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## Analysis of a $p$ -version finite volume method for 1D elliptic problems

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### ABSTRACT

In this work, we present and analyze a  $p$ -version finite volume method (FVM) for elliptic problems in the one dimensional setting. Under some regularity assumptions of the exact solution, it is shown that the  $p$ -version FV solution converges with exponential rates under  $H^1$ ,  $L^2$  and  $L^\infty$ -norms. Superconvergence properties at nodal, Lobatto and Gauss points have been also discussed. Numerical results are presented to support our theoretical findings.

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## 1. Introduction

As a popular numerical method for partial differential equations in practice, the FVM has been intensively studied during the past several decades, see [1–13] for an incomplete list of references. However, all theoretical studies of polynomial based FVM belong to the catalog of the  $h$ -version, i.e., convergence is achieved by decreasing the mesh size, while the polynomial degree of the trial space is fixed. To the best of our knowledge, no paper on the  $p$ -version finite volume scheme has been published in the literature yet. Given the fact that many works have been done on the  $p$ -version finite element method (FEM) and spectral element methods (cf., [14–20]), it is meaningful to study the  $p$ -version FVM. The main advantage in introducing the  $p$ -version FVM is in the flux conservation of the FVM. In many applications, flux conservation is also desired. Thanks to the equivalent variational formulation, the energy is also preserved the same as the FEM.

Very recently, FV schemes of any fixed order ( $h$ -version) have been constructed and analyzed for one dimensional second order differential equations (cf., [21,22]). In this paper, we present and analyze a  $p$ -version FV scheme for 1D elliptic problems. We construct a primal partition which contains a fixed finite number of subintervals. The corresponding trial space is then defined as the space of piecewise polynomials with respect to the primal partition. The dual partition consists of control volumes which are constructed with the Gauss points in each subinterval. The test space consists of piecewise constants with respect to the dual partition. A fundamental difference between the  $p$ -version scheme and the  $h$ -version scheme in [21] is that the convergence is realized by increasing the polynomial degree  $p$ .

Our analysis is under the framework of the Petrov–Galerkin method. We first establish the inf–sup condition and the continuity property for the FV bilinear form with the help of a one-to-one mapping from the trial space to the test space introduced in [21]. We then show that our FV solution converges to the exact solution exponentially under some regularity assumptions.

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We further investigate superconvergence properties of the  $p$ -version FVM. Note that studies on superconvergence properties of the  $p$ -version are not as many as those of the  $h$ -version. We refer to [23] for the spectral finite element method and [20] for the spectral collocation method. In this paper, superconvergence properties of the derivative approximation at Gauss points and the function values approximation at Lobatto points are considered. In general, the error bounds we obtain for our  $p$ -version FV scheme at Gauss and interior Lobatto points are the same as those for the counterpart spectral collocation method proposed in [20]. While in some special cases, the convergence rate of the derivative approximation at Gauss points can be higher than the counterpart  $p$ -version FEM, similar as in [21] for the  $h$ -version case.

The rest of the paper is organized as follows. In Section 2, we develop a  $p$ -version finite volume method for 1D general elliptic problems. In Section 3, we prove the inf-sup condition and the continuity property of our FV scheme and use them to establish exponential convergence rates under  $H^1$ ,  $L^2$  and  $L^\infty$ -norms. In Section 4, we investigate superconvergence properties at Gauss and Lobatto points. Numerical results supporting our theoretical findings are provided in Section 5.

Throughout this paper, we adopt standard notations for Sobolev spaces such as  $W^{m,q}(D)$  on sub-domain  $D \subset \Omega$  equipped with the norm  $\|\cdot\|_{m,q,D}$  and semi-norm  $|\cdot|_{m,q,D}$ . When  $D = \Omega$ , we omit the index  $D$ ; and if  $q = 2$ , we set  $W^{m,q}(D) = H^m(D)$ ,  $\|\cdot\|_{m,q,D} = \|\cdot\|_{m,D}$ , and  $|\cdot|_{m,q,D} = |\cdot|_{m,D}$ . Notation “ $A \lesssim B$ ” implies that  $A$  can be bounded by  $B$  multiplied by a constant independent of the polynomial degree. “ $A \sim B$ ” stands for “ $A \lesssim B$ ” and “ $B \lesssim A$ ”.

