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THE SPECTRAL COLLOCATION METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS

Can Huang

Department of Mathematics, Michigan State University
East Lansing, MI 48824, USA

Zhimin Zhang

Department of Mathematics, Wayne State University
Detroit, MI 48202, USA
Beijing Computational Science Research Center
No. 3 Heqing Road, Haidian District, Beijing 100084, China

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Abstract. In this paper, we use the Chebyshev spectral collocation method to solve a certain type of stochastic differential equations (SDEs). We also use this method to estimate parameters of stochastic differential equations from discrete observations by maximum likelihood technique and Kessler technique. Our numerical tests shows that the spectral method gives better results than the Euler's method and the Shoji-Ozaki method.

1. **Introduction.** Numerical solutions of stochastic differential equations (SDEs) have caught the attention of researchers for a very long time, a few examples are [9, 10, 15, 18, 19, 20]. These methods are classical finite difference schemes, based upon Ito-Taylor expansion. Runge-Kutta methods are also taken into account, see [5, 6, 7] etc, where Butcher tableau for Stratonovich SDEs are developed and rooted tree analysis is applied for the convergence theory. The higher order the convergence rate is, the larger amount of terms are involved in both finite difference schemes and Runge-Kutta schemes. Natural extensions for these schemes to SDEs with jump-diffusion can be found in [11, 12, 3, 4, 23]. Recently, Galerkin methods are also used to solve ODEs or PDEs with random coefficients [1, 2, 13, 17]. They tried to use Karhunen-Loeve expansion to approximate the noise coefficients by certain orthogonal polynomials, thus convert the problem into a coupled-deterministic system. A typical method of this idea is the polynomial chaos expansion method [26, 27]. The size of the system totally depends on the number of terms that they use, which leads to an very expensive computational cost.

In this paper, we attempt to use the spectral collocation method to solve stochastic differential equations of the form

$$dX_t = f(t, X_t)dt + g(X_t)dW_t, \quad X(0) = x_0 \tag{1}$$

where $f(t, x)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t, $g(x)$ only depends on x and continuously

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differentiable, W_t is the standard Wiener process. It is well-known that spectral collocation methods gain exponential rate of convergence if the solution of a deterministic equation is analytic and thus much more accurate than finite difference schemes and Runge-Kutta schemes. Our numerical experiments show that the spectral collocation method is rather accurate for some SDEs as well. Furthermore, the differentiation matrix, which plays an essential role in the method, can be obtained before a problem setting. This makes the spectral collocation method concise and neat. Our numerical examples also confirm that spectral collocation methods perform better than the Euler's method and the Shoji-Ozaki method, which is a local linearization method proposed for SDE parameter estimation [24]. However, it should be pointed out that the spectral collocation method requires the deterministic part of the solution to the SDE after the Lamperti transformation is smooth enough to obtain an accurate approximation.

We organize the paper as follows. In section 2, the Lamperti transform, the spectral method, Maximum likelihood estimator and Kessler estimator are introduced; in section 3, numerical tests are implemented and the behaviors of these three methods above are compared.

2. **Some preliminary knowledge and the spectral collocation method.**

In this section, we introduce some crucial components which will be used in the whole process for SDE. Since our method require that the diffusion term has a constant coefficient, equation (1) undergoes the Lamperti transform [22] first. After that, spectral collocation method is applied. Two parameter estimators are also introduced in this section to compare the performance of the spectral method and some other methods.

2.1. **The Lamperti transform.**

For equation of the form (1), set

$$Y_t = F(X_t) = \int_z^{X_t} \frac{1}{g(u)} du, \tag{2}$$

where z is any arbitrary value in the state space of X. Then, the process Y_t solves the stochastic differential equation

$$dY_t = b(t, Y_t)dt + dW_t, \tag{3}$$

where

$$b(t,y) = \frac{f(t, F^{-1}(y))}{g(F^{-1}(y))} - \frac{1}{2} g_x(F^{-1}(y)) \tag{4}$$

which can also be written as

$$dY_t = (\frac{f(t, X_t)}{g(X_t)} - \frac{1}{2} g_x(X_t))dt + dW_t. \tag{5}$$

One can directly use the Ito formula to obtain the result.

