{P}}.$$

Namely

$$\left| A(u - u_{\mathcal{P}}, G_{\mathcal{P}}) \right| \lesssim h^{r+2} \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\mathcal{P}}.$$

Combining this estimate with (4.12), we obtain

$$|(u_r - u_{\mathcal{P}})(x)| \lesssim h^{r+2} \|u\|_{2r+1,\infty,\mathcal{P}}. \tag{4.15}$$

By the inverse inequality,

$$|(u_r - u_{\mathcal{P}})'(x)| \lesssim h^{-1} \|u_r - u_{\mathcal{P}}\|_{\infty} \lesssim h^{r+1} \|u\|_{2r+1,\infty,\mathcal{P}}.$$

Choosing $x = g_{i,j}$, $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_r$ in the above inequality and noting (4.10), we obtain (4.13). Choosing $x = l_{i,j}$, $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{r-1}$ in (4.15), we obtain (4.14). □

As a direct consequence , we have

$$|u - u_{\mathcal{P}}|_{G,1} \lesssim h^{r+1}, \quad |u - u_{\mathcal{P}}|_{aver,1} \lesssim h^{r+1}, \tag{4.16}$$

and

$$|u - u_{\mathcal{P}}|_{L,0} \lesssim h^{r+2}, \quad |u - u_{\mathcal{P}}|_{aver,0} \lesssim h^{r+2}, \tag{4.17}$$

where

$$|v|_{\text{aver},1} = \left(\frac{1}{Nr} \sum_{i=1}^{N} \sum_{j=1}^{r} v'(g_{i,j})^2 \right)^{\frac{1}{2}}$$

and

$$|v|_{L,0} = \left(\sum_{i=1}^{N} \sum_{j=0}^{r} W_{i,j} v(l_{i,j})^2 \right)^{\frac{1}{2}}, \quad |v|_{\text{aver},0} = \left(\frac{1}{Nr} \sum_{i=1}^{N} \sum_{j=0}^{r} v(l_{i,j})^2 \right)^{\frac{1}{2}},$$

here $W'_{i,j}s$ are weights of the Lobatto quadrature.

Next we improve the estimate (4.13) for two special cases in which $\beta = 0$ and $\beta = \gamma = 0$. To this end, we need to estimate first $|e(x_i) - e(x_{i-1})|$, $i \in \mathbb{Z}_N$.

Lemma 4.6 *If the conditions of Theorem 4.4 hold, then for all $i \in \mathbb{Z}_N$,*

$$|e(x_i) - e(x_{i-1})| \lesssim h^{2r+1} \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\mathcal{P}}. \tag{4.18}$$

Proof By the same arguments as in Theorem 4.4, we obtain

$$e(x_i) - e(x_{i-1}) = \sum_{k=1}^{N} E'_{1,k},$$

where

$$E'_{1,k} = \frac{h_k^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3} \left[\left((\alpha(y)e'(y) - \epsilon(y)\right) \left(\frac{\partial G}{\partial y}(x_i, y) - \frac{\partial G}{\partial y}(x_{i-1}, y) \right) \right]^{(2r)} \Bigg|_{y=\xi_k}$$

with $\xi_k \in \tau_k$.

Recall the construction of the Green function $G(x_i, \cdot)$, there holds for all $j \in \mathbb{N}_{2r}$,

$$\left\| \frac{\partial G}{\partial y}(x_i, y) - \frac{\partial G}{\partial y}(x_{i-1}, y) \right\|_{j,\infty,\Omega\backslash\tau_i} \lesssim h\|G\|_{j+1,\infty,\Omega\backslash\tau_i},$$
$$\left\| \frac{\partial G}{\partial y}(x_i, y) - \frac{\partial G}{\partial y}(x_{i-1}, y) \right\|_{j,\infty,\tau_i} \lesssim h\|G\|_{j+1,\infty,\tau_i}.$$