## 2. A $p$ -version finite volume method

Consider the two-point boundary value problem:

$$\begin{aligned} -(\alpha(x)u'(x))' + \beta(x)u'(x) + \gamma(x)u(x) &= f(x), \quad \forall x \in \Omega = (a, b), \\ u(a) = u(b) &= 0, \end{aligned} \tag{2.1}$$

where the coefficients  $\alpha, \beta, \gamma \in L^\infty(\bar{\Omega})$  satisfy  $\alpha \geq \alpha_0 > 0$ ,  $\gamma - \frac{1}{2}\beta' \geq \lambda_0 > 0$ ,  $f$  is a real-valued function defined on  $\bar{\Omega}$ .

Let  $a = x_0 < x_1 < \dots < x_N = b$  be  $N + 1$  distinct points on  $\bar{\Omega}$  and  $\mathbb{Z}_k = \{1, \dots, k\}$  for all positive integers  $k$ . Let  $h_i = x_i - x_{i-1}$ ,  $h = \max_i h_i$ ,  $i \in \mathbb{Z}_N$  and

$$\mathcal{P} = \{\tau_i : \tau_i = (x_{i-1}, x_i), i \in \mathbb{Z}_N\}$$

be a partition generated by  $x_i$ ,  $i \in \mathbb{Z}_N$ .

We define a function space

$$U_p = U_{\mathcal{P}, p_1, \dots, p_N} = \{v \in C(\Omega) : v|_{\tau_i} \in \mathbb{P}_{p_i}, \forall i \in \mathbb{Z}_N, v(a) = v(b) = 0\},$$

where for all nonnegative integers  $r \geq 0$ ,  $\mathbb{P}_r$  is the set of polynomials of degree no more than  $r$  and  $p_i$ ,  $i \in \mathbb{Z}_N$  is a nonnegative integer,  $p = \min_{i \in \mathbb{Z}_N} p_i$ .

For any integer  $r \geq 0$ , let  $G_{r,1}, \dots, G_{r,r}$  be  $r$  Gauss points, i.e., zeros of Legendre polynomial of  $r$ th degree, in the interval  $[-1, 1]$ . The  $p_i$  Gauss points in each interval  $\tau_i$  are defined as the affine transformations of  $G_{p_i,j}$  to  $\tau_i$ , that is,

$$g_{i,j} = \frac{1}{2}(x_i + x_{i-1} + h_i G_{p_i,j}), \quad j \in \mathbb{Z}_{p_i}.$$

With these Gauss points, we construct another partition

$$\mathcal{P}' = \{\tau'_{1,0}\} \cup \{\tau'_{i,j} : (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}\},$$

where

$$\tau'_{1,0} = [a, g_{1,1}], \quad \tau'_{i,j} = [g_{i,j}, g_{i,j+1}], \quad (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}$$

with

$$g_{i,p_i+1} = g_{i+1,1}, \quad \forall i \in \mathbb{Z}_{N-1}, \quad g_{N,p_N+1} = b.$$

We define the space of piecewise constants as

$$V_p = \text{Span} \{ \psi_{i,j} : (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{\tilde{p}_i} \},$$

where  $\psi_{i,j} = \chi_{[g_{i,j}, g_{i,j+1}]}$  is the characteristic function on the interval  $\tau'_{i,j}$ ,  $\tilde{p}_i = p_i$ ,  $i \in \mathbb{Z}_{N-1}$  and  $\tilde{p}_N = p_N - 1$ .

