DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **18**, Number **3**, May **2013**

doi:10.3934/dcdsb.2013.18.797

pp. **797–819**

THE LONG TIME BEHAVIOR OF A SPECTRAL COLLOCATION METHOD FOR DELAY DIFFERENTIAL EQUATIONS OF PANTOGRAPH TYPE

Jie Tang

College of Science, Hunan University of Technology,
Zhuzhou, Hunan 412007, China

Ziqing Xie

Key Laboratory of High Performance Computing
and Stochastic Information Processing,
College of Mathematics and Computer Science
Hunan Normal University, Changsha, Hunan 410081, China

Zhimin Zhang

Department of Mathematics
Wayne State University Detroit, MI 48202, USA
and No.3 HeQing Road, Haidian District
Beijing Computational Science Research Center
Beijing, 100084, China

(Communicated by Jie Shen)

Abstract. In this paper, we propose an efficient numerical method for delay differential equations with vanishing proportional delay qt $(0 < q < 1)$. The algorithm is a mixture of the Legendre-Gauss collocation method and domain decomposition. It has global convergence and spectral accuracy provided that the data in the given pantograph delay differential equation are sufficiently smooth. Numerical results demonstrate the spectral accuracy of this approach and coincide well with theoretical analysis.

1. **Introduction.** We consider the delay differential equation (DDE):

$$\begin{aligned} y'(x) &= a(x)y(x) + b(x)y(qx) + f(x), \qquad x \in J := [0,T], \\ y(0) &= y_0, \end{aligned} \tag{1}$$

where $q \in (0,1)$ is a given constant and $a,\ b,\ f$ are smooth functions on $[0,T]$. Eq. (1) belongs to a class of so-called pantograph delay differential equations, which arise in diverse scientific and engineering applications (see [10], [14]-[16] for details). Considerable progress has been made in this field recently. However, many questions (especially regarding the asymptotic behavior of numerical solutions) remain yet to be answered. The monographs by Bellen and Zennaro [4] and Brunner [6], as well as the survey paper [7] convey good pictures on the present state-of-the-art in the numerical analysis of DDEs (and related more general functional integral and integro-differential equations) with vanishing delay functions.

2010 *Mathematics Subject Classification.* 34K28, 65L05, 65Q10, 41A10.

Key words and phrases. Pantograph delay differential equations, spectral collocation method, exponential convergence, domain decomposition, vanishing proportional delay.

The existing numerical methods for solving (1) include Runge-Kutta type methods (see, e.g., the monograph [4]), collocation methods (cf. [3, 5, 6, 9]) and discontinuous Galerkin methods [8]. It was shown in [8] that despite the underlying Galerkin structure, the DG time-stepping method is related to a certain implicit scheme of the Runge-Kutta type. The main difficulty in Runge-Kutta methods to (1) is the lack of information at the grid points for the function on the second term of the right-hand side of (1); these numerical data have to be generated by some local interpolation process. While collocation methods yield globally defined approximations, the collocation solutions are not globally smooth. Moreover, it has been shown in [9] that for arbitrarily smooth solutions of (1) the optimal order at the grid points of collocation methods using piecewise polynomials of degree m cannot exceed $p = m+2$ when $m \geq 2$ (in contrast to their application to ordinary differential equations where collocation at the Gauss points leads to $\mathcal{O}(h^{2m})$-convergence).

Over the years, the spectral method has become increasingly popular and been widely used in spatial discretizations of PDEs owing to its high accuracy. The solution of a DDE globally depends on its history due to the delay variable, so a global spectral method could be a good candidate for the numerical approximation of DDEs. Some work has been done along this line, e.g., T. Tang, X. Xu and J. Cheng [17] proposed and analyzed a spectral collocation method for the Volterra integral equations of the second kind, and their idea was extended to pantograph DDEs with a single delay and multiple delays in [1] and [2], respectively. X. Tao, Z. Q. Xie and X. J. Zhou [18] provided general spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches for the second kind Volterra integro-differential equations. All above methods are designed for one time interval, which is relatively small. From a practical point of view, it is more interesting and challenging to develop and analyze high-order methods for pantograph-type DDEs in a long time period.

At the same time, Guo and Wang [12], and Guo and Yan [13] developed Legendre-Gauss collocation methods for ODEs. More recently, Z. Q. Wang and L. L. Wang [19] introduced an efficient Legendre-Gauss collocation method for nonlinear DDEs. It should be noted that the authors in [19] had investigated Legendre-Gauss collocation method for nonlinear DDEs. they analyzed the convergence of the single-step and multi-domain versions of the proposed method, and showed that the scheme enjoys high order accuracy and can be implemented in a stable and efficient manner. Our technique and ideas are similar to but different from those of [19]. This paper, motivated by [12], is devoted to construct a novel ‘time stepping’ scheme based on uniform meshes for large T. It is desirable to partition the time interval $[0, T]$ and use moderate mode N to construct numerical solution on each subinterval, step by step. This process has not only the spectral accuracy, but also the global convergence. Therefore, it has some remarkable advantages over Runge-Kuta type methods and is more appropriate to predict long time behaviors of dynamical systems.

The outline of the paper is as follows: in section 2, we introduce the spectral collocation method for (1) and discuss the existence and uniqueness of the spectral collocation solution. The convergence result and analysis are stated and proved in section 3. Numerical results in section 4 validate the theoretical prediction in section 3.

2. **Legendre-Gauss collocation method with domain decomposition.**

In this section, we will introduce the spectral collocation method for the above pantograph-type DDE with vanishing proportional delay and describe the corresponding computational ('time-stepping') scheme.

2.1. **Preliminaries.** In this subsection, we derive the numerical approach by using the Legendre-Gauss interpolation. Let $L_l(t)$ be the Legendre polynomial of degree l defined on $[-1,1]$. The Legendre polynomials satisfy the three-term recurrence relation

$$\begin{aligned} &L_0(t)=1, \qquad L_1(t)=t, \\ &(l+1)L_{l+1}(t)-(2l+1)tL_l(t)+lL_{l-1}(t)=0, \qquad l\geq 1. \end{aligned} \tag{2}$$

Moreover, there holds the recursive relation

$$\frac{d}{dt}L_{l+1}(t)-\frac{d}{dt}L_{l-1}(t)=(2l+1)L_l(t), \qquad l\geq 1. \tag{3}$$

The set of $L_l(t)$ forms a complete $L^2(-1,1)$-orthogonal system, namely,

$$\int_{-1}^{1} L_l(t)L_m(t)dt=\frac{2}{2l+1}\delta_{l,m}, \tag{4}$$

where $\delta_{l,m}$ is the Kronecker symbol. Thus for any $v\in L^2(-1,1)$, we write

$$v(t)=\sum_{l=0}^{\infty}\widehat{v}_l L_l(t), \qquad \widehat{v}_l=\frac{2l+1}{2}\int_{-1}^{1}v(t)L_l(t)dt. \tag{5}$$

We now turn to the Legendre-Gauss interpolation. Let $\{t_j,\omega_j\}_{j=0}^N$ be the Legendre-Gauss quadrature nodes (in $(-1,1)$ and arranged in ascending order) and weights, and $\mathcal{P}_N$, the space of (real) polynomials of degree not exceeding $N(\geq 0)$. Then, we recall that for any $\phi\in\mathcal{P}_{2N+1}(-1,1)$,

$$\int_{-1}^{1}\phi(t)dt=\sum_{j=0}^{N}\phi(t_j)\omega_j. \tag{6}$$

Let (u,v) and $\|v\|$ be the inner product and norm in $L^2(-1,1)$, respectively. We also introduce the following discrete inner product and norm,

$$(u,v)_N=\sum_{j=0}^{N}u(t_j)v(t_j)\omega_j, \quad \|v\|_N=(v,v)_N^{\frac{1}{2}}.$$

Thanks to (6), for any $\phi\psi\in\mathcal{P}_{2N+1}(-1,1)$ and $\varphi\in\mathcal{P}_N(-1,1)$,

$$(\phi,\psi)=(\phi,\psi)_N, \quad \|\varphi\|=\|\varphi\|_N. \tag{7}$$

For any $v\in C(-1,1)$, the Legendre-Gauss interpolation $I_Nv(t)\in\mathcal{P}_N(-1,1)$ is determined by

$$I_Nv(t_j)=v(t_j), \quad 0\leq j\leq N.$$

In view of (7), we have that for any $\phi\in\mathcal{P}_{N+1}(-1,1)$,

$$(I_Nv,\phi)=(I_Nv,\phi)_N=(v,\phi)_N. \tag{8}$$

$I_Nv(t)$ can be expanded as

$$I_Nv(t)=\sum_{l=0}^{N}\widetilde{v}_l L_l(t), \tag{9}$$

where by (5) and (8),

$$\widetilde{v}_l = \frac{2l+1}{2}(I_N v, L_l) = \frac{2l+1}{2}(v, L_l)_N, \quad 0 \le l \le N. \tag{10}$$

Furthermore, one verifies from (5), (7) and (10) that for any $\psi \in \mathcal{P}_{N+1}(-1,1)$,

$$\psi(t) = \sum_{l=0}^{N+1} \widehat{\psi}_l L_l(t), \quad I_N\psi(t) = \sum_{l=0}^{N} \widetilde{\psi}_l L_l(t) \implies \widetilde{\psi}_l = \widehat{\psi}_l, \quad 0 \le l \le N,$$

which implies that (cf. [12])

$$\|\psi\|_N \le \|\psi\|, \quad \forall \psi \in \mathcal{P}_{N+1}(-1,1). \tag{11}$$

Let r be a nonnegative integer, $H^r(-1,1)$ be the usual Sobolev space, and denote its corresponding norm and semi-norm by $\|\cdot\|_r$ and $|\cdot|_r$, respectively. There hold the following estimates (see, e.g., formulas (5. 4. 33) and (5. 4. 34) of [11]).