Since the Green function $G(x_i, \cdot) \in C^{2r}(\tau_k)$, $k \in \mathbb{Z}_N$ is bounded, then

$$\left[\left((\alpha(y)e'(y) - \epsilon(y)\right) \left(\frac{\partial G}{\partial y}(x_i, y) - \frac{\partial G}{\partial y}(x_{i-1}, y) \right) \right]^{(2r)} \Bigg|_{y=\xi_k}$$
$$\lesssim \sum_{j=0}^{2r} \binom{2r}{j} \|\alpha e' - \epsilon\|_{j,\infty,\tau_k} \left\| \frac{\partial G}{\partial y}(x_i, y) - \frac{\partial G}{\partial y}(x_{i-1}, y) \right\|_{2r-j,\infty,\tau_k}$$
$$\lesssim h_k \left(\sum_{j=0}^{r} |e|_{j,\infty,\tau_k} + \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\tau_k} \right).$$

Following the same estimate for $\sum_{j=0}^{r} |e|_{j,\infty,\tau_k} + \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\tau_k}$ as in Theorem 4.4, we obtain

$$|E'_{1,k}| \lesssim h_k^{2r+2} \sum_{j=r+1}^{2r+1} |u|_{j,\infty,\mathcal{P}},$$

from which the inequality (4.18) follows. □

Theorem 4.7 *Let the conditions of Theorem 4.4 be satisfied. If $\beta(x) = 0, \forall x \in \Omega$, then for all $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_r$,*

$$|(u - u_{\mathcal{P}})'(g_{i,j})| \lesssim h^{\min\{r+2,2r\}} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}. \tag{4.19}$$

If $\beta(x) = \gamma(x) = 0, \forall x \in \Omega$, then for all $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_r$,

$$|u'(g_{i,j}) - u'_{\mathcal{P}}(g_{i,j})| \lesssim h^{2r} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}. \tag{4.20}$$

Proof On one hand, both u and $u_{\mathcal{P}}$ satisfy (2.2), there holds for all $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_r$ that

$$\alpha(g_{i,j})e'(g_{i,j}) - \alpha(g_{i,1})e'(g_{i,1}) = \int_{g_{i,1}}^{g_{i,j}} \gamma(x)e(x)dx. \tag{4.21}$$

On the other hand,

$$e(x_i) - e(x_{i-1}) = \int_{x_{i-1}}^{x_i} e'(y)dy = \sum_{j=1}^{r} A_{i,j}e'(g_{i,j}) + \tilde{E}_i,$$

where by [14, p. 98 (2.7.12)],

$$|\tilde{E}_i| = \left| \frac{h_i^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}(e')^{(2r)}(\xi_i) \right| \lesssim h^{2r+1}|u|_{2r+1,\infty,\tau_i}, \quad \xi_i \in \tau_i.$$

By the estimate (4.18) in Theorem 4.6, we have

$$\left| \sum_{j=1}^{r} A_{i,j}e'(g_{i,j}) \right| \lesssim h_i^{2r+1} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}. \tag{4.22}$$

Substituting (4.21) into (4.22), we obtain

$$\left| \sum_{j=1}^{r} A_{i,j}\alpha^{-1}(g_{i,j})(\alpha(g_{i,1})e'(g_{i,1}) + \int_{g_{i,1}}^{g_{i,j}} \gamma(x)e(x)dx) \right| \lesssim h_i^{2r+1} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}.$$

Note that $\sum_{j=1}^{r} A_{i,j} \sim h_i$, then

$$|e'(g_{i,1})| \lesssim \int_{x_i}^{x_{i+1}} |\gamma(x)e(x)|\,dx + h_i^{2r} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}.$$

Moreover,

$$\int_{x_{i-1}}^{x_i} |\gamma(x)e(x)|\,dx \lesssim h_i\|e\|_{0,\infty,\mathcal{P}} \lesssim h^{r+2}\left(|u|_{r+2,\mathcal{P}} + |u|_{r+1,\infty,\mathcal{P}}\right). \tag{4.23}$$

Therefore,

$$|e'(g_{i,1})| \lesssim h^{\min\{r+2,2r\}} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}.$$

Then (4.19) follows from the above inequality, (4.21) and (4.23).