The finite volume solution of (2.1) is a function  $u_p \in U_p$  satisfying local conservation laws on all volumes  $\tau'_{i,j}$ ,  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{\tilde{p}_i}$ , that is:

$$\alpha(g_{i,j})u'_p(g_{i,j}) - \alpha(g_{i,j+1})u'_p(g_{i,j+1}) + \int_{\tau'_{i,j}} (\beta(x)u'_p(x) + \gamma(x)u_p(x))dx = \int_{\tau'_{i,j}} f(x)dx. \tag{2.2}$$

Note that any  $w_p \in V_p$  can be represented as

$$w_p = \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} w_{i,j} \psi_{i,j},$$

where  $w'_{i,j}$ s are constants. Multiplying (2.2) with  $w_{i,j}$  and then summing up for all  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{\tilde{p}_i}$ , we obtain

$$\sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} w_{i,j} \left( \alpha(g_{i,j})u'_p(g_{i,j}) - \alpha(g_{i,j+1})u'_p(g_{i,j+1}) + \int_{\tau'_{i,j}} (\beta(x)u'_p(x) + \gamma(x)u_p(x))dx \right) = \int_a^b f(x)w_p(x)dx,$$

or equivalently,

$$\sum_{i=1}^N \sum_{j=1}^{p_i} [w_{i,j}] \alpha(g_{i,j})u'_p(g_{i,j}) + \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} w_{i,j} \left( \int_{\tau'_{i,j}} (\beta(x)u'_p(x) + \gamma(x)u_p(x))dx \right) = \int_a^b f(x)w_p(x)dx,$$

where  $[w_{i,j}] = w_{i,j} - w_{i,j-1}$ ,  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}$  is the jump of  $w$  at the point  $g_{i,j}$  with  $w_{1,0} = 0$ ,  $w_{N,p_N} = 0$  and  $w_{i,0} = w_{i-1,p_{i-1}}$ ,  $2 \leq i \leq N$ .

We define the FV bilinear form for all  $v \in H_0^1(\Omega)$ ,  $w_p \in V_p$  by

$$a_p(v, w_p) = \sum_{i=1}^N \sum_{j=1}^{p_i} [w_{i,j}] \alpha(g_{i,j})v'(g_{i,j}) + \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} w_{i,j} \left( \int_{g_{i,j}}^{g_{i,j+1}} (\beta(x)v'(x) + \gamma(x)v(x))dx \right). \tag{2.3}$$

The  $p$ -version finite volume method for solving (2.1) reads as: find  $u_p \in U_p$  such that

$$a_p(u_p, w_p) = (f, w_p), \quad \forall w_p \in V_p. \tag{2.4}$$

**Remark 2.1.** Naturally, there is another possibility for the dual partition, namely, using the Gauss–Lobatto points, in which case, one more subinterval is created at each vertex (a special Lobatto point). In order to balance the total degrees of freedom, a discontinuous Galerkin method would be a natural choice, which will be investigated in a separate work.

**Remark 2.2.** In our  $p$ -version scheme, the degrees  $p_1, \dots, p_N$  are variable while the mesh size  $h$  is fixed. However, if we increase the polynomial degree and refine the mesh at the same time, it leads to an  $hp$ -version finite volume method.

### 3. Convergence

This section is devoted to an analysis of convergence properties of the FV solution. It will be shown that under  $H^1$ ,  $L^2$  and  $L^\infty$ -norms, the FV solution converges with exponential rates. The analysis will be done under the convectonal framework of the Petrov–Galerkin method in which the inf–sup condition and the continuity property of the bilinear form (2.3) play important roles.

For simplicity, in this section, we suppose  $\beta(x) = \text{constant} = \beta_0$ ,  $\gamma(x) = \text{constant} = \gamma_0 \geq 0$ .