Lemma 2.1. *Assume that $u \in H^r(-1,1)$ and denote $I_N u$ its interpolation polynomial associated with the $(N+1)$-points Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{t_j\}_{j=0}^N$, namely, $I_N u = \sum_{j=0}^{N} u(t_j)h_j(t)$, where h_j is the j-th Lagrange basis function. Then the following estimates hold,*

$$\|u - I_N u\|_{L^2(-1,1)} \le cN^{-r}|u|_r, \tag{12}$$

$$\|u' - (I_N u)'\|_{L^2(-1,1)} \le cN^{\frac{3}{2}-r}|u|_r, \tag{13}$$

with integer $1 \le r \le N+1$.

The following lemma is necessary for our convergence analysis (see formula (3.9) of [12]).

Lemma 2.2. *For any $v \in H^1(-1,1)$, we have:*

$$\max_{t\in[-1,1]} |v(t)|^2 \le \|v\|^2 + 4\|\frac{dv}{dt}\|^2. \tag{14}$$

2.2. **The discrete scheme of the spectral collocation method.** Let $J_h := \{x_m | x_m = mh : m = 0, 1, \cdots, M \ (Mh = T)\}$ be a uniform mesh for the given interval $J = [0,T]$, and set $I_m = (x_{m-1}, x_m]$ $(m = 1, \cdots, M)$. In the spectral collocation method, we shall use $y_m^N(x) \in \mathcal{P}_{N+1}(I_m)$ to approximate the solution y in the subinterval $(x_{m-1}, x_m]$. To be more precise, we first apply a spectral collocation method on $(0,h]$ to get $y_1^N(x)$ with the initial condition $y_1^N(0) = y_0$. Then we use $y_1^N(h)$ as the new initial condition to obtain y_2^N on $(h, 2h]$, and so on. The global numerical solution of (1.1) is given by

$$y^N(x) = y_m^N(x), \quad x \in I_m, \quad 1 \le m \le M.$$

We now present the numerical scheme for y_m^N. The Legendre-Gauss points of interval $I_m = (x_{m-1}, x_m]$, are defined by

$$x_{m,i} = \frac{x_m + x_{m-1}}{2} + \frac{x_m - x_{m-1}}{2}t_i = \frac{h}{2}(t_i + 2m - 1) = x_{m-1} + c_i h, \tag{15}$$

where $x_{m,i}$ and $c_i := \frac{t_i+1}{2}$ are the collocation points and the collocation parameters of subinterval I_m, respectively. For simplicity, we denote the delay term $\theta(x_{m,i}) = qx_{m,i} \in I_\nu$, for some ν satisfying $1 \le \nu \le m$.

The spectral collocation method for (1) is to seek $y_m^N \in \mathcal{P}_{N+1}(I_m)$, such that

$$\begin{cases} \frac{d}{dx} y_m^N(x_{m,i}) = a(x_{m,i}) y_m^N(x_{m,i}) + b(x_{m,i}) y_\nu^N(\theta(x_{m,i})) \\ \qquad\qquad + f(x_{m,i}), & 0 \le i \le N, 1 \le m \le M, \\ y_m^N(x_{m-1}) = y_{m-1}^N(x_{m-1}), & 2 \le m \le M, \\ y_1^N(0) = y_0. \end{cases} \tag{16}$$

We transfer the interval $[x_{m-1}, x_m]$ into the reference element $I := [-1, 1]$. Hence, by (15), we can rewrite the equation (16) on I_m as follows:

$$\begin{cases} \frac{2}{h}\frac{d}{dt} y_m^N(\frac{h}{2}(t_i + 2m - 1)) = a(\frac{h}{2}(t_i + 2m - 1)) y_m^N(\frac{h}{2}(t_i + 2m - 1)) \\ \qquad\qquad + b(\frac{h}{2}(t_i + 2m - 1)) y_\nu^N(\frac{qh}{2}(t_i + 2m - 1)) \\ \qquad\qquad + f(\frac{h}{2}(t_i + 2m - 1)), \quad 0 \le i \le N, \ 1 \le m \le M, \\ y_m^N((m-1)h) = y_{m-1}^N((m-1)h), & 2 \le m \le M, \\ y_1^N(0) = y_0. \end{cases} \tag{17}$$

Set

$$u_m^N(t_i) = y_m^N(\frac{h}{2}(t_i + 2m - 1)); \qquad a_m(t_i) = a(\frac{h}{2}(t_i + 2m - 1));$$
$$b_m(t_i) = b(\frac{h}{2}(t_i + 2m - 1)); \qquad f_m(t_i) = f(\frac{h}{2}(t_i + 2m - 1)).$$

Inserting the above expression into (17), we obtain the discrete scheme as follows:

$$\begin{cases} \frac{d}{dt} u_m^N(t_i) = \frac{h}{2} a_m(t_i) u_m^N(t_i) + \frac{h}{2} b_m(t_i) w_\nu^N(t_i) \\ \qquad\qquad + \frac{h}{2} f_m(t_i), & 0 \le i \le N, \quad 1 \le m \le M, \\ u_m^N(-1) = u_{m-1}^N(1), & 2 \le m \le M, \\ u_1^N(-1) = y_0, \end{cases} \tag{18}$$

where $w_\nu^N(t) = u_\nu^N(qt + q_m^\nu)$ and $q_m^\nu = q(2m-1) - (2\nu - 1)$. The global numerical solution of (1) is given by

$$y^N(x) = y_m^N(x) = u_m^N(t), \quad x \in I_m, \ t \in I, \ 1 \le m \le M. \tag{19}$$

2.3. **The computational form of the spectral collocation equation.** We next describe the numerical implementation of the collocation scheme (18). For this purpose, we expand the collocation solution as

$$u_m^N(t) = \sum_{l=0}^{N+1} \widetilde{u}_{m,l} L_l(t) \in \mathcal{P}_{N+1}(-1, 1), \qquad -1 < t \le 1. \tag{20}$$

Clearly, $u_m^N(t)L_l(t) \in \mathcal{P}_{2N+1}(-1,1)$, for $0 \le l \le N$. Using (4) and (7), we have

$$\begin{aligned}\widetilde{u}_{m,l} &= \frac{2l+1}{2}(u_m^N, L_l) = \frac{2l+1}{2}(u_m^N, L_l)_N \\ &= \frac{2l+1}{2}\sum_{j=0}^{N} u_m^N(t_j)L_l(t_j)\omega_j, \qquad 0 \le l \le N,\ 1 \le m \le M.\end{aligned} \tag{21}$$

On the other hand, $L_l(-1) = (-1)^l$. Therefore, we obtain from (20) and (21) that

$$\begin{aligned}\widetilde{u}_{m,N+1} &= (-1)^{N+1}u_m^N(-1) + \sum_{l=0}^{N}(-1)^{N+l}\widetilde{u}_{m,l} \\ &= (-1)^{N+1}u_m^N(-1) + \frac{1}{2}\sum_{l=0}^{N}\sum_{j=0}^{N}(-1)^{N+l}(2l+1)u_m^N(t_j)L_l(t_j)\omega_j, \\ &\quad 1 \le m \le M.\end{aligned} \tag{22}$$

Furthermore, let $[l]$ be the integer part of l. According to [12], we have

$$\frac{d}{dt}L_l(t) = \sum_{k=0}^{[\frac{l-1}{2}]}(2l-4k-1)L_{l-2k-1}(t).$$

By (20)-(22), we have

$$\begin{aligned}\frac{d}{dt}u_m^N(t) &= \sum_{l=1}^{N+1}\widetilde{u}_{m,l}\frac{d}{dt}L_l(t) = \sum_{l=1}^{N+1}\widetilde{u}_{m,l}\sum_{k=0}^{[\frac{l-1}{2}]}(2l-4k-1)L_{l-2k-1}(t) \\ &= \sum_{j=0}^{N}\left[\frac{1}{2}\omega_j\sum_{l=1}^{N}(2l+1)L_l(t_j)\left(\sum_{k=0}^{[\frac{l-1}{2}]}(2l-4k-1)L_{l-2k-1}(t)\right)\right. \\ &\quad \left. + \frac{1}{2}\omega_j\sum_{l=0}^{N}(-1)^{N+l}(2l+1)L_l(t_j)\left(\sum_{k=0}^{[\frac{N}{2}]}(2N-4k+1)L_{N-2k}(t)\right)\right]u_m^N(t_j) \\ &\quad + (-1)^{N+1}u_m^N(-1)\sum_{k=0}^{[\frac{N}{2}]}(2N-4k+1)L_{N-2k}(t).\end{aligned} \tag{23}$$

For simplicity of statements, we shall use the following notations,

$$\begin{aligned}D_{i,j} =& \frac{1}{2}\omega_j\sum_{l=1}^{N}(2l+1)L_l(t_j)\left(\sum_{k=0}^{[\frac{l-1}{2}]}(2l-4k-1)L_{l-2k-1}(t_i)\right) \\ &+ \frac{1}{2}\omega_j\sum_{l=0}^{N}(-1)^{N+l}(2l+1)L_l(t_j)\left(\sum_{k=0}^{[\frac{N}{2}]}(2N-4k+1)L_{N-2k}(t_i)\right), \\ &0 \le i,\ j \le N, \\ c(t_i) =& (-1)^{N+1}\sum_{k=0}^{[\frac{N}{2}]}(2N-4k+1)L_{N-2k}(t_i), \quad 0 \le i \le N.\end{aligned}$$

It is worthwhile to point out that both $D_{i,j}$ and $c(t_i)$ are independent of m, then (23) reads

$$\frac{d}{dt}u_m^N(t_i) = \sum_{j=0}^{N} D_{i,j}u_m^N(t_j) + u_m^N(-1)c(t_i), \quad 0 \le i \le N. \tag{24}$$