Obviously, if $\gamma = 0$, we have

$$|e'(g_{i,1})| \lesssim h_i^{2r} \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}.$$

The inequality (4.20) is a direct consequence of the above inequality and (4.21). □

Remark 4.8 We see that at the Gauss points, when $\beta = \gamma = 0$, the derivative convergence rate h^{2r} doubles the global optimal rate h^r, which is much better than the counterpart finite element method's h^{r+1} rate, when $\beta = 0$, the derivative convergence rate h^{r+2} is one order higher than the counterpart finite element method's h^{r+1}; and at the nodal points, the convergence rate h^{2r} almost doubles the global optimal rate h^{r+1} and equals to the counterpart finite element method's h^{2r} rate; and at the Lobatto points, the convergence rate h^{r+2} is one order higher than the optimal global rate h^{r+1}, which is the same as the counterpart finite element method.

5 Post Processing

We observe from (4.13), (4.19) and (4.20) that $u'_{\mathcal{P}}$ approximates the derivative of the exact solution u pretty well at the Gauss points. In this subsection, we will recover u' in the whole domain Ω.

For all $i = 1, \dots, N-1$, we construct a function $v_i \in \mathbb{P}_{2r-1}([x_{i-1}, x_{i+1}])$ by letting

$$v_i(g_{l,k}) = u'_{\mathcal{P}}(g_{l,k}), \quad l = i, i+1;\ k = 1, 2, \dots, r.$$

Then we define for all $x \in \tau_i = [x_{i-1}, x_i]$, $i = 1, \dots, N$,

$$v(x) = \begin{cases} v_1(x), & i = 1, \\ \frac{1}{2}\big(v_i(x) + v_{i-1}(x)\big), & 2 \le i \le N-1, \\ v_{N-1}(x), & i = N. \end{cases}$$

To study the approximation property of u, we note that in each $[x_{i-1}, x_{i+1}]$,

$$u'(x) = (L_{2r-1}u')(x) + \frac{u^{(2r+1)}(\xi)}{(2r)!}\prod_{j=1}^{r}(x - g_{i,j})(x - g_{i+1,j}), \xi \in [x_{i-1}, x_{i+1}]$$

where the Lagrange interpolant

$$(L_{2r-1}u')(x) = \sum_{l=i}^{i+1}\sum_{j=1}^{r} u'(g_{l,j})w_{l,j}(x), w_{l,j}(x) = \prod_{l' \ne l, j' \ne j} \frac{x - g_{l',j'}}{g_{l,j} - g_{l',j'}}.$$

Noting that

$$v_i(x) = \sum_{l=i}^{i+1}\sum_{j=1}^{r} u'_{\mathcal{P}}(g_{l,j})w_{l,j}(x),$$

we have

$$u'(x) - v_i(x) = \sum_{l=i}^{i+1}\sum_{j=1}^{r}(u' - u'_{\mathcal{P}})(g_{l,j})w_{l,j}(x) + \frac{u^{(2r+1)}(\xi)}{(2r)!}\prod_{j=1}^{r}(x - g_{i,j})(x - g_{i+1,j}). \tag{5.1}$$