#### 3.1. Inf–sup property and continuity

We begin with a description of some discrete norms and semi-norms. We first define a discrete semi-norm for all  $v \in H_0^1(\Omega)$  by

$$|v|_G^2 = \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} (v'(g_{i,j}))^2 \tag{3.1}$$

and a discrete norm by

$$\|v\|_G^2 = |v|_G^2 + \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} v^2(g_{i,j}) + (v, v),$$

where  $A_{i,j}$  are weights for the  $p_i$ -point Gauss quadrature rule on the interval  $\tau_i$ . Note that the  $r$ -point Gauss quadrature rule is exact for polynomials of degree  $2r - 1$ , we have

$$|v|_G = |v|_1, \quad \forall v \in U_p. \tag{3.2}$$

Moreover, we have the following equivalence.

**Lemma 3.1.** For all  $v \in U_p$ ,

$$\|v\|_G \sim |v|_1, \tag{3.3}$$

where the hidden constant is independent of  $p$ .

**Proof.** By (3.2) and the Poincaré inequality, we only need to show that

$$\sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} v^2(g_{i,j}) \lesssim |v|_1^2. \tag{3.4}$$

Let

$$E_i = \int_{\tau_i} v^2(x) dx - \sum_{j=1}^{p_i} A_{i,j} v^2(g_{i,j}), \quad \forall i \in \mathbb{Z}_N$$

be the error of Gauss quadrature. It is shown in [24, (p. 98, (2.7.12))] that

$$E_i = h_i^{2p_i+1} \frac{(p_i!)^4}{(2p_i + 1)[(2p_i)!]^3} (v^2)^{(2p_i)}(\xi_i), \quad \xi_i \in \tau_i.$$

Note that  $v|_{\tau_i} \in \mathbb{P}_{p_i}$  and

$$(v^2)^{(2p_i)}(\xi_i) = \frac{(2p_i)!}{(p_i!)^2} |v|_{p_i, \infty, \tau_i}^2 \geq 0,$$

we have

$$E_i \geq 0$$

and thus

$$\sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} v^2(g_{i,j}) \leq \|v\|_0^2 \lesssim |v|_1^2.$$

Then (3.4) is valid, which implies the statement of the lemma.  $\square$

Next, we introduce a semi-norm and a norm in the test space. For all  $w_p = \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} w_{ij} \psi_{i,j} \in V_p$ , let

$$|w_p|_{\mathcal{P}'}^2 = \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j}^{-1} [w_{i,j}]^2, \quad \|w_p\|_{\mathcal{P}'}^2 = |w_p|_{\mathcal{P}'}^2 + \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} p_i^{-1} h_i w_{i,j}^2.$$

It is shown in [25] that for any positive integer  $r$  and  $j \in \mathbb{Z}_r$

$$1 - \frac{1}{8(r + 1/2)^2(1 - G_{r,j}^2)} \leq \pi^{-1} \left(r + \frac{1}{2}\right) (1 - G_{r,j}^2)^{-\frac{1}{2}} W_{r,j} \leq 1,$$

where  $W_{r,j}$  are weights for the  $r$ -point Gauss quadrature rule on the interval  $[-1, 1]$ . Since  $A_{i,j} = \frac{h_i}{2} W_{p_i,j}$  and for all  $w_p \in V_p$ ,

$$w_{i,j} = \sum_{l=1}^{i-1} \sum_{k=1}^{p_l} [w_{l,k}] + \sum_{k=1}^j [w_{i,k}], \quad \forall (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i},$$

we have

$$\sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} p_i^{-1} h_i w_{i,j}^2 \leq (b - a) \max_{i,j} w_{i,j}^2 \lesssim \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j}^{-1} [w_{i,j}]^2,$$

where the hidden constant in the last inequality is independent of the polynomial degrees  $p_1, \dots, p_N$ . Consequently,

$$|w_p|_{\mathcal{P}'} \sim \|w_p\|_{\mathcal{P}'}, \quad \forall w_p \in V_p. \tag{3.5}$$

To study the inf-sup property of FV bilinear form (2.3), we recall the special transformation from the trial space to the test space introduced in [21]. Let  $\Pi : U_p \rightarrow V_p$  be the mapping defined for all  $w \in U_p$  by

$$\Pi w := w_p = \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} w_{i,j} \psi_{i,j},$$

where the coefficients  $w_{i,j}$  are determined by the constraints

$$[w_{i,j}] = A_{i,j} w'(g_{i,j}), \quad \forall (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{\tilde{p}_i}. \tag{3.6}$$