We now need to represent $w_\nu^N(t_i)$ using $u_\nu^N(t_i)$, i.e., the values at all the grid points. By (20)-(22), we have

$$\begin{aligned} w_\nu^N(t_i) =& u_\nu^N(qt_i + q_m^\nu) = \sum_{l=0}^{N+1} \widetilde{u}_{\nu,l}L_l(qt_i + q_m^\nu) \\ =& \sum_{j=0}^{N}\Big[\frac{1}{2}\omega_j \sum_{l=0}^{N}(2l+1)L_l(t_j)\left(L_l(qt_i+q_m^\nu)\right. \\ & \left.+(-1)^{N+l}L_{N+1}(qt_i+q_m^\nu)\right)\Big]u_\nu^N(t_j) \\ & + (-1)^{N+1}u_\nu^N(-1)L_{N+1}(qt_i+q_m^\nu). \end{aligned} \tag{25}$$

Denote

$$\begin{aligned} (M_\nu)_{i,j} =& \frac{1}{2}\omega_j \sum_{l=0}^{N}(2l+1)L_l(t_j)(L_l(qt_i+q_m^\nu) \\ & + (-1)^{N+l}L_{N+1}(qt_i+q_m^\nu)), \qquad 0 \le i,\ j \le N, \\ r_\nu(t_i) =& (-1)^{N+1}u_\nu^N(-1)L_{N+1}(qt_i+q_m^\nu), \qquad 0 \le i \le N. \end{aligned}$$

Obviously $(M_\nu)_{i,j}$ and $r_\nu(t_i)$ are dependent on m. For simplicity of the notions, we will not show the dependence of them on m explicitly. Then (25) reads

$$w_\nu^N(t_i) = \sum_{j=0}^{N}(M_\nu)_{i,j}u_\nu^N(t_j) + r_\nu(t_i), \qquad 0 \le i \le N. \tag{26}$$

Further, let

$$\begin{aligned} \mathbf{U}_m^N =& (u_m^N(t_0), u_m^N(t_1), \cdots, u_m^N(t_N))^T, \qquad \mathbf{C} = (c(t_0), c(t_1), \cdots, c(t_N))^T, \\ \mathbf{F}_m =& (f_m(t_0), f_m(t_1), \cdots, f_m(t_N))^T, \qquad \mathbf{R}_\nu = (r_\nu(t_0), r_\nu(t_1), \cdots, r_\nu(t_N))^T, \end{aligned}$$

and $D,\ M_\nu$ be the matrix with the entries $D_{i,j},\ (M_\nu)_{i,j},\ 0 \le i,\ j \le N$, respectively. Then (18) can be rewritten as the following matrix form,

$$\begin{aligned} D\mathbf{U}_m^N =& \frac{h}{2}A_m\mathbf{U}_m^N + \frac{h}{2}B_m(M_\nu\mathbf{U}_\nu^N + \mathbf{R}_\nu) \\ & + \frac{h}{2}\mathbf{F}_m - u_m^N(-1)\mathbf{C}, \quad 1 \le m \le M, \end{aligned} \tag{27}$$

where

$$\begin{aligned} A_m =& diag([a_m(t_0), a_m(t_1), \cdots, a_m(t_N)]), \\ B_m =& diag([b_m(t_0), b_m(t_1), \cdots, b_m(t_N)]). \end{aligned}$$

The structure of these M systems depends strongly on the delay term $\frac{h}{2}b_m(t_i)w_\nu^N(t_i)$ in the spectral collocation equation (18), and changes for each value of m as we pass from Phase I to Phase III, as described below. Actually the contribution of the delay term $\frac{h}{2}b_m(t_i)w_\nu^N(t_i)$ in (18) to the coefficient matrices and the right-hand side of the linear algebraic system for $\mathbf{U}_m^N$ is governed by certain relationships between m and the value of q in the delay function qx. We will discuss the situation in three phases below.

• Phase I: $m = 1$. In this initial phase we have complete overlap: for any $x \in I_1$ the image qx lies in I_1, that is $\nu = 1$. Hence, we define the matrix $M^I = M_1$. The vector $\mathbf{U}_1^N$ is determined by the linear algebraic system:

$$\left(D - \frac{h}{2}A_1 - \frac{h}{2}B_1M^I\right)\mathbf{U}_1^N = \frac{h}{2}B_1\mathbf{R_1} + \frac{h}{2}\mathbf{F_1} - y_0\mathbf{C}. \tag{28}$$

• Phase II: $1 < m \le q^{II} := \lfloor \frac{1}{1-q} \rfloor$. In this transition phase we encounter partial overlap: for some $x \in I_m$ the image qx is still in I_m, in this case, $\nu = m$, while for other (smaller) $x \in I_m$, we have $qx \in I_{m-1}$, $\nu = m - 1$. (For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer equal to or less than x.)

Let $s^* := (m-1)\frac{1-q}{q} \in (0,1)$, and k be the maximum subscript of c_i, such that $c_i \le s^*$ $(i = 0, 1, \cdots, k)$. Then define:

$$(M_1^{II})_{i,j} = \begin{cases} (M_{m-1})_{i-1,j-1}, & if \quad 1 \le i \le k+1, \quad 1 \le j \le N+1, \\ 0, & if \quad k+1 < i \le N+1, \quad 1 \le j \le N+1. \end{cases}$$

$$(M_2^{II})_{i,j} = \begin{cases} 0, & if \quad 1 \le i \le k+1, \quad 1 \le j \le N+1, \\ (M_m)_{i-1,j-1}, & if \quad k+1 < i \le N+1, \quad 1 \le j \le N+1. \end{cases}$$

$$(B_m^1)_{i,j} = \begin{cases} (B_m)_{i,j}, & if \quad 1 \le i,j \le k+1, \\ 0, & if \quad k+1 < i,j \le N+1. \end{cases}$$

$$(B_m^2)_{i,j} = \begin{cases} 0, & if \quad 1 \le i,j \le k+1, \\ (B_m)_{i,j}, & if \quad k+1 < i,j \le N+1. \end{cases}$$

In this phase $\mathbf{U}_m^N$ $(1 < m \le q^{II})$ is given by the linear algebraic system:

$$\begin{aligned}\left(D - \frac{h}{2}A_m - \frac{h}{2}B_m^2M_2^{II}\right)\mathbf{U}_m^N =& \frac{h}{2}B_m^1(M_1^{II}\mathbf{U}_{m-1}^N + \mathbf{R}_{m-1}) \\ &+ \frac{h}{2}B_m^2\mathbf{R}_m + \frac{h}{2}\mathbf{F}_m - u_{m-1}^N(1)\mathbf{C}.\end{aligned} \tag{29}$$

• Phase III: $m \ge q^{III} := \lceil \frac{1}{1-q} \rceil$. In this pure delay phase, the image qx is outside I_m for all $x \in I_m$(For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the least integer equal to or larger than x). In this situation, we will have to deal with two cases:

(a). If there exist $\mu^* \in [0,1]$ such that $\theta(x_{m-1} + \mu^* h) := m^*h$ $(1 \le m^* < m-1)$ is one of the grid points. Let l be the maximum subscript of c_i, such that $c_i \le \mu^*$ $(i = 0, 1, \cdots, l)$. Similarly, for $c_i \le \mu^* : \theta(x_{m,i}) \in I_{m^*}$, that is, $\nu = m^*$; while for

$c_i > \mu^* : \theta(x_{m,i}) \in I_{m^*+1}$, $\nu = m^*+1$. We define:

$$(M_1^{III})_{i,j} = \begin{cases} (M_{m*})_{i-1,j-1}, & if \quad 1 \le i \le l+1, \quad 1 \le j \le N+1, \\ 0, & if \quad l+1 < i \le N+1, \quad 1 \le j \le N+1. \end{cases}$$

$$(M_2^{III})_{i,j} = \begin{cases} 0, & if \quad 1 \le i \le l+1, \quad 1 \le j \le N+1, \\ (M_{m^*+1})_{i-1,j-1}, & if \quad l+1 < i \le N+1, \quad 1 \le j \le N+1. \end{cases}$$

$$(B_m^1)_{i,j} = \begin{cases} (B_m)_{i,j}, & if \quad 1 \le i,j \le l+1, \\ 0, & if \quad l+1 < i,j \le N+1. \end{cases}$$

$$(B_m^2)_{i,j} = \begin{cases} 0, & if \quad 1 \le i,j \le l+1, \\ (B_m)_{i,j}, & if \quad l+1 < i,j \le N+1. \end{cases}$$

Then, $\mathbf{U}_m^N$ is the solution of the linear algebraic system:

$$\begin{aligned}\left(D - \frac{h}{2}A_m\right)\mathbf{U}_m^N =& \frac{h}{2}B_m^1(M_1^{III}\mathbf{U}_{m^*}^N + \mathbf{R}_{m^*}) \\ &+ \frac{h}{2}B_m^2(M_2^{III}\mathbf{U}_{m^*+1}^N + \mathbf{R}_{m^*+1}) + \frac{h}{2}\mathbf{F}_m - u_{m-1}^N(1)\mathbf{C}.\end{aligned} \tag{30}$$

(b). If there is no $\mu^* \in [0,1]$, such that $\theta(x_{m-1}+\mu^* h)$ is a grid point, let $m^{**} = \lceil qm \rceil$ (actually, in this case, $\lceil qm \rceil = \lceil q(m-1) \rceil$). Then, $\theta(x_{m,i}) \in I_{m^{**}}$, for all $0 \le i \le N$. So we have the system as follows:

$$\left(D - \frac{h}{2}A_m\right)\mathbf{U}_m^N = \frac{h}{2}B_m(M^{III}\mathbf{U}_{m^{**}}^N + \mathbf{R}_{m^{**}}) + \frac{h}{2}\mathbf{F}_m - u_{m-1}^N(1)\mathbf{C}, \tag{31}$$

where

$$(M^{III})_{i,j} = (M_{m^{**}})_{i-1,j-1}, \quad 1 \le i, \ j \le N+1.$$

In the numerical implementation, we first use (28)-(31) to evaluate $u_m^N(t_j)$, $0 \le j \le N$, and then use (20)-(22) to obtain

$$\begin{aligned}u_m^N(1) = \sum_{l=0}^{N+1}\widetilde{u}_{m,l} = \frac{1}{2}\sum_{j=0}^{N}\left(\sum_{l=0}^{N}(1+(-1)^{N+l})(2l+1)L_l(t_j)\right) \times u_m^N(t_j)\omega_j \\ + (-1)^{N+1}u_m^N(-1).\end{aligned} \tag{32}$$

2.4. **Existence and uniqueness of the spectral collocation solution.** We briefly discuss the existence and uniqueness of the spectral collocation solution defined by the linear algebraic systems (28)-(31).