Since for all $l = i, i+1$, $j = 1, \dots, r$, we have

$$|w_{l,j}(x)| \le c_r, \forall x \in [x_{i-1}, x_{i+1}],$$

Table 1 $r = 4$

N	$\|u - u_{\mathcal{P}}\|_0$	$\|u - u_{\mathcal{P}}\|_1$	$\|u_I - u_{\mathcal{P}}\|_1$	$\|u - u_{\mathcal{P}}\|_{L,0}$
2	1.8618e−03	5.1201e−02	5.2554e−03	3.3420e−04
4	1.4386e−04	7.2801e−03	3.1271e−04	9.8931e−06
8	5.9282e−06	5.9099e−04	1.1758e−05	1.8624e−07
16	1.9882e−07	3.9516e−05	3.8485e−07	3.0490e−09
32	6.3240e−09	2.5119e−06	1.2166e−08	4.8197e−11
64	1.9850e−10	1.5766e−07	3.8129e−10	7.5536e−13
N	$\|u - u_{\mathcal{P}}\|_{aver,0}$	$\|u - u_{\mathcal{P}}\|_{G,1}$	$\|u - u_{\mathcal{P}}\|_{aver,1}$	E_{node}
2	2.1895e−04	8.0770e−04	5.4962e−04	1.1874e−05
4	6.2680e−06	5.3025e−05	3.5877e−05	5.9186e−08
8	1.1716e−07	1.9692e−06	1.3338e−06	2.3666e−10
16	1.9150e−09	6.3947e−08	4.3328e−08	9.2827e−13
32	3.0260e−11	2.0170e−09	1.3667e−09	–
64	4.7425e−13	6.3175e−11	4.2809e−11	–

where c_r is a constant depends only on r, we obtain by (4.13), (4.19) and (4.20) that

$$|u'(x) - v_i(x)| \lesssim h^m \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}},$$

where $m = r + 1$ for general elliptic equations, $m = r + 2$ if $\beta = 0$, and $m = 2r$ if $\beta = \gamma = 0$. Consequently, we have

$$|u'(x) - v(x)| \lesssim h^m \sum_{k=r+1}^{2r+1} |u|_{k,\infty,\mathcal{P}}, \forall x \in \Omega.$$

6 Numerical Experiments

In this section, we present numerical examples to demonstrate the method and to verify the theoretical results proved in this paper.

In our experiments, we solve the two-point boundary value problem (2.1) by the FVM scheme (2.4) with $r = 4$ or $r = 5$. The underlying meshes are obtained by subdividing $\Omega = (0, 1)$ to $N = 2, 4, 8, 16, 32, 64$ subintervals with equal sizes.

Example 1 We consider the two-point boundary value problem (2.1) with

$$\alpha(x) = e^x, \quad \beta(x) = \cos x, \quad \gamma(x) = x, \quad \forall x \in \Omega,$$

and f is chosen so that the exact solution of this problem is

$$u(x) = \sin x(x^{12} - x^{11}).$$

We list approximate errors under various (semi-)norms in Table 1 (for the scheme $r = 4$) and Table 2 (for the scheme $r = 5$).

To explicitly show the convergence rate of different approximate errors, we plot the error curves in Figs. 1 and 2. We observe from Fig. 1 that the convergence rate $|u - u_{\mathcal{P}}|_1$ is r and

Table 2 $r = 5$

N	$\|u - u_{\mathcal{P}}\|_0$	$\|u - u_{\mathcal{P}}\|_1$	$\|u_I - u_{\mathcal{P}}\|_1$	$\|u - u_p\|_{L,0}$
2	4.8206e−04	1.5546e−02	8.5017e−04	4.0891e−05
4	1.5627e−05	9.6503e−04	2.1627e−05	5.2643e−07
8	2.9713e−07	3.6434e−05	3.8065e−07	4.6413e−09
16	4.8711e−09	1.1927e−06	6.1190e−09	3.7318e−11
32	7.7022e−11	3.7707e−08	9.6282e−11	2.9365e−13
64	1.2073e−12	1.1817e−09	1.5081e−12	–
N	$\|u - u_{\mathcal{P}}\|_{aver,0}$	$\|u - u_{\mathcal{P}}\|_{G,1}$	$\|u - u_{\mathcal{P}}\|_{aver,1}$	E_{node}
2	2.6075e−05	2.2179e−04	1.4819e−04	4.6819e−08
4	3.2965e−07	5.8493e−06	3.9162e−06	3.0508e−11
8	2.8971e−09	1.0085e−07	6.7553e−08	2.6318e−14
16	2.3277e−11	1.6089e−09	1.0779e−09	–
32	1.8311e−13	2.5266e−11	1.6928e−11	–
64	–	3.9473e−13	2.6482e−13	–