Thus, here and hereafter, we only focus on the following stochastic differential equation

$$dX_t = b(t, X_t)dt + dW_t, X_0 = x_0. \tag{6}$$

where $b(t, x)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

2.2. **Maximum likelihood estimator.** Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution that depends on one or more unknown parameters $\theta_1, \theta_2, \ldots, \theta_m$ with density $f(x; \theta_1, \theta_2, \ldots, \theta_m)$. Then the joint probability density function of $X_1, X_2, \ldots, X_n$, namely

$$\begin{aligned} L(\theta_1, \theta_2, \ldots, \theta_m) &= f(X_1; \theta_1, \theta_2, \ldots, \theta_m) f(X_2; \theta_1, \theta_2, \ldots, \theta_m) \\ &\quad \ldots f(X_m; \theta_1, \theta_2, \ldots, \theta_m) \end{aligned} \tag{7}$$

is called the likelihood function. Then

$$\hat{\theta}_i = u_i(X_1, X_2, \ldots, X_n), \quad i = 1, 2, \ldots m$$

that maximize the likelihood function are called maximum likelihood estimators of $\theta_1, \theta_2, \ldots, \theta_m$.

Example. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with density

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty, \ \theta \in R^+.$$

The Likelihood function is given by

$$\begin{aligned} L(\theta) &= L(\theta; X_1, X_2, \ldots, X_n) \\ &= (\frac{1}{\theta} e^{-X_1/\theta})(\frac{1}{\theta} e^{-X_2/\theta}) \ldots (\frac{1}{\theta} e^{-X_n/\theta}) \\ &= \frac{1}{\theta^n} \exp\Big(-\frac{1}{\theta} \sum_{i=1}^{n} X_i \Big). \end{aligned} \tag{8}$$

Therefore,

$$\ln L(\theta) = -n \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^{n} X_i.$$

A simple calculation shows that $\theta = \frac{1}{n} \sum_{i=1}^{n} X_i$ maximizes $\ln L(\theta)$, so

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i. \tag{9}$$

2.3. **Kessler estimator.** For parametric estimation, the maximum likelihood estimator is not practicable since the conditional density function is unknown, thus the likelihood function is unknown in general, and we have to turn back on other estimators which don't require conditional density to estimate parameters. Kessler estimator [16] is one of them.

Let $H = \{\phi \in \ C^2(R)$ such that $f\phi' \in L^2(\mu_\theta)$ and such that $L_\theta \phi \in L^2(\mu_\theta)\}$, where μ is the invariant measure of the ergodic stochastic process, f is the drift coefficient of the SDE and L_θ is the generator of the process. Then Kessler estimator is the one such that

$$\sum_{k=1}^{n} L_\theta h(X_{i-1}) = 0, \ h \in H,$$

where $X_i = X(t_i)$.

Example[16]**.** A generalization of the Cox-Ingersoll-Ross model is given by

$$dX_t = (\alpha + \theta X_t)dt + \sigma X_t^\gamma dW_t, \ X_0 = x_0. \tag{10}$$

We will consider here α, θ and σ to be known, and we are interested in estimating γ. Considering an estimating function $h(x) = x^2 \in H$, we obtain

$$L_\theta h(X_{i-1}) = 2X_{i-1}(\alpha + \theta X_{i-1}) + X_{i-1}^{2\gamma}. \tag{11}$$

Then the Kessler estimator is the zero of

$$2\sum_{k=1}^{n} X_{i-1}(\alpha + \theta X_{i-1}) + \sum_{k=1}^{n} X_{i-1}^{2\gamma} = 0. \tag{12}$$

2.4. **The spectral collocation method.** Spectral method is one of the big three technologies for numerical solutions of PDEs(Lloyd N. Trefethen)[25], which came in 1970s. The other two are finite difference methods (FDMs) and finite element methods (FEMs). For an ODE or PDE with smooth coefficients and simple domain, spectral methods are usually the best tool. They can often achieve ten digits of accuracy while FDMs and FEMs would get two or three. At lower accuracy, they demand less computer costs. Interested readers are referred to [8, 14, 25].

In this paper, we use the Chebyshev Spectral method solely. The idea of this method is as follows. First, we choose a certain type of Chebyshev points as our collocation points. Then interpolate a polynomial $p(x)$ on these points. At last, take derivative of the polynomial and evaluate at these points to obtain a Chebyshev differentiation matrix D_N, which plays the role of operator $\frac{d}{dt}$.