Theorem 2.3. *Assume that the given functions a, b, and f in (1) are continuous on J. Then for any $q \in (0,1)$, there exists $\bar{h} > 0$ (depending on q) such that for all $h \in (0, \bar{h})$ each of the linear algebraic systems (28)-(31) possesses a unique solution $U_m^N \in \mathbb{R}^N$.*

Proof. Owing to the structure of the matrices

$$D - \frac{h}{2}A_1 - \frac{h}{2}B_1 M^I, \quad D - \frac{h}{2}A_m - \frac{h}{2}B_m^2 M_2^{II}, \quad \text{and} \quad D - \frac{h}{2}A_m,$$

which are the left-hand sides of linear algebraic systems (28)-(31), respectively, and the fact that the given functions a and b are in $C(J)$, it suffices to show that D is nonsingular.

Consider the ordinary differential equation:

$$\begin{cases} u'(t) = f(t), & -1 < t \le 1, \\ u(-1) = 0. \end{cases} \tag{33}$$

The spectral collocation method for (33) is to find $u^N \in \mathcal{P}_{N+1}(I)$, such that

$$\begin{cases} {u^N}'(t_i) = f(t_i), & 0 \le i \le N, \\ u^N(-1) = 0. \end{cases} \tag{34}$$

Since ${u^N}' \in \mathcal{P}_N(I)$, we have $I_N {u^N}'(t) = {u^N}'(t)$. Therefor (34) is equivalent to

$$\begin{cases} {u^N}'(t) = I_N f(t), \; -1 < t \le 1, \\ u^N(-1) = 0. \end{cases} \tag{35}$$

A well-known fact is that the solution of (35), or equivalently (34), is existent and unique.

Denote $\mathbf{U}^N = [u^N(t_0), u^N(t_1), \cdots, u^N(t_N)]^T$, and $\mathbf{F} = [f(t_0), f(t_1), \cdots, f(t_N)]^T$. Similar to (24), the matrix form of (34) is as follows:

$$D\mathbf{U}^N = \mathbf{F} - u^N(-1)\mathbf{C} = \mathbf{F}.$$

This implies that D is nonsingular and the non-singularity leads to the non-singularity of $D - \frac{h}{2}A_1 - \frac{h}{2}B_1 M^I$ whenever h is sufficiently small.

A similar argument establishes the non-singularity of the matrices in Phase II and Phase III. Therefore, for any $q \in (0,1)$ there exists a positive $\bar{h}$ so that for all $h \in (0, \bar{h})$ and $1 \le m \le M$, (18) defines a unique spectral collocation solution for (1). □

Remark 1. By transferring the interval $I_m := [x_{m-1}, x_m]$ into the reference element $I := [-1, 1]$, every term on the right-hand side of numerical scheme (18) has a factor $\frac{h}{2}$, this is important and critical for our proof.

3. Convergence analysis.

We first state regularity result for the pantograph DDE (1). In contrast to DDEs with non-vanishing (e.g., constant) delays, the solution of a DDE with the vanishing delay function qx ($0 < q < 1$; $x \in J$) essentially inherits the regularity of the given functions a, b *and* f. More precisely, we have the following regularity result.

Lemma 3.1. (see Theorem 5.1.8 on p. 262 in [6]). *Assume that, for some $r \ge 1$, the given functions in (1) satisfy a, b, $f \in C^{r-1}(J)$. Then for any $q \in (0,1)$ and any $y_0 \in \mathbb{R}$, the initial-value problem for the DDE (1) has a unique solution $y \in C^r(J)$.*

As a result, it is natural to employ spectral-type methods. Actually, the resulting errors of our approach inherit the typical property of the exponential decay of spectral methods as we will prove both theoretically and numerically.

In the following, we analyze the numerical errors. Denote by

$$y(x) = y_m(x) = u_m(t), \quad x \in I_m, \quad t \in I, \quad 1 \le m \le M. \tag{36}$$

Then by (1),

$$\begin{cases} \dfrac{d}{dx}u_m(t_i) = \dfrac{h}{2}a_m(t_i)u_m(t_i) + \dfrac{h}{2}b_m(t_i)W_\nu(t_i) \\ \qquad\qquad + \dfrac{h}{2}f_m(t_i), & 0 \le i \le N,\ 1 \le m \le M, \\ u_m(-1) = u_{m-1}(1), & 2 \le m \le M, \\ u_1(-1) = y_0, \end{cases} \tag{37}$$

where $W_\nu(t) = u_\nu(qt + q_m^\nu)$.

We see from (18) and (37) that the local numerical solution $u_m^N(t)$ is actually an approximation to the local exact solution $u_m(t)$, with the approximate initial data $u_m^N(-1) = u_{m-1}^N(1)$.

The results stated in the following lemma play a key role in the proof of error estimate.

Lemma 3.2. *If $u_m \in H^r(-1,1)$, with integer $2 \le r \le N+1$, and β is a fixed constant satisfying*

$$2\bar{a}h + \frac{1}{2}\bar{b}h + 6\bar{b}hq^{-1} \le \beta < \frac{1}{3}, \tag{38}$$

then for any $1 \le m \le M$,

$$\|u_m - u_m^N\|^2 \le c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2, \tag{39}$$

$$\max_{t\in[-1,1]} |u_m(t) - u_m^N(t)|^2 \le c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2, \tag{40}$$

where $\bar{a} = \max_{x\in[0,T]} |a(x)|$, $\bar{b} = \max_{x\in[0,T]} |b(x)|$, and c_β is a positive constant depending only on β.

Remark 2. In condition (3.3), 1/3 can be replaced by any constant less than 1/2. For simplicity, we just fix it as 1/3.

The proof of this lemma is presented in the following subsection.

3.1. **Proof of Lemma 3.2.** Denote $E_m^N(t) = u_m^N(t) - I_N u_m(t)$. We prove (39) and (40) by induction.

Step 1: First we analyze the error estimate on I_1, by (18) and (37),

$$\begin{cases} \dfrac{d}{dt}u_1^N(t_i) = \dfrac{h}{2}a_1(t_i)u_1^N(t_i) + \dfrac{h}{2}b_1(t_i)w_1^N(t_i) + \dfrac{h}{2}f_1(t_i), & 0 \le i \le N, \\ u_1^N(-1) = y_0, \end{cases} \tag{41}$$

$$\begin{cases} \dfrac{d}{dt}u_1(t_i) = \dfrac{h}{2}a_1(t_i)u_1(t_i) + \dfrac{h}{2}b_1(t_i)W_1(t_i) + \dfrac{h}{2}f_1(t_i), & 0 \le i \le N, \\ u_1(-1) = y_0. \end{cases} \tag{42}$$

Set

$$G_1^N(t) = I_N \frac{du_1}{dt}(t) - \frac{d}{dt}I_N u_1(t).$$

We have from (42)

$$\frac{d}{dt}I_N u_1(t_i) = \frac{h}{2}a_1(t_i)I_N u_1(t_i) + \frac{h}{2}b_1(t_i)I_N W_1(t_i) + \frac{h}{2}f_1(t_i) - G_1^N(t_i). \tag{43}$$

Further, subtracting (43) from (41) yields

$$\begin{cases} \dfrac{d}{dt}E_1^N(t_i) = \dfrac{h}{2}a_1(t_i)E_1^N(t_i) + \dfrac{h}{2}b_1(t_i)G_2^N(t_i) + G_1^N(t_i), \quad 0 \le i \le N, \\ E_1^N(-1) = u_1^N(-1) - I_N u_1(-1) = u_1(-1) - I_N u_1(-1), \end{cases} \tag{44}$$

where $G_2^N(t_i) = w_1^N(t_i) - I_N W_1(t_i)$. We now multiply the first identity of (44) by $2E_1^N(t_i)\omega_i$, and sum the resulting equation for $0 \le i \le N$. Then it follows that

$$\begin{aligned} 2(E_1^N, \frac{d}{dt}E_1^N)_N &= h(a_1 E_1^N, E_1^N)_N + h(b_1 G_2^N, E_1^N)_N + 2(G_1^N, E_1^N)_N \\ &\le \bar{a}h\|E_1^N\|_N^2 + \bar{b}hA_1^N + A_2^N. \end{aligned} \tag{45}$$

where

$$A_1^N = |(G_2^N, E_1^N)_N|, \quad A_2^N = 2(G_1^N, E_1^N)_N.$$

Obviously, $\frac{d}{dt}E_1^N(t) \in \mathcal{P}_N(-1,1)$. Thus by (7)

$$2(E_1^N, \frac{d}{dt}E_1^N)_N = 2(E_1^N, \frac{d}{dt}E_1^N) = (E_1^N(1))^2 - (E_1^N(-1))^2, \tag{46}$$