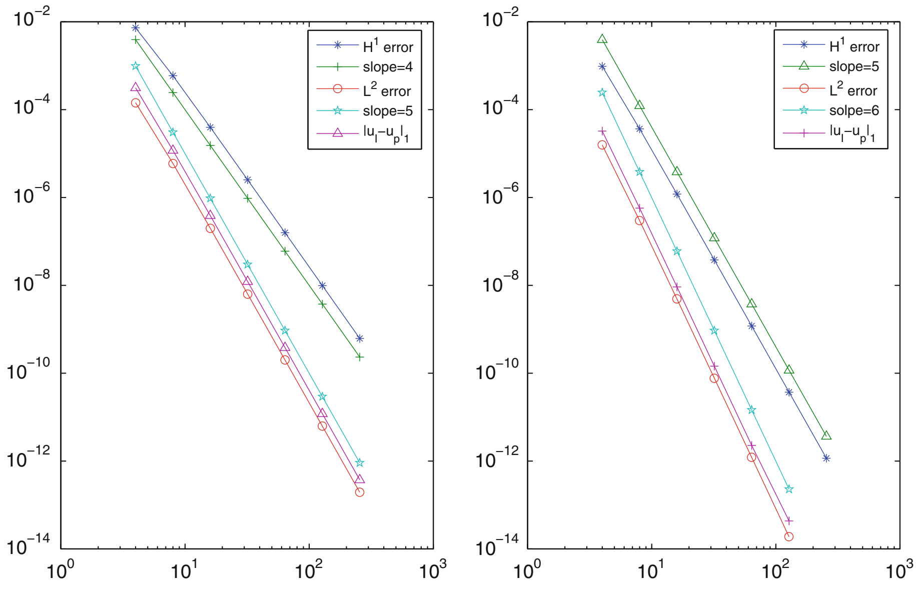

Fig. 1 left: $r = 4$, right: $r = 5$

the convergence rate of $\|u - u_{\mathcal{P}}\|_0$ is $r + 1$. In other words, the FVM approximate solution converges to the exact solution with optimal convergence rates under both for H^1 and L^2 norms, as predicted in (3.9) and (4.4). We also observe that the error $|u_I - u_{\mathcal{P}}|_1$ is of order $r+1$, which confirms the convergence result in (4.3). The errors $|u - u_{\mathcal{P}}|_{aver,0}$, $|u - u_{\mathcal{P}}|_{L,0}$ and E_{node} are presented in Fig. 2. It is observed that $|u - u_{\mathcal{P}}|_{aver,0}$ and $|u - u_{\mathcal{P}}|_{L,0}$ converge with a degree $r + 2$ which confirm the superconvergence property at Labatto points given in Theorem 4.5. Since E_{node} converges with a rate $2r$, it confirms our theory in Theorem 4.4.