Example. Consider a simple linear equation

$$\frac{dx_t}{dt} = b(t, x_t), \quad x(-1) = 0,\ t \in [-1, 1]. \tag{13}$$

Assume the uniqueness and existence of the solution. Spectral differentiation is carried out like this.

- *Choose $N+1$ Chebyshev Gauss-Lobatto nodes*

$$t_{N-j} = \cos\frac{j\pi}{N}, j = 0, , \cdots, N. \tag{14}$$

- *Let $p(t)$ be the unique polynomial of degree N with $p(-1) = 0$ and $p(t_j) = x_j, 1 \le j \le N$, where the vector $X = (x_0, \cdots, x_N)$ is the unknown.*
- Set $w_j = p^{'}(t_j)$.

Then, $w = D_N X$. A closed form of D_N is available in [25] (page 53). In particular, the Chebyshev differentiation matrix on any interval $[a, b]$ is of the form $\frac{2}{b-a} D_N$.

For this example, the differentiation matrix is D_N itself. Let us find D_2 as an illustration. For $N = 2$, collocation points are $t_0 = -1, t_1 = 0$ and $t_2 = 1$. Therefore, the interpolant is

$$p(t) = \frac{1}{2}t(t-1)x_0 + (1+t)(1-t)x_1 + \frac{1}{2}t(1+t)x_2.$$

Then

$$p'(t) = (t - \frac{1}{2})x_0 - 2tx_1 + (t + \frac{1}{2})x_2.$$

Hence, the differentiation matrix is

$$\begin{aligned} D_2 &= \begin{pmatrix} t-\frac{1}{2}|_{t=-1} & -2t|_{t=-1} & t+\frac{1}{2}|_{t=-1} \\ t-\frac{1}{2}|_{t=0} & -2t|_{t=0} & t+\frac{1}{2}|_{t=0} \\ t-\frac{1}{2}|_{t=1} & -2t|_{t=1} & t+\frac{1}{2}|_{t=1} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{pmatrix}. \end{aligned} \tag{15}$$

In the original differential equation, however, x_0 is fixed at zero. This implies that the first column of D_N has no effect (since multiply by zeros) and first row has no effect either (since ignored).

$$\text{ignored} \rightarrow \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix} = \left(\begin{array}{c|c} & \\ \hline & \tilde{D}_N \end{array}\right) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} \leftarrow \text{zeroed} \tag{16}$$

Here, $\tilde{D}_N$ is obtained by removing the first row and first column of D_N. For simplicity of notation, we denote $X = (X_1, \cdots, X_N)^T$ again. Then, we solve the nonlinear system

$$\tilde{D}_N X = b(T, X), \tag{17}$$

where $T = [t_1, \cdots, t_N]^T$.

Now, we use this method to solve SDEs of the form (6). First, we take a transform $Y_t = X_t - x_0$ to vanish the initial condition. Since D_N plays the role of $\frac{d}{dt}$, we rewrite the equation as

$$\frac{d}{dt} y_t = b(t, y_t + x_0) + \frac{d}{dt} W_t. \tag{18}$$

Let Y be the spectral approximation and replace $\frac{d}{dt}$ with D_N, then one achieves

$$\tilde{D}_N Y = b(t, Y + x_0) + \tilde{D}_N W. \tag{19}$$

Multiplying both sides by $\tilde{D}_N^{-1}$, one easily obtain

$$Y = \tilde{D}_N^{-1} b(t, Y + x_0) + W. \tag{20}$$

Remark 1. D_N itself is singular, but $\tilde{D}_N$ is invertible.

Remark 2. To find the derivative of function $u(t)$ on Chebyshev points, $D_N u$ is of high accuracy only if $u(t)$ is smooth enough. But $W(t)$ is a nowhere differentiable process, $D_N W$ behaves very bad. However, if the coefficient of diffusion term of SDE is a constant, we can avoid $D_N W$ as above.

The numerical approximation for the Euler's method/the spectral method is the same as the numerical approximation for the equation,

$$dX_t = b(t, X_t)dt + dh_t, X_0 = x_0, \quad t \in [0, T]. \tag{21}$$

where h_t is the interpolated polynomial on equidistant/Chebyshev points on the interval $[0, T]$ for W_t. It is well-known that the spectral approximation for the above equation is much more accurate than that for the Euler 's method. Thereby, we can gain a highly accurate result.