By young's inequality,

$$A_1^N \le \|G_2^N\|_N\|E_1^N\|_N \le \|G_2^N\|_N^2 + \frac{1}{4}\|E_1^N\|_N^2. \tag{47}$$

Since $G_1^N(t) \in \mathcal{P}_N(-1,1)$, we use (7) to obtain that for any $\varepsilon > 0$,

$$|A_2^N| \le \varepsilon\|E_1^N\|_N^2 + \varepsilon^{-1}\|G_1^N\|_N^2 = \varepsilon\|E_1^N\|_N^2 + \varepsilon^{-1}\|G_1^N\|^2. \tag{48}$$

Inserting (46)-(48) into (45) and using (11), we have

$$\begin{aligned} (E_1^N(1))^2 \le &(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)\|E_1^N\|^2 + \bar{b}h\|G_2^N\|_N^2 \\ &+ \varepsilon^{-1}\|G_1^N\|^2 + (E_1^N(-1))^2. \end{aligned} \tag{49}$$

On the other hand, for any $t \in [-1,1]$,

$$(E_1^N(t))^2 = (E_1^N(1))^2 - \int_t^1 \frac{d}{ds}(E_1^N(s))^2 ds \le (E_1^N(1))^2 + 2\|E_1^N\|\|\frac{d}{dt}E_1^N\|.$$

Integrating the above with respect to t yields that

$$\|E_1^N\|^2 \le 2(E_1^N(1))^2 + 4\|E_1^N\|\|\frac{d}{dt}E_1^N\|. \tag{50}$$

Moreover, by (7), (11) and (44)

$$\begin{aligned} \|\frac{d}{dt}E_1^N\| = \|\frac{d}{dt}E_1^N\|_N &\le \frac{1}{2}\bar{a}h\|E_1^N\|_N + \frac{1}{2}\bar{b}h\|G_2^N\|_N + \|G_1^N\|_N \\ &\le \frac{1}{2}\bar{a}h\|E_1^N\| + \frac{1}{2}\bar{b}h\|G_2^N\|_N + \|G_1^N\|. \end{aligned} \tag{51}$$

This with (50) implies that

$$(E_1^N(1))^2 \ge [\frac{1}{2} - (\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_1^N\|^2 - \bar{b}h\|G_2^N\|_N^2 - \varepsilon^{-1}\|G_1^N\|^2, \tag{52}$$

where young's inequality is used accordingly. The above with (49) leads to

$$[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_1^N\|^2 \le 2\bar{b}h\|G_2^N\|_N^2 + 2\varepsilon^{-1}\|G_1^N\|^2 + (E_1^N(-1))^2. \tag{53}$$

Using Lemma 2.1, for integer $r \geq 2$, we have

$$\|\frac{d}{dt}u_1 - I_N\frac{d}{dt}u_1\|_{L^2(-1,1)} \leq cN^{1-r}|u_1|_r. \tag{54}$$

$$\|\frac{d}{dt}(I_N u_1 - u_1)\|_{L^2(-1,1)} \leq cN^{\frac{3}{2}-r}|u_1|_r. \tag{55}$$

Therefore

$$\|G_1^N\|^2 \leq 2\|\frac{d}{dt}(I_N u_1 - u_1)\|^2 + 2\|\frac{d}{dt}u_1 - I_N\frac{d}{dt}u_1\|^2 \leq cN^{3-2r}|u_1|_r^2. \tag{56}$$

By Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}\max_{t\in[-1,1]}|I_N u_1(t) - u_1(t)|^2 &\leq \|I_N u_1 - u_1\|^2 + 4\|\frac{d}{dt}(I_N u_1 - u_1)\|^2 \\ &\leq cN^{3-2r}|u_1|_r^2.\end{aligned} \tag{57}$$

Then

$$(E_1^N(-1))^2 = |I_N u_1(-1) - u_1(-1)|^2 \leq cN^{3-2r}|u_1|_r^2. \tag{58}$$

By virtue of (11), (12) and a direct calculation, we deduce that, for any $\varepsilon > 0$,

$$\begin{aligned}\|G_2^N\|_N^2 \leq \|G_2^N\|^2 &= \|w_1^N - I_N W_1\|^2 \\ &\leq (1+\varepsilon)\|w_1^N - W_1\|^2 + (1+\varepsilon^{-1})\|W_1 - I_N W_1\|^2 \\ &\leq (1+\varepsilon)q^{-1}\|u_1 - u_1^N\|^2 + c\varepsilon^{-1}N^{-2r}|W_1|_r^2 \\ &\leq (1+\varepsilon)q^{-1}\|u_1 - u_1^N\|^2 + c\varepsilon^{-1}q^{2r-1}N^{-2r}|u_1|_r^2.\end{aligned} \tag{59}$$

Substituting (56), (58), (59) into (53) yields

$$[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_1^N\|^2 \leq 2\bar{b}h(1+\varepsilon)q^{-1}\|u_1 - u_1^N\|^2 + c\varepsilon^{-1}N^{3-2r}|u_1|_r^2. \tag{60}$$

Thanks to (12), we have that

$$\begin{aligned}\|u_1 - u_1^N\|^2 &\leq (1+\varepsilon)\|E_1^N\|^2 + (1+\varepsilon^{-1})\|u_1 - I_N u_1\|^2 \\ &\leq (1+\varepsilon)\|E_1^N\|^2 + c\varepsilon^{-1}N^{-2r}|u_1|_r^2.\end{aligned} \tag{61}$$

This with (60) implies that

$$[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|u_1 - u_1^N\|^2 \leq 2\bar{b}h(1+\varepsilon)^2q^{-1}\|u_1 - u_1^N\|^2 + c\varepsilon^{-1}N^{3-2r}|u_1|_r^2,$$

or equivalently,

$$[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon) - 2\bar{b}h(1+\varepsilon)^2q^{-1}]\|u_1 - u_1^N\|^2 \leq c\varepsilon^{-1}N^{3-2r}|u_1|_r^2. \tag{62}$$

Under the assumption $2\bar{a}h + \frac{1}{2}\bar{b}h + 6\bar{b}hq^{-1} \le \beta < \frac{1}{3}$, we select $\varepsilon = \sqrt{\frac{7}{3(\beta+2)}} - 1$ to make $(1+\varepsilon)^2(\beta+2) - 2 = \frac{1}{3}$, and therefore

$$\begin{aligned}
&2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon) + 2\bar{b}h(1+\varepsilon)^2 q^{-1} \\
=&2\bar{a}h + \frac{1}{2}\bar{b}h + 2(\varepsilon+1) + 2\bar{b}hq^{-1}(1+\varepsilon)^2 - 2 \\
\le&2\bar{a}h + \frac{1}{2}\bar{b}h + 2(\varepsilon+1) + 6\bar{b}hq^{-1}(1+\varepsilon)^2 - 2 \\
<&(1+\varepsilon)^2(2\bar{a}h + \frac{1}{2}\bar{b}h + 6\bar{b}hq^{-1} + 2) - 2 \\
\le&(1+\varepsilon)^2(\beta+2) - 2 = \frac{1}{3}.
\end{aligned} \tag{63}$$

This estimate, combined with (62), implies that for certain constant $c_\beta > 0$,

$$\|u_1 - u_1^N\|^2 \le c_\beta N^{3-2r}|u_1|_r^2. \tag{64}$$

By (12) and (64),

$$\|E_1^N\|^2 \le 2\|u_1 - I_N u_1\|^2 + 2\|u_1 - u_1^N\|^2 \le c_\beta N^{3-2r}|u_1|_r^2. \tag{65}$$

Direct calculation shows that

$$\begin{aligned}
\|\frac{d}{dt}(u_1 - u_1^N)\|^2 &\le 2\|\frac{d}{dt}E_1^N\|^2 + 2\|\frac{d}{dt}(u_1 - I_N u_1)\|^2 \\
&= 2\|\frac{d}{dt}E_1^N\|_N^2 + 2\|\frac{d}{dt}(u_1 - I_N u_1)\|^2 \\
&\le \frac{3}{2}\bar{a}^2h^2\|E_1^N\|^2 + \frac{3}{2}\bar{b}^2h^2\|G_2^N\|_N^2 \\
&\quad + 6\|G_1^N\|^2 + cN^{3-2r}|u_1|_r^2,
\end{aligned} \tag{66}$$

where (7), (11), (13) and (44) are used. Then, using (56), (59), (64)-(66) yields

$$\|\frac{d}{dt}(u_1 - u_1^N)\|^2 \le c_\beta N^{3-2r}|u_1|_r^2. \tag{67}$$

Finally, by virtue of (14), (64), and (67), we obtain

$$\max_{t\in[-1,1]} |u_1(t) - u_1^N(t)|^2 \le c_\beta N^{3-2r}|u_1|_r^2. \tag{68}$$

We now turn to the error estimates for $PhaseII$ and $PhaseIII$ $(m \ge 2)$.