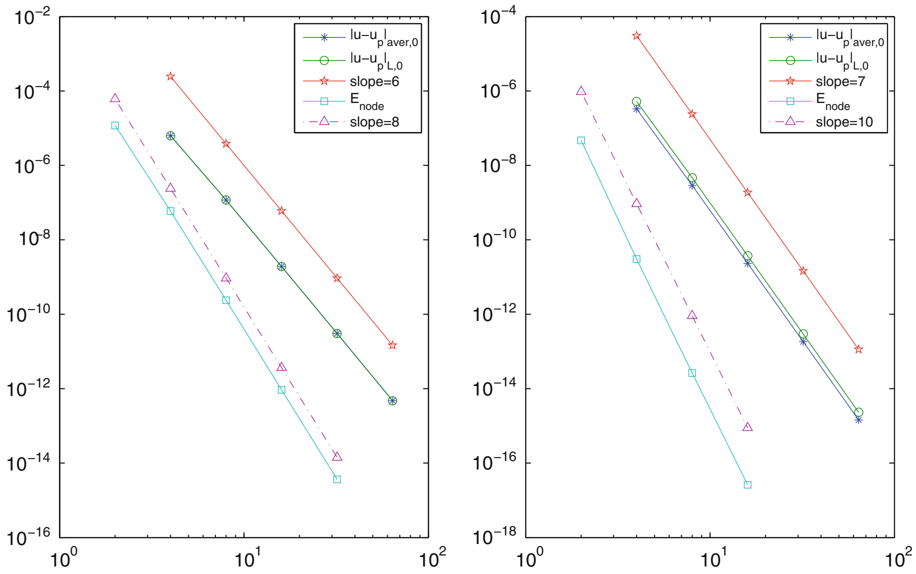

Fig. 2 left: $r=4$, right: $r=5$

Table 3 Gauss points

N	$r=4$			$r=5$		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
1	4.7633e−03	4.1098e−03	4.1493e−03	1.5667e−03	1.1105e−04	1.0183e−04
2	5.4962e−04	3.1457e−05	2.9493e−05	2.6075e−05	1.9562e−06	8.6701e−09
4	3.5877e−05	4.2964e−07	1.1316e−07	3.9162e−06	2.8751e−08	3.1732e−11
8	1.3338e−06	8.4296e−09	4.3677e−10	6.7553e−08	2.6812e−10	3.6386e−14
16	4.3328e−08	1.4052e−10	1.7002e−12	1.0779e−09	2.1878e−12	–
32	1.3667e−09	2.2319e−12	6.6291e−15	1.6928e−11	1.7284e−14	–

Example 2 In this example, we test the convergence behavior of derivative error at Gauss points. We consider three cases of Eq. (2.1), they are

$$\begin{aligned}
&\text{Case 1 }: \alpha(x)=e^x,\ \beta(x)=\cos x,\ \gamma(x)=x;\\
&\text{Case 2 }: \alpha(x)=e^x,\ \beta(x)=0,\ \gamma(x)=x;\\
&\text{Case 3 }: \alpha(x)=e^x,\ \beta(x)=0,\ \gamma(x)=0.
\end{aligned}$$

The exact solution is always $u(x)=\sin x(x^{12}-x^{11})$ and the right-hand function f changes according to the coefficients in different cases.

Listed in Table 3 are errors in the derivative approximation at Gauss points for three different cases for $r=4$ and $r=5$, respectively. Plotted in Fig. 3 are corresponding error curves. We observe that the convergence rate is $r+1$ for Case 1, $r+2$ for Case 2 and $2r$ for Case 3. These numerical results are consistent with our theories derived in Sect. 4.