3. **Numerical experiment.** In this part, we compare the behavior of the spectral method with the Shoji-Ozaki method and the Euler's method. First, we introduce the definition of strong convergence and the Shoji-Ozaki scheme.

Definition 3.1. Let x_n be the numerical approximation to true solution $x(t_n)$ at time t_n; then x_n is said to converge strongly to x if there exists a $C > 0$ independent of the mesh size and $\delta > 0$ such that

$$\mathbb{E}(|x(t_n) - x_n|) \to 0, \quad h \in (0, \delta). \tag{22}$$

The Shoji-Ozaki scheme for the equation

$$dx_t = f(x_t, t)dt + g(x_t)dW_t,$$

is [24] for $t \in [s, s + \Delta s)$

$$x_t = x_s + \frac{f(x_s, s)}{L_s}(e^{L_s(t-s)} - 1) + \frac{M_s}{L_s^2}\{e^{L_s(t-s)} - 1 - L_s(t-s)\} + \sigma \int_s^t e^{L_s(t-u)} dw(u), \tag{23}$$

where,

$$\begin{aligned} L_s &= \frac{\partial f}{\partial x}(x_s, s) \\ M_s &= \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x_s, s) + \frac{\partial f}{\partial t}(x_s, s). \end{aligned} \tag{24}$$

Note that the stochastic term in the above scheme follows the Normal distribution with mean 0 and variance $\mathrm{Var}(x_t)$, where

$$\mathrm{Var}(x_t) = \sigma^2\left(\frac{e^{2L_s(t-s)} - 1}{2L_s}\right). \tag{25}$$

We substitute the stochastic integration term by the Normal distribution in simulation. This is a tractable trick in parameter estimation, however, the Normal distribution can not be expressed by the original Wiener process, which leads to the failure of strong approximation of the solution process. Therefore, we exclude this method for Example 1.

Since we are able to express the exact solutions of the first example numerically, we find errors of our approximation for this case. On the contrary, we can not express the exact solutions of example 2-4 numerically, and we have to turn to parameter estimation to compare the accuracy of different methods. In the first example, we solve the equation on the interval $[0, 2]$. For both methods (the Euler method and the spectral method), we collocate $N+1$ points on the interval. Hence, there exists N subintervals. The difference is that for the Euler's method, these subintervals are equidistant and for the spectral method, they are not (the distance between successive Chebyshev points is $\sqrt{1-x^2}/N$, $x \in [-1, 1]$ [21]), see Figure 1. In Example 2-4, we solve equations on the interval [0,1000] respectively with mesh size 1. Then we partition each subinterval $[i-1, i]$ $(i = 1, 2, \dots, 1000)$ into N pieces as in Example 1 and collect discrete observations at the right endpoint of each subinterval, see Figure 1.

Example 1. We solve the SDE

$$\begin{cases} dX(t) = e^t dt + dW(t) \\ X(0) = 0. \end{cases} \tag{26}$$

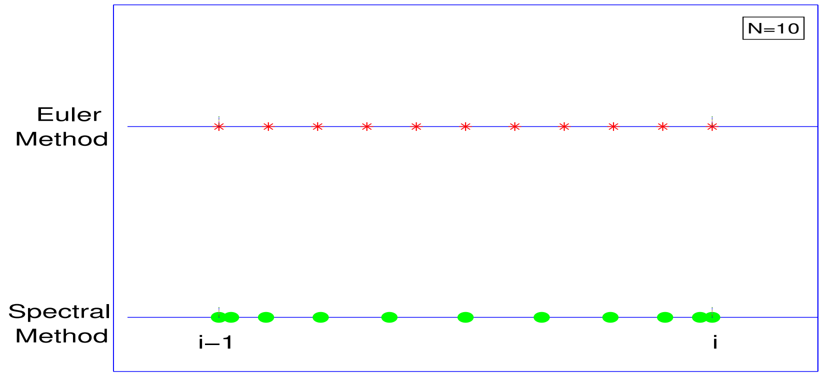


FIGURE 1. An illustration of collocation points for the Euler method and the spectral method on $[i-1,i]$

on $[0,2]$ with solution $X(t)=e^t-1+W(t)$. We repeat our code 10000 times and take the arithmetic mean of $\max|X_{t_i}-X(t_i)|,\ i=1,\cdots,N+1$ as its expectation. Numerical results by different methods are shown in Table 1 and right part of Figure 1. Clearly, the numerical solution of the spectral method converges strongly to the true solution and much faster than the Euler's method.