Step 2: In order to use an induction argument, assume that the estimates

$$\|u_k - u_k^N\|^2 \le c_\beta N^{3-2r}\sum_{j=1}^{k}|u_j|_r^2, \tag{69}$$

$$\max_{t\in[-1,1]} |u_k(t) - u_k^N(t)|^2 \le c_\beta N^{3-2r}\sum_{j=1}^{k}|u_j|_r^2, \tag{70}$$

are valid for $k = 1, 2, \cdots, m-1$. Consider the case $k = m$. Let $G_{m,1}^N(t) = I_N\frac{d}{dt}u_m(t) - \frac{d}{dt}I_N u_m(t)$ and

$$\Lambda_m^\nu = \{x_{m,i}|\theta(x_{m,i}) = qx_{m,i} \in I_\nu,\ 0 \le i \le N\},\ 1 \le \nu \le m,\ 2 \le m \le M.$$

Following the same line as in the derivation of (44), we obtain from (18) and (37) that

$$\begin{cases} \frac{d}{dt}E_m^N(t_i) = \frac{h}{2}a_m(t_i)E_m^N(t_i) + \frac{h}{2}b_m(t_i)G_{\nu,2}^N(t_i) \\ \qquad + G_{m,1}^N(t_i), & 0 \le i \le N,\ 2 \le m \le M, \\ E_m^N(-1) = u_m^N(-1) - I_N u_m(-1), & 2 \le m \le M, \end{cases} \tag{71}$$

here $G_{\nu,2}^N(t_i) = w_\nu^N(t_i) - I_N W_\nu(t_i)$. With a similar argument as in the derivation of (49) and using (11), we deduce that

$$(E_m^N(1))^2 \le (\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)\|E_m^N\|^2 + \bar{b}h\|G_{\nu,2}^N\|_N^2 + \varepsilon^{-1}\|G_{m,1}^N\|^2 + (E_m^N(-1))^2. \tag{72}$$

On the other hand, like (50), we get

$$\|E_m^N\|^2 \le 2(E_m^N(1))^2 + 4\|E_m^N\|\|\frac{d}{dt}E_m^N\|. \tag{73}$$

Moreover, by (7), (11) and (71),

$$\|\frac{d}{dt}E_m^N\| = \|\frac{d}{dt}E_m^N\|_N \le \frac{1}{2}\bar{a}h\|E_m^N\| + \frac{1}{2}\bar{b}h\|G_{\nu,2}^N\|_N + \|G_{m,1}^N\|. \tag{74}$$

By (73) and (74), we can derive

$$(E_m^N(1))^2 \ge [\frac{1}{2} - (\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_m^N\|^2 - \bar{b}h\|G_{\nu,2}^N\|_N^2 - \varepsilon^{-1}\|G_{m,1}^N\|^2. \tag{75}$$

This, along with (72) leads to

$$[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_m^N\|^2 \le 2\bar{b}h\|G_{\nu,2}^N\|_N^2 + 2\varepsilon^{-1}\|G_{m,1}^N\|^2 + (E_m^N(-1))^2. \tag{76}$$

Similar to (56), we have

$$\|G_{m,1}^N\|^2 \le cN^{3-2r}|u_m|_r^2. \tag{77}$$

Thanks to (14), an argument similar to (58) yields

$$\begin{aligned} (E_m^N(-1))^2 &= |u_m^N(-1) - I_N u_m(-1)|^2 \\ &\le 2|u_{m-1}^N(1) - u_{m-1}(1)|^2 + 2|u_m(-1) - I_N u_m(-1)|^2 \\ &\le 2|u_{m-1}^N(1) - u_{m-1}(1)|^2 + cN^{3-2r}|u_m|_r^2. \end{aligned} \tag{78}$$

Substituting (77) and (78) into (76) leads to

$$\begin{aligned} [\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_m^N\|^2 \le & 2\bar{b}h\|G_{\nu,2}^N\|_N^2 + 2|u_{m-1}^N(1) - u_{m-1}(1)|^2 \\ & + c\varepsilon^{-1}N^{3-2r}|u_m|_r^2. \end{aligned} \tag{79}$$

We next estimate $\|G_{\nu,2}^N\|_N^2$. Clearly, a glimpse of Phase II and Phase III will help in making the following analysis obvious:

Case I. If m lies in Phase II, then for $0 \le i \le N$, the images $\theta(x_{m,i})$ are in I_m and I_{m-1}. Therefore, noting the fact $\sum_{i=0}^{N}\omega_i = 2$, we deduce that for any $1 < m \le$

$\lfloor \frac{1}{1-q} \rfloor$.

$$\begin{aligned}\|G_{\nu,2}^N\|_N^2 &= \sum_{\nu=m-1}^{m} \sum_{x_{m,i}\in\Lambda_m^\nu} (u_\nu^N(qt_i+q_m^\nu) - u_\nu(qt_i+q_m^\nu))^2\omega_i \\ &\le \sum_{\nu=m-1}^{m} \max_{-1\le t\le 1} |u_\nu^N(t) - u_\nu(t)|^2 \sum_{x_{m,i}\in\Lambda_m^\nu} \omega_i \\ &\le 2\max\{\|u_{m-1}^N - u_{m-1}\|_{L^\infty}^2, \|u_m^N - u_m\|_{L^\infty}^2\} \\ &\le 2\|u_{m-1}^N - u_{m-1}\|_{L^\infty}^2 + 2\|u_m^N - u_m\|_{L^\infty}^2.\end{aligned} \tag{80}$$

By using (13), (71) and a result similar to (66), we can derive

$$\begin{aligned}\|\frac{d}{dt}(u_m - u_m^N)\|^2 \le &\frac{3}{2}\bar{a}^2h^2\|E_m^N\|^2 + \frac{3}{2}\bar{b}^2h^2\|G_{\nu,2}^N\|_N^2 \\ &+ 6\|G_{m,1}^N\|^2 + cN^{3-2r}|u_m|_r^2.\end{aligned} \tag{81}$$

Then, by (14), (77) and (81) yields

$$\begin{aligned}\|u_m - u_m^N\|_{L^\infty}^2 \le &\|u_m - u_m^N\|^2 + 6\bar{a}^2h^2\|E_m^N\|^2 \\ &+ 6\bar{b}^2h^2\|G_{\nu,2}^N\|_N^2 + cN^{3-2r}|u_m|_r^2.\end{aligned} \tag{82}$$

Substituting (82) into (80), we obtain

$$\begin{aligned}(1-12\bar{b}^2h^2)\|G_{\nu,2}^N\|_N^2 \le &2\|u_{m-1}^N - u_{m-1}\|_{L^\infty}^2 + 2\|u_m - u_m^N\|^2 \\ &+ 12\bar{a}^2h^2\|E_m^N\|^2 + cN^{3-2r}|u_m|_r^2.\end{aligned} \tag{83}$$

Due to (38), we have that $1 - 12\bar{b}^2h^2 > \frac{4}{5}$. Thus

$$\begin{aligned}\|G_{\nu,2}^N\|_N^2 \le &\frac{5}{2}\|u_{m-1}^N - u_{m-1}\|_{L^\infty}^2 + \frac{5}{2}\|u_m - u_m^N\|^2 \\ &+ 15\bar{a}^2h^2\|E_m^N\|^2 + cN^{3-2r}|u_m|_r^2.\end{aligned} \tag{84}$$

Substituting (84) into (79), we have

$$\begin{aligned}&[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon) - 30\bar{a}^2\bar{b}h^3]\|E_m^N\|^2 \\ \le &5\bar{b}h\|u_{m-1}^N - u_{m-1}\|_{L^\infty}^2 + 5\bar{b}h\|u_m - u_m^N\|^2 \\ &+ 2|u_{m-1}^N(1) - u_{m-1}(1)|^2 + c\varepsilon^{-1}N^{3-2r}|u_m|_r^2.\end{aligned} \tag{85}$$

Furthermore, like (61), we have

$$\|u_m - u_m^N\|^2 \le (1+\varepsilon)\|E_m^N\|^2 + c\varepsilon^{-1}N^{-2r}|u_m|_r^2.$$

The above two inequalities lead to that for any $1 < m \le \lfloor \frac{1}{1-q} \rfloor$

$$\begin{aligned}&[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon) - 30\bar{a}^2\bar{b}h^3 - 5\bar{b}h(1+\varepsilon)]\|u_m - u_m^N\|^2 \\ \le &5\bar{b}h(1+\varepsilon)\|u_{m-1}^N - u_{m-1}\|_{L^\infty}^2 + 2(1+\varepsilon)|u_{m-1}^N(1) - u_{m-1}(1)|^2 \\ &+ c\varepsilon^{-1}N^{3-2r}|u_m|_r^2.\end{aligned} \tag{86}$$

According to (38), we have $\bar{a}h < \frac{1}{6}$, this inequality, along with (38), leads to that

$$2\bar{a}h + \frac{1}{2}\bar{b}h + 30\bar{a}^2\bar{b}h^3 + 5\bar{b}h \le 2\bar{a}h + \frac{1}{2}\bar{b}h + 6\bar{b}hq^{-1} \le \beta < \frac{1}{3}.$$

Let $\varepsilon = \sqrt{\frac{7}{3(\beta+2)}} - 1$, then $(1+\varepsilon)(\beta+2) - 2 < (1+\varepsilon)^2(\beta+2) - 2 = \frac{1}{3}$. Hence,

$$\begin{aligned}&2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon) + 30\bar{a}^2\bar{b}h^3 + 5\bar{b}h(1+\varepsilon)\\ =&2\bar{a}h + \frac{1}{2}\bar{b}h + 2(\varepsilon+1) + 30\bar{a}^2\bar{b}h^3 + 5\bar{b}h(1+\varepsilon) - 2\\ \leq&(1+\varepsilon)(2\bar{a}h + \frac{1}{2}\bar{b}h + 30\bar{a}^2\bar{b}h^3 + 5\bar{b}h + 2) - 2\\ \leq&(1+\varepsilon)(\beta+2) - 2\\ <&\frac{1}{3}.\end{aligned} \tag{87}$$

A combination of the above estimate, (70) and (86) leads to, for any $1 < m \leq \lfloor \frac{1}{1-q} \rfloor$

$$\|u_m - u_m^N\|^2 \leq c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2. \tag{88}$$

By (12) and (88),

$$\|E_m^N\|^2 \leq 2\|u_m - u_m^N\|^2 + 2\|u_m - I_N u_m\|^2 \leq c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2. \tag{89}$$