It is also proved in [21] that

$$[w_{N,p_N}] = A_{N,p_N} w'(g_{N,p_N}). \tag{3.7}$$

Then we have

$$\sum_{j=1}^{p_i} A_{i,j}^{-1} [w_{i,j}]^2 = |w|_{1, \tau_i}^2, \quad \forall i \in \mathbb{Z}_N.$$

Therefore, we conclude from the definition of  $\|\cdot\|_{\mathcal{P}'}$  and  $\|\cdot\|_G$

$$\|\Pi w\|_{\mathcal{P}'} \sim |w|_1 \sim \|w\|_G, \quad \forall w \in U_p. \tag{3.8}$$

We are now ready to present and prove the inf–sup property of  $a_p(\cdot, \cdot)$ .

**Theorem 3.2.** *Suppose  $\beta(x) = \beta_0, \gamma(x) = \gamma_0$  are constants. Then*

$$\inf_{v_p \in U_p} \sup_{w_p \in V_p} \frac{a_p(v_p, w_p)}{\|v_p\|_G \|w_p\|_{\mathcal{P}'}} \geq c_0, \tag{3.9}$$

where  $c_0 > 0$  is a constant independent of  $p$ .

**Proof.** Recall the bilinear form (2.3), then we have

$$a_p(v_p, \Pi v_p) = J_1 + J_2, \quad \forall v_p \in U_p,$$

where

$$J_1 = \sum_{i=1}^N \sum_{j=1}^{p_i} [v_{i,j}] (\alpha(g_{i,j}) v_p'(g_{i,j}) - \beta(g_{i,j}) v_p(g_{i,j})),$$

$$J_2 = \sum_{i=1}^N \sum_{j=1}^{\tilde{p}_i} v_{i,j} \int_{g_{i,j}}^{g_{i,j+1}} (\gamma - \beta')(x) v_p(x) dx.$$

By (3.6) and (3.7), we have

$$J_1 = \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} \alpha(g_{i,j}) (v_p'(g_{i,j}))^2 - \beta_0 \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} v_p(g_{i,j}) v_p'(g_{i,j})$$

$$\geq \alpha_0 |v_p|_1^2 - \beta_0 \int_a^b v_p(x) v_p'(x) dx = \alpha_0 |v_p|_1^2.$$

We turn to estimating  $J_2$ . Let  $V(x) = \int_a^x v_p(s) ds, x \in (a, b)$ , then we have

$$J_2 = -\gamma_0 \sum_{i=1}^N \sum_{j=1}^{p_i} [v_{i,j}] V(g_{i,j}) = -\gamma_0 \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} v_p'(g_{i,j}) V(g_{i,j}).$$

Let the quadrature error

$$E_i = \int_{\tau_i} v_p'(x) V(x) dx - \sum_{j=1}^{p_i} A_{i,j} v_p'(g_{i,j}) V(g_{i,j}).$$

For all  $x \in \tau_i$ , note that  $v_p'(x) \in \mathbb{P}_{2p_i}$ , we have

$$(v_p' V)^{(2p_i)}(x) = \frac{p_i}{p_i + 1} (v_p^2)^{(2p_i)}(x) \geq 0.$$

Then  $E_i \geq 0, \forall i \in \mathbb{Z}_N$  and thus

$$J_2 = -\gamma_0 \int_a^b v_p'(x) V(x) dx + \gamma_0 \sum_{i \in \mathbb{Z}_N} E_i \geq \gamma_0 \|v_p\|_0^2 \geq 0.$$