TABLE 1. Errors for Example 1

	E max $\lvert X_{t_i}-X(t_i)\rvert$	
N	The Euler's method	The Spectral method
3	2.3646	1.5283e-01
4	1.7298	1.4428e-02
5	1.3628	1.1560e-03
6	1.1239	8.0267e-05
7	0.9561	4.9105e-06

Example 2.(Ornstein-Uhlenbeck Model)[22] We consider the process solves the stochastic differential equation

$$dX_t=(\theta_1-\theta_2X_t)dt+\theta_3dW_t,\quad X(0)=x_0, \tag{27}$$

where $\theta_1,\theta_2\in R,\theta_3\in R^+$ and it is ergodic for $\theta_2>0$. In particular, the conditional density $P_\theta(t,\dot{|}x_0)$ is the density of a Gaussian law with mean and variance respectively

$$m(t,x)=\frac{\theta_1}{\theta_2}+(x_0-\frac{\theta_1}{\theta_2})e^{-\theta_2t}, \tag{28}$$

$$v(t,x)=\frac{\theta_3^2(1-e^{-2\theta_2t})}{2\theta_2}. \tag{29}$$

Here, $(\theta_1,\theta_2,\theta_3)=(3,1,1)$ are parameters to be estimated. The Euler's method and the Shoji-Ozaki's method are the same for this case.

Different methods together with the maximum likelihood estimation with step size 1 on the interval [0, 1000] give Table 2. Profiles for the three parameters of

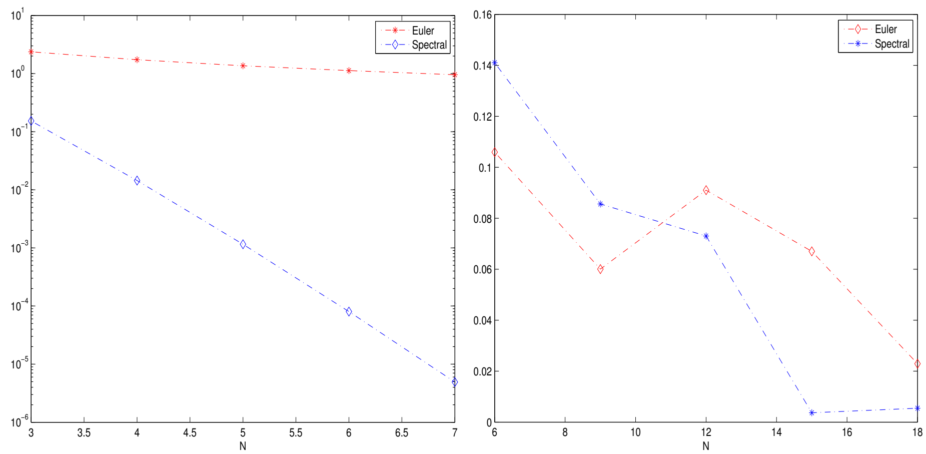


FIGURE 2. (left) Error for example 1; (right) Error for example 5.

TABLE 2. Estimates of $(\theta_1, \theta_2, \theta_3)$ in Example 2

95% confidence interval	The Euler's method	The spectral method
θ_1	[2.1012, 2.9009]	[2.5546, 3.4989]
θ_2	[0.7070, 0.9626]	[0.8569, 1.1705]
θ_3	[NA, 1.9574]	[0.9378, 1.0771]