In particular, by virtue of (14), (77) and (81), we have

$$\begin{aligned}\max_{t\in[-1,1]} |u_m(t) - u_m^N(t)|^2 &\leq \|u_m - u_m^N\|^2 + 4\|\frac{d}{dt}(u_m - u_m^N)\|^2\\ &\leq \|u_m - u_m^N\|^2 + 6\bar{a}^2h^2\|E_m^N\|^2\\ &\quad + 6\bar{b}^2h^2\|G_{\nu,2}^N\|_N^2 + cN^{3-2r}|u_m|_r^2.\end{aligned}$$

The above with (84), (70), (88), (89) and (38) leads to

$$\max_{t\in[-1,1]} |u_m(t) - u_m^N(t)|^2 \leq c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2. \tag{90}$$

Case II. If m lies in Phase III, then $\Lambda_m^m = \Phi$ for any $m \geq \lceil \frac{1}{1-q} \rceil$. As discussed before, in this pure delay phase, we will have to deal with two cases:
(a). For $1 \leq i \leq N$, the images $\theta(x_{m,i})$ are in I_{m^*} and I_{m^*+1}, $m^*+1 < m$. Then

$$\begin{aligned}\|G_{\nu,2}^N\|_N^2 &= \sum_{\nu=m^*}^{m^*+1} \sum_{x_{m,i}\in\Lambda_m^\nu} (u_\nu^N(qt_i + q_m^\nu) - u_\nu(qt_i + q_m^\nu))^2\omega_i\\ &\leq 2\max_{m^*\leq\nu\leq m^*+1} \|u_\nu^N - u_\nu\|_{L^\infty}^2.\end{aligned} \tag{91}$$

Inserting the above inequality into (79), we obtain that

$$\begin{aligned}[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_m^N\|^2 \leq&4\bar{b}h \max_{m^*\leq\nu\leq m^*+1} \|u_\nu^N - u_\nu\|_{L^\infty}^2\\ &+ 2|u_{m-1}^N(1) - u_{m-1}(1)|^2 + c\varepsilon^{-1}N^{3-2r}|u_m|_r^2.\end{aligned} \tag{92}$$

Furthermore

$$\|u_m - u_m^N\|^2 \leq (1+\varepsilon)\|E_m^N\|^2 + c\varepsilon^{-1}N^{-2r}|u_m|_r^2.$$

The above two inequalities with (38) and (70) lead to

$$\|u_m - u_m^N\|^2 \le c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2. \tag{93}$$

(b). For $0 \le i \le N$, the images $\theta(x_{m,i})$ are all in $I_{m^{**}}$, we have

$$\begin{aligned}\|G_{\nu,2}^N\|_N^2 &= \sum_{x_{m,i}\in\Lambda_m^{m^{**}}} (u_{m^{**}}^N(qt_i + q_m^{m^{**}}) - u_{m^{**}}(qt_i + q_m^{\nu}))^2 \omega_i \\ &\le 2\|u_{m^{**}}^N - u_{m^{**}}\|_{L^\infty}^2.\end{aligned} \tag{94}$$

Inserting the above inequality into (79), we obtain that

$$\begin{aligned}[\frac{1}{2} - 2(\bar{a}h + \frac{1}{4}\bar{b}h + \varepsilon)]\|E_m^N\|^2 \le& 4\bar{b}h\|u_{m^{**}}^N - u_{m^{**}}\|_{L^\infty}^2 \\ &+ 2|u_{m-1}^N(1) - u_{m-1}(1)|^2 + c\varepsilon^{-1}N^{3-2r}|u_m|_r^2.\end{aligned} \tag{95}$$

Furthermore

$$\|u_m - u_m^N\|^2 \le (1+\varepsilon)\|E_m^N\|^2 + c\varepsilon^{-1}N^{-2r}|u_m|_r^2.$$

The above two inequalities with (87) and (70) leads to (93).

Similarly, by virtue of (14), (93) and a result similar to (66), we get

$$\max_{t\in[-1,1]} |u_m(t) - u_m^N(t)|^2 \le c_\beta N^{3-2r} \sum_{j=1}^{m} |u_j|_r^2. \tag{96}$$

This completes the proof of the lemma. □

3.2. **Main results.** By virtue of the relations between the local and global solutions (19) and (36), and a direct calculation, we can readily rewrite the above lemma as follows:

Theorem 3.3. *If $y \in H^r(0,T)$, with integer $2 \le r \le N+1$, β is a fixed constant satisfying (38), then for any $1 \le m \le M$,*

$$\begin{aligned}\|y - y^N\|_{L^2(x_{m-1},x_m)}^2 &\le c_\beta N^{3-2r}(\frac{h}{2})^{2r}|y|_{r,(0,x_m)}^2, \\ \max_{x\in[x_{m-1},x_m]} |y(x) - y^N(x)|^2 &\le c_\beta N^{3-2r}(\frac{h}{2})^{2r-1}|y|_{r,(0,x_m)}^2,\end{aligned}$$

Further,

$$\|y - y^N\|_{L^2(J)}^2 \le c_\beta T N^{3-2r}(\frac{h}{2})^{2r-1}|y|_{r,J}^2,$$

where $h = T/M$ and c_β is a constant depending only on β.

Remark 3. The condition (38) is only technically necessary for the proof, and it is not essential. As we will demonstrate in Section 4, some numerical examples do not satisfy this condition, but the numerical scheme still converges.

TABLE 1. The maximum point-wise error of long time behavior ($T = 10^4$).

N	$Maximum\ error\ (q = 0.01)$	$Maximum\ error\ (q = 0.5)$	$Maximum\ error\ (q = 0.99)$
2	8.984e-03	7.632e-03	8.293e-03
4	5.816e-05	1.961e-05	1.747e-05
6	2.394e-07	4.879e-08	6.369e-08
8	6.657e-10	9.181e-11	1.371e-10
10	1.291e-12	2.598e-13	4.943e-13

4. **Numerical experiments.** In this section, we present some numerical results using scheme (18) to illustrate the efficiency of our algorithm.

Example 4.1. Consider the pantograph-type delay differential equation (1) with variable coefficients,

$$a(x) = sin2x - 2, \qquad b(x) = cos\left(\frac{x}{3}\right),$$
$$f(x) = (2 - sin2x)sinx - cos\left(\frac{x}{3}\right) sinqx + cosx, \qquad y_0 = 0.$$

The exact solution of the problem is then $y(x) = sinx$.

In our computations, we take $T = 10^4$ and choose uniform meshes J_h with mesh size $h = 2$. The maximum point-wise errors of long time behavior between the numerical solution obtained by the numerical scheme (18) and the exact solution are given in Table 1 for $q = 0.01,\ q = 0.5$ and $q = 0.99$, respectively. The errors are also shown in Fig. 1 for $q = 0.01$ and $q = 0.99$, respectively. These numerical results exhibit the well-known exponential convergence property of spectral methods.

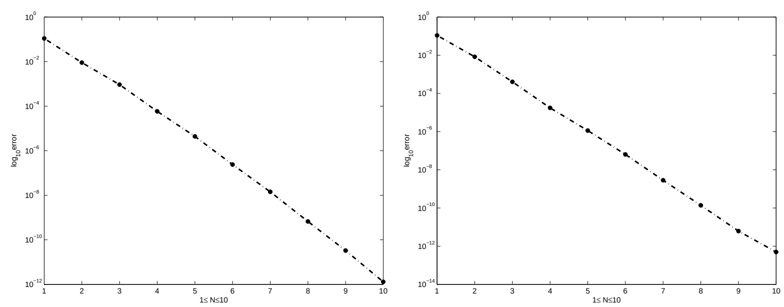

FIGURE 1. The numerical errors for $T = 10^4,\ h = 2$ in example 4.1 of $q = 0.01$ (left) and $q = 0.99$ (right).

Fig. 2 shows the errors based on the uniform meshes J_h with various values of mesh sizes h for $T = 1000,\ q = 0.01$. They indicate that the numerical errors decay exponentially as N increases and h decreases.

In Table 2, we compare the errors of our method LCMDD (the Legendre collocation method and domain decomposition) with the method MDLGC (the multi-domain Legendre-Gauss collocation method presented in [19]). In our computations, we take $T = 20, h = \tau_m = 1$ and uniform N_m. the two methods have the same mesh and the same number of interpolation nodes, the maximum point-wise errors are given in Table 2 for $q = 0.01,\ q = 0.5$ and $q = 0.99$, respectively. Since we present an algorithm to evaluate the nodal values of unknown function, and the authors

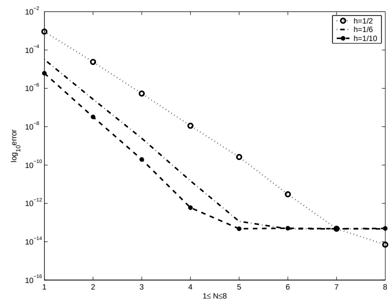

FIGURE 2. The maximum point-wise errors with different h in example 4.1.

TABLE 2. Numerical results for example 4.1.

N	$q = 0.01$		$q = 0.5$		$q = 0.99$	
	LCMDD	MDLGC	LCMDD	MDLGC	LCMDD	MDLGC
2	1.536e-04	1.536e-04	1.845e-04	1.845e-04	1.361e-04	1.361e-04
4	2.525e-07	2.525e-07	1.001e-07	1.001e-07	5.556e-08	5.556e-08
6	1.982e-10	1.982e-10	6.511e-11	6.511e-11	3.034e-11	3.034e-11
8	2.123e-13	2.123e-13	3.059e-14	2.937e-14	3.275e-14	3.098e-14
10	2.109e-15	2.498e-16	2.332e-15	2.220e-16	2.887e-15	3.331e-16

of [19] designed an algorithm to resolve the unknown expansion coefficients. the similar technique lead to almost the same numerical results.