Combining  $J_1$  with  $J_2$ , we obtain

$$a_p(v_p, \Pi v_p) \geq \alpha_0 |v_p|_1^2.$$

By (3.3) and (3.8), for all  $v_p \in U_p$ ,

$$\sup_{w_p \in V_p} \frac{a_p(v_p, w_p)}{\|w_p\|_{\mathcal{P}'}} \geq \frac{a_p(v_p, \Pi v_p)}{\|\Pi v_p\|_{\mathcal{P}'}} \geq c_0 \|v_p\|_G,$$

where  $c_0$  is independent of  $p$ . The inf–sup property (3.9) follows.  $\square$

**Remark 3.3.** The inf–sup condition (3.9) is different from that in [21] since here the lower bound  $c_0$  is independent of the polynomial degree  $p$ , while the lower bound of the inf–sup condition in [21] is independent of the mesh size  $h$ .

Next, we analyze the continuity of  $a_p(\cdot, \cdot)$ .

**Theorem 3.4.** *The finite volume bilinear form  $a_p(\cdot, \cdot)$  is variationally exact:*

$$a_p(u, w_p) = (f, w_p), \quad \forall w_p \in V_p \tag{3.10}$$

and continuous:

$$a_p(v, w_p) \leq C \|v\|_G \|w_p\|_{\mathcal{P}'}, \quad \forall v \in H_0^1(\Omega), w_p \in V_p, \tag{3.11}$$

where  $C$  is independent of  $p$ .

**Proof.** Multiplying (2.1) with any function  $w_p \in V_p$  and then using the Newton–Leibniz formula on each control volume  $[g_{i,j}, g_{i,j+1}]$ ,  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{\bar{p}_i}$ , we obtain (3.10) directly.

On the other hand, by the Cauchy–Schwarz inequality, for all  $v \in H_0^1(\Omega)$ ,  $w_p \in V_p$ , we have

$$\begin{aligned} a_p(v, w_p) &= \sum_{i=1}^N \sum_{j=1}^{p_i} [w_{i,j}] (\alpha(g_{i,j}) v'(g_{i,j}) - \beta(g_{i,j}) v(g_{i,j})) + \sum_{i=1}^N \sum_{j=1}^{\bar{p}_i} w_{i,j} \left( \int_{g_{i,j}}^{g_{i,j+1}} (\gamma(x) - \beta'(x)) v(x) dx \right) \\ &\leq C \|v\|_G \|w_p\|_{\mathcal{P}'}. \end{aligned}$$

The proof of (3.11) is completed.  $\square$

### 3.2. Estimates under $H^1, L^2$ and $L^\infty$ -norms

We begin with some preliminaries. Let  $P_k, k \geq 1$  be the Legendre polynomial of degree  $k$  and denote

$$\phi_j(t) = \int_{-1}^t P_{j-1}(s) ds, \quad \forall j \geq 2, t \in [-1, 1]$$

with  $\phi_0(t) = 1, \phi_1(t) = t$ . For all  $v(x) \in H_0^1(\Omega)$  and  $x \in \tau_i, i \in \mathbb{Z}_N, v(x)$  has the following expansion

$$v(x) = \sum_{j=0}^{\infty} v_j \phi_j(t),$$

where  $t = (2x - x_i - x_{i-1})/h_i$  and

$$v_0 = \frac{v(x_i) + v(x_{i-1})}{2}, \quad v_1 = \frac{v(x_i) - v(x_{i-1})}{2}, \tag{3.12}$$

$$v_j = \frac{2j-1}{2} \int_{x_{i-1}}^{x_i} v'(x) P_{j-1}(t) dx, \quad j \geq 2. \tag{3.13}$$