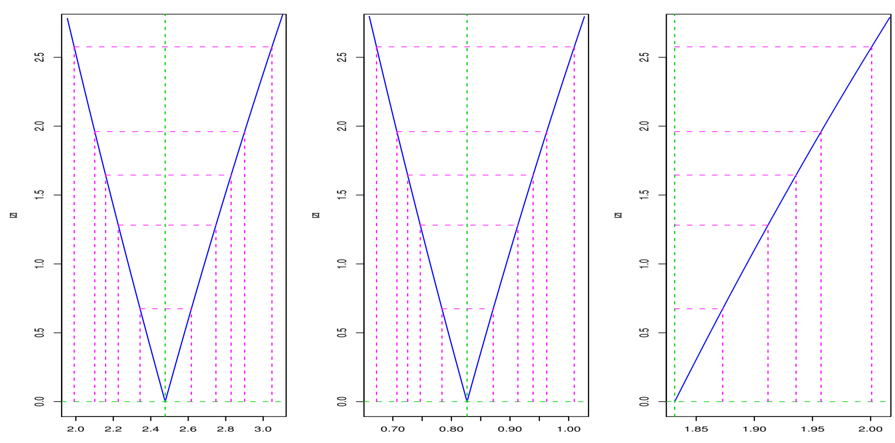


FIGURE 3. Profile likelihood for θ of Example 2: the Euler's method

different methods is in Figure 3 and Figure 4. Clearly, the confidence interval of the Euler's method does not capture the true parameters. Moreover, the confidence interval for θ_3 does not exist. But the spectral catches each one of them.

Example 3.(The Cox-Ingrsoll-Ross Model(CIR))[22] Our third example is the CIR model solution to the equation

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 \sqrt{X_t} dW_t, \quad X(0) = x_0 > 0, \tag{30}$$

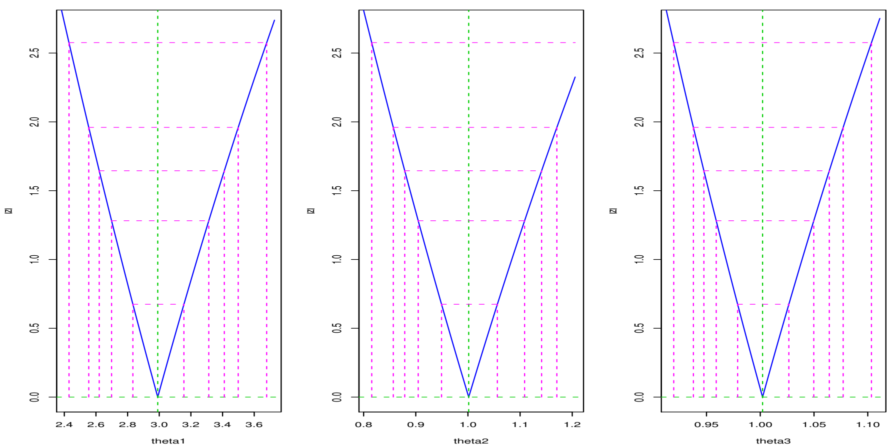


FIGURE 4. Profile likelihood for θ of Example 2: the Spectral method

where $\theta_1, \theta_2, \theta_3 \in R^+$. After the Lamperti transform $F(X_t) = \frac{2}{\theta_3}\sqrt{X_t}$, we obtain

$$dY_t = \left(\frac{2\theta_1}{\theta_3^2 Y_t} - \frac{\theta_2}{2} Y_t - \frac{1}{2Y_t}\right)dt + dW_t. \tag{31}$$

If $2\theta_1 > \theta_3^2$, the process is strictly positive. Under this condition, the conditional density $P_\theta(t, |x_0)$ follows χ^2 distribution with

$$P_\theta(t, y|x) = ce^{-u-v}\left(\frac{u}{v}\right)^{q/2} I_q(2\sqrt{uv})$$

where

$$c = \frac{2\theta_2}{\theta_3^2(1 - e^{-\theta_2 t})}, \quad q = \frac{2\theta_1}{\theta_3^2} - 1$$

$$u = cxe^{-\theta_2 t}, \quad v = cy$$

$$I_q(x) = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+q}\frac{1}{k!\Gamma(k+q+1)}, \quad x \in R$$

We use meshes with step size 1 on [0, 1000] to estimate $(\theta_1, \theta_2, \theta_3)$ by maximum likelihood estimation for all three methods. Here, we take $(\theta_1, \theta_2, \theta_3) = (0.2, 0.05, 0.3)$.

TABLE 3. Estimates of $(\theta_1, \theta_2, \theta_3)$ in Example 3

The 95% confidence interval	The Euler's method	The Shoji-Ozaki method	The spectral method
θ_1	[0.1907, 0.3649]	[0.3133, 0.5412]	[0.1841, 0.3605]
θ_2	[0.0480, 0.0966]	[0.0617 ,0.1164]	[0.0424, 0.0888]
θ_3	[0.2767, 0.3031]	[0.4598 ,0.5040]	[0.2912, 0.3188]

Both the Euler's method and the spectral collocation method captures all parameters for this example except the Shoji-Ozaki method.