Example 4.2. Consider the pantograph-type delay differential equation (1.1) with: $a(x) = -2e^{-\frac{3}{2}x} - \frac{3}{2}$, $b(x) = e^{-x} + 1$, $f(x) = 5(1 - \frac{x}{2})e^{-\frac{1}{2}x} + \frac{5}{2}x(3 + 4e^{-\frac{3}{2}x})e^{-\frac{1}{2}x} - 5qx(1 + e^{-x})e^{-\frac{1}{2}qx}$, $y_0 = 0$. The exact solution is $y(x) = 5xe^{-\frac{1}{2}x}$.

The error for $q = 0.5$ using the numerical scheme (18) is shown in Fig. 3, in which we have taken $T = 10^4$ and choose uniform meshes J_h with various values of mesh sizes h. It is observed that with at most 10 spectral collocation points the errors drop to the machine accuracy. The numerical errors decay exponentially as N increases and h decreases.

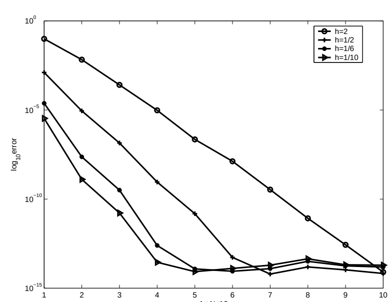

FIGURE 3. The maximum point-wise errors with different h in example 4.2.

TABLE 3. Numerical results for example 4.3.

N	$q = 0.01$		$q = 0.5$		$q = 0.9$	
	LCMDD	MDLGC	LCMDD	MDLGC	LCMDD	MDLGC
2	1.194e-04	1.194e-04	2.107e-05	2.107e-05	1.858e-05	1.858e-05
4	2.508e-07	2.508e-07	2.052e-08	2.052e-08	1.698e-08	1.698e-08
6	2.711e-10	2.711e-10	1.385e-11	1.385e-11	5.461e-12	5.461e-12
8	2.046e-13	2.041e-13	3.997e-15	4.941e-15	4.718e-15	4.607e-15
10	8.882e-16	1.110e-16	2.602e-16	1.110e-16	8.882e-16	5.551e-17

Example 4.3. Consider the pantograph equation (1) with constant coefficients,

$$a(x) = -1, \quad b(x) = \frac{1}{2}, \quad f(x) = -\frac{1}{2}e^{-qx}, \quad y_0 = 1.$$

The exact solution of the problem is $y(x) = e^{-x}$.

In Table 3, we compare the errors of our method LCMDD with the method MDLGC (presented in [19]). In our computations, we take $T = 20, h = \tau_m = 1$ and uniform N_m. the two methods have the same mesh and the same number of interpolation nodes, the maximum point-wise errors are given in Table 3 for $q = 0.01$, $q = 0.5$ and $q = 0.9$, respectively. we can observe that two methods provide again almost the same numerical results.

In Fig. 4, we compare the LCM (the Legendre-collocation method presented in [1]) with our method LCMDD under $T = 2$, $q = 0.9$. Though the two methods have the same time interval and the same number of interpolation nodes, it is noted that our method provides more rapid convergence rate than LCM.

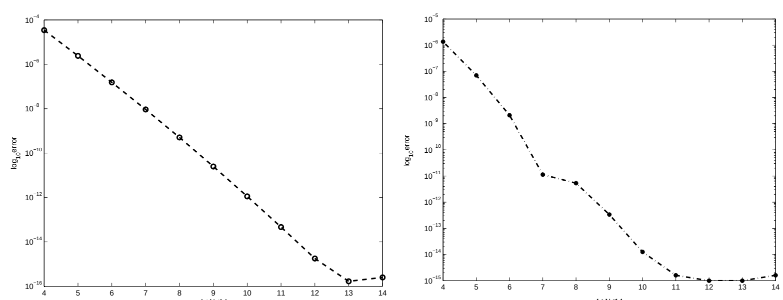

FIGURE 4. The numerical errors of LCM (left) and LCMDD (right)

.

Fig. 5 shows the errors of LCM and LCMDD when $q = 0.01$, $T = 10$ and $T = 100$. In our method, we choose uniform meshes J_h with mesh size $h = 2$. It is observed that our method provides again much better numerical results than LCM. In particular, our method is still valid even for large time interval (see Figs. 1-3).

As pointed out, in actual computation, even if the conditions for Theorems 3.3 are not satisfied, i.e., h is not sufficiently small, the numerical solutions of our method match the exact solutions very well.

5. **Concluding remark.** In this paper, we proposed a spectral collocation method combined with domain decomposition (with uniform mesh) for pantograph-type

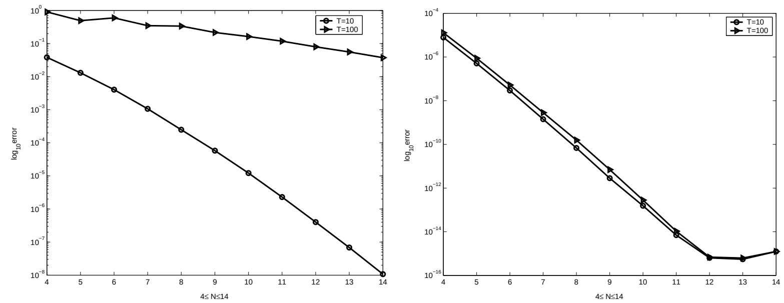

FIGURE 5. The numerical errors of LCM (left) and LCMDD (right).

DDEs. We are able to demonstrate rigorously that the global errors of the numerical approximations decay exponentially, which is a desired feature for a spectral method. Our numerical results demonstrated the spectral accuracy of the proposed algorithm and coincided well with the theoretical analysis. More importantly, both theoretically and numerically our approach have a so-called long-time behaviors, i.e., it is valid even for large time interval.

Acknowledgments. The first author is supported in part by Scientific Research Fund of Hunan Provincial Education Department of China (10C0654), the NSF of China (10971059, 11101136), the NSF of Hunan Province, China (10JJ6003), the Grant of Science and Technology Commission of Hunan Province, China (2012FJ411 6), and the NSF of Hunan University of Technology (2011HZX17). The second author is supported in part by the NSF of China (11171104, 10871066). The third author is supported in part by the US National Science Foundation through grant DMS-1115530.

REFERENCES

[1] I. Ali, H. Brunner and T. Tang, *A spectral method for pantograph-type delay differential equations and its convergence analysis*, J. Comput. Math., **27** (2009), 254–265.

[2] I. Ali, H. Brunner and T. Tang, *Spectral methods for pantograph-type differential and integral equations with multiple delays*, Front. Math. China, **4** (2009), 49–61.

[3] A. Bellen, *Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay*, IMA J. Numer. Anal., **22** (2002), 529–536.

[4] A. Bellen and M. Zennaro, "Numerical Methods for Delay Differential Equations," Oxford University Press, Oxford, 2003.

[5] A. Bellen, H. Brunner, S. Maset and L. Torelli, *Superconvergence in collocation methods on quasi-graded meshes for functional differential equations with vanishing delays*, BIT, **46** (2006), 229–247.

[6] H. Brunner, "Collocation Methods for Volterra Integral and Related Functional Differential Equations," Cambridge University Press, Cambridge, 2004.

[7] H. Brunner, *Current work and open problems in the numerical analysis of Volterra functional equations with vanishing delays*, Front. Math. China, **4** (2009), 3–22.

[8] H. Brunner, Q. M. Huang and H. H. Xie, *Discontinuous Galerkin methods for delay differential equations of pantograph type*, SIAM J. Numer. Anal., **48** (2010), 1944–1967.

[9] H. Brunner and Q. Y. Hu, *Optimal superconvergence results for delay integro-differential equations of pantograph type*, SIAM J. Numer. Anal., **45** (2007), 986–1004.

[10] L. Fox, D. F. Mayers, J. R. Ockendon and A. B. Tayler, *On a functional differential equation*, J. Inst. Math. Appl., **8** (1971), 271–307.

[11] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, "Spectral Methods. Fundamentals in Single Domains," Springer-Verlag, Berlin, 2006.

[12] B. Y. Guo and Z. Q. Wang, *Legendre-Gauss collocation methods for ordinary differential equations*, Adv. Comp. Math., **30** (2009), 249–280.

[13] B. Y. Guo and J. P. Yan, *Legendre-Gauss collocation methods for initial value problems of second ordinary differential equations*, App. Numer. Math., **59** (2009), 1386–1408.

[14] A. Iserles, *On the generalized pantograph functional differential equation*, Europ J. Appl. Math., **4** (1993), 1–38.

[15] A. Iserles, *On nonlinear delay-differential equations*, Trans. Amer. Math. Soc., **344** (1994), 441–477.

[16] T. Kato and J. B. Mcleod, *The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$*, Bull. Amer. Math. Soc., **77** (1971), 891–937.

[17] T. Tang, X. Xu, and J. Cheng, *On spectral methods for Volterra type integral equations and the convergence analysis*, J. Comput. Math., **26** (2008), 825–837.

[18] X. Tao, Z. Q. Xie, and X. J. Zhou, *Spectral Petrov-Galerkin methods for the second kind Volterra type integro-differential equations*, Numer. Math. Theor. Meth. Appl., **4** (2011), 216–236.

[19] Z. Q. Wang and L. L. Wang, *A Legendre-Gauss collocation method for nonlinear delay differential equations*, Dis. Cont. Dyn. Sys. B., **13** (2010), 685–708.

Received October 2011; revised September 2012.

E-mail address: tj049@163.com
E-mail address: ziqingxie@yahoo.com.cn
E-mail address: ag7761@wayne.edu