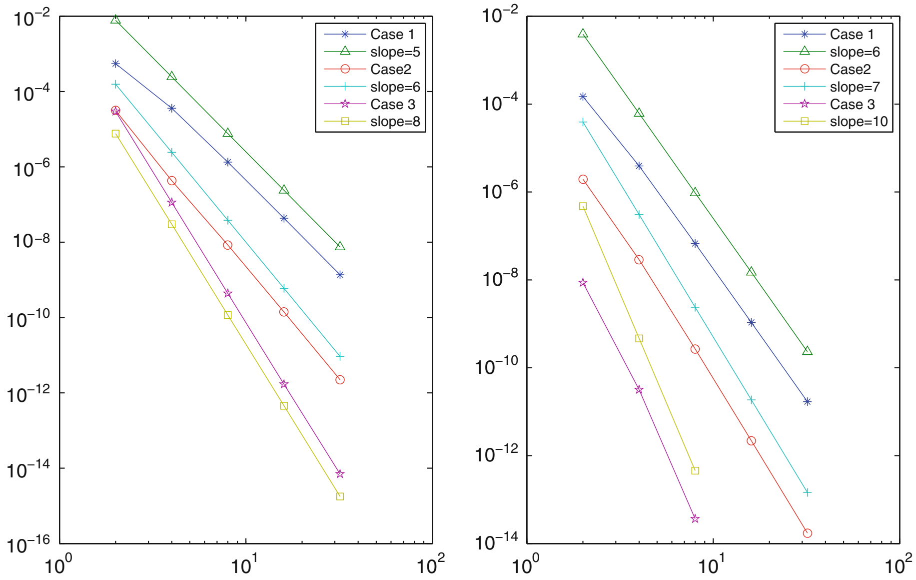

Fig. 3 left: $r = 4$, right: $r = 5$

7 Concluding Remarks

In this article, we provide a unified proof of the inf-sup condition for a family of any order FVM schemes in the one dimensional setting. Based on this, we show that the FVM solution converges to the exact solution with optimal order, both in the H^1 and L^2 norms. This unified approach is different from that of the standard argument in the FVM literature in that we formulate our FVM as the Petrov–Galerkin finite element method and introduce an unconventional projection $\Pi_{\mathcal{P}}$ from the trial space to the test space and thereby to make a unified argument possible. Technically, we have been used the property of the Gauss-quadrature repeatedly to establish the best possible error bounds.

We also studied the superconvergence properties of our FVM schemes. It is shown both theoretically and numerically that at the nodal and interior Lobatto points, the superconvergence behavior of FVM is similar to that of the counterpart finite element method. Moreover, in some special cases, the superconvergence property of the derivative of the FVM solution at the Gauss points maybe much better than that of the counterpart finite element method. For instances, when $\beta = 0$, the convergence rate of the derivative of the FVM solution is h^{r+2} which is one order higher than the counterpart finite element method's h^{r+1}; when $\beta = \gamma = 0$, the order is h^{2r} which doubles the global optimal rate h^r, and it is much better than the counterpart finite element method's h^{r+1} rate. It is not strange that the FVM solution may have higher derivative convergence rate than that of FEM since from the design principle, FEM is based on energy minimization and FVM is based on conservation of flux, which is more suitable for the Sturm-Liouville system, especially, when $\beta = \gamma = 0$. In this special case, the numerical flux approximation is exact at the Gauss points when the solution is a polynomial of degree no more than $2N$, which is based on the fact that the Gauss-quadrature is exact for polynomials of degree up to $2N - 1$. Therefore, selecting the dual mesh with grid points at the Gauss points is crucial for this design.

Finally, we would like to mention that the current paper is our first attempt on the *more in depth* mathematical investigation of the FVM. The essential idea behind our analysis for one dimensional FV schemes can be applied to analyze higher dimensional FV schemes. For examples, we have studied any order FV schemes on rectangle meshes in [29] and any order FV schemes on arbitrary quadrilateral meshes in [30]. For rectangle meshes, we proved the optimal convergence rates of FV solutions and the supercloseness between the FV solution and a suitable interpolation of the exact solution. For arbitrary quadrilateral meshes, we show the optimal convergence rates of any order FV solutions. Note that the results for arbitrary quadrilateral meshes are better than previous FVM analysis in the literature in two aspects : (1) the previous analysis in the literature are often done case-by-case for low-order schemes, our analysis are uniformly done for any order schemes. (2) the mesh conditions required in our analysis are much weaker than the sufficient but might not be necessary mesh conditions in the literature, instead, they are very much similar to the most relaxed mesh condition in finite element methods. Apparently the techniques for higher dimensional any order FV schemes are more complex than that for one dimensional case. For instance, we have not successfully provided a unified proof for any order FV schemes on triangular mesh yet. In summary, more works are called for to find a better understanding of the FVM in a mathematical view.

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