Example 4.(A generalized Cox-Ingrsoll-Ross Model)[16] We consider the same model as in subsection 2.3 with $\alpha = 10, \theta = -1$ and $\sigma = 1$. We assume that

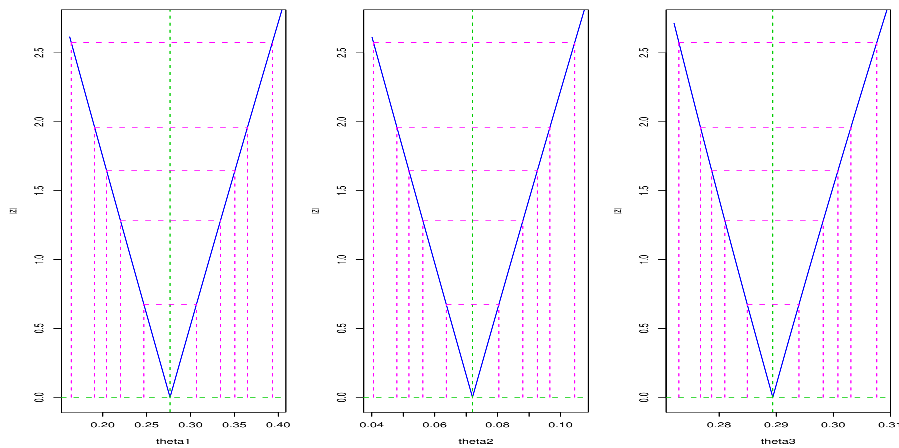


FIGURE 5. Profile likelihood for θ of Example 3: the Euler's method

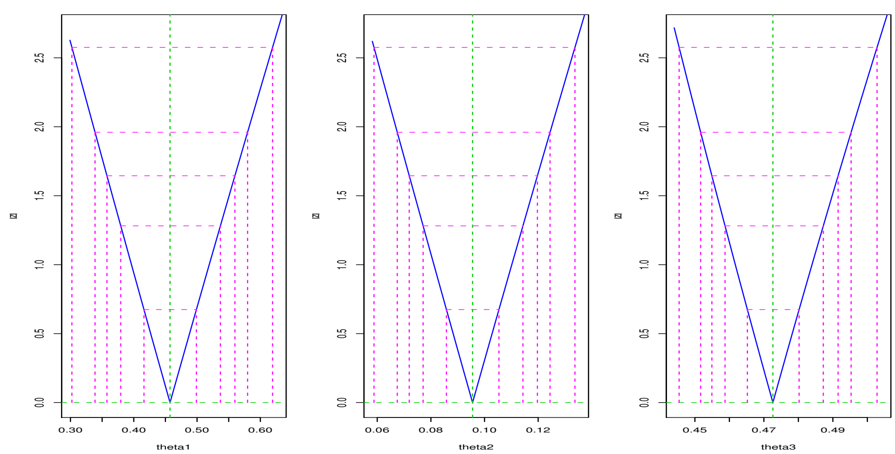


FIGURE 6. Profile likelihood for θ of Example 3: the Shoji-Ozaki method

the true value for γ is $1/2$. As in [16], we apply bisection method for 200 realizations of the trajectory to solve the equation

$$2\sum_{k=1}^{n} X_{i-1}(\alpha+\theta X_{i-1})+\sum_{k=1}^{n} X_{i-1}^{2\gamma}=0. \tag{32}$$

By these three methods, Kessler estimator gives Table 4, from which we can conclude that the spectral method yields the most accurate mean though the standard deviation is a little bigger than that of the Euler's method while the Shoji-Ozaki method converges very slowly compared with the other two.

Example 5. Consider the Langevin equation of a particle of unit mass,

$$\begin{cases} dq = pdt, \\ dp = -\gamma pdt - \nabla F(q)dt + \sigma dw, t \in [0, 64]. \end{cases} \tag{33}$$

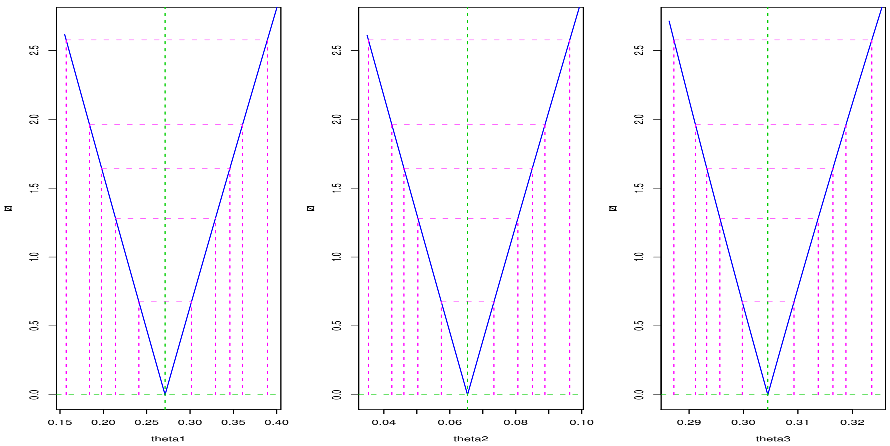


FIGURE 7. Profile likelihood for θ of Example 3: the Sectral method

TABLE 4. Estimation of γ in Example 4

	The Euler's Method		The Shoji-Ozaki Method		The Spectral Method	
N	Mean	S.D.	Mean	S.D.	Mean	S.D.
3	0.5348	0.0459	0.4535	0.0599	0.4983	0.0532
4	0.5199	0.0452	0.4500	0.0601	0.4946	0.0551
5	0.5171	0.0499	0.4579	0.0569	0.4905	0.0540
6	0.5114	0.0467	0.4521	0.0674	0.4944	0.0492
7	0.5097	0.0540	0.4683	0.0629	0.4924	0.0517

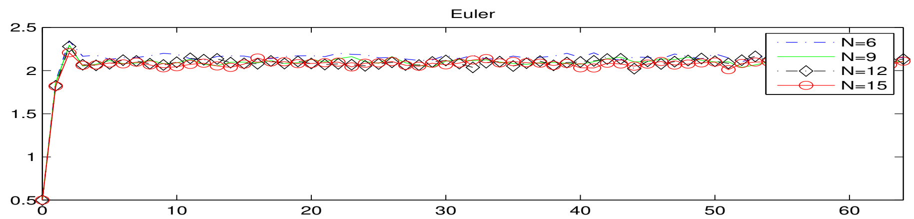
Here $\gamma > 0$ to ensure a damped-driven Hamiltonian. For $\sigma = \sqrt{2}$, there exists an invariant measure for (p, q) with density

$$\rho(p,q) \propto \exp\left\{ -\gamma\left[\frac{p^2}{2} + F(q)\right]\right\}. \tag{34}$$

We specify the Langevin equation with $F(q) = \frac{1}{4}(q^2-1)^2, \gamma = 1$, and initial data $p_0 = q_0 = 1/2$. After a simple calculation, we can obtain that for sufficiently large T, $\mathbb{E}(p(T)^2 + q(T)^2) \approx 2.0418$. In this example, we approximate this quantity by averaging over 4000 paths at $T = 64$ for the Euler method and the spectral method. The error of this quantity compared with 2.0418 for different N is given in Figure 2, and the time evolution of averaged $(p(t_n)^2 + q(t_n)^2)$ is given in Figure 8.

4. **Conclusion.** We apply the Chebyshev spectral collocation method to solve stochastic differential equations of the form (1) and compare the performance of this method with those of the Euler's method and the Shoji-Ozaki method. It is clear from our illustrations that,

- It can gain high accuracy for the true solution even if the number of points used is small;
- In the whole process, only the difference matrix D_N is introduced after the Lamperti transformation. Thus, the method is rather concise and neat.



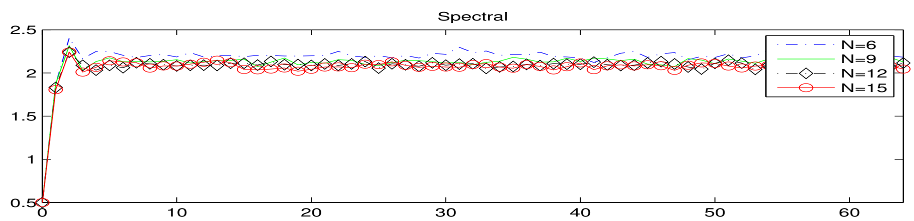


FIGURE 8. Example 5: $\mathbb{E}(p(t_n)^2 + q(t_n)^2)$ against t_n.

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E-mail address: canhuang@math.msu.edu
E-mail address: ag7761@wayne.edu