We define the truncated expansion of  $v$  on each interval  $\tau_i$  by

$$\pi_{p_i}^i v := \sum_{j=0}^{p_i} v_j \phi_j(t)$$

and the projection operator  $\pi_p : H_0^1(\Omega) \longrightarrow U_p$  as

$$\pi_p v|_{\tau_i} := \pi_{p_i}^i v, \quad i \in \mathbb{Z}_N.$$

When  $v$  satisfies the regularity condition

$$\|v\|_{k,\infty} \lesssim M^k, \quad \forall k \geq 0 \tag{3.14}$$

for some constant  $M > 0$  and the polynomial degree  $p$  satisfies

$$(2p+1)(2p+3) > 2M^2, \tag{3.15}$$

by the same argument as in [20], we derive

$$\|v - \pi_p v\|_0 \lesssim p^{-\frac{1}{2}} \left( \frac{heM}{4p} \right)^{p+1}, \quad |v - \pi_p v|_1 \lesssim p^{\frac{1}{2}} \left( \frac{heM}{4p} \right)^{p+1}, \tag{3.16}$$

$$\|v - \pi_p v\|_{0,\infty} \lesssim \left( \frac{heM}{4p} \right)^{p+1}, \quad |v - \pi_p v|_{1,\infty} \lesssim p \left( \frac{heM}{4p} \right)^{p+1}, \tag{3.17}$$

and

$$|(v - \pi_p v)'(g_{i,j})| \lesssim \left(\frac{heM}{4p}\right)^{p+2}, \quad (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}. \quad (3.18)$$

We are ready to present convergence properties of the FV solution.

**Theorem 3.5.** Suppose  $\beta(x) = \beta_0$ ,  $\gamma(x) = \gamma_0$  are constants. Let  $u$  and  $u_p$  be the solutions of (2.1) and (2.4), respectively. Then

$$\|u - u_p\|_G \leq \frac{C}{c_0} \inf_{v_p \in U_p} \|u - v_p\|_G, \quad (3.19)$$

where  $c_0, C$  are the same as in (3.9) and (3.11).

Consequently, if  $u$  satisfies the regularity condition (3.14) and  $p$  satisfies (3.15), we have

$$|u - u_p|_1 \lesssim p^{-\frac{1}{2}} e^{-\sigma p}, \quad \|u - u_p\|_0 \lesssim e^{-\sigma(p+1)} \quad (3.20)$$

and

$$\|u - u_p\|_{0,\infty} \lesssim e^{-\sigma(p+1)}, \quad (3.21)$$

where  $\sigma = -\ln \frac{heM}{4p}$ .

**Proof.** By (3.9)–(3.11), we have for all  $v_p \in U_p$

$$\|u - u_p\|_G \leq \|u - v_p\|_G + \|u_p - v_p\|_G \leq \left(1 + \frac{C}{c_0}\right) \|u - v_p\|_G.$$

Then (3.19) follows by a similar argument as in Xu and Zikatanov [26].

By the triangular inequality, (3.2) and (3.19), we obtain

$$|u - u_p|_1 \leq |u - v_p|_1 + |v_p - u_p|_G \lesssim |u - v_p|_1 + \|u - v_p\|_G, \quad \forall v_p \in U_p.$$

Choosing  $v_p = \pi_p u$ , we have, from (3.16)–(3.18)

$$\begin{aligned} |u - u_p|_1 &\lesssim |u - \pi_p u|_1 + \|u - \pi_p u\|_G \\ &\lesssim p^{-\frac{1}{2}} \left(\frac{heM}{4p}\right)^p = p^{-\frac{1}{2}} e^{-\sigma p}. \end{aligned}$$

On the other hand, by (3.19) and the fact that  $\|\cdot\|_0 \leq \|\cdot\|_G$ , we obtain

$$\|u - u_p\|_0 \lesssim \|u - \pi_p u\|_G \lesssim \left(\frac{heM}{4p}\right)^{p+1}.$$

Then (3.20) is completed. We now consider  $\|u - u_p\|_{0,\infty}$ . For all  $x \in \Omega$ , there holds

$$\begin{aligned} |(\pi_p u - u_p)(x)| &= \left| \int_a^x (\pi_p u - u_p)'(t) dt \right| \lesssim \|\pi_p u - u_p\|_G \\ &\lesssim \|u - \pi_p u\|_G \lesssim \left(\frac{heM}{4p}\right)^{p+1}. \end{aligned}$$

Then (3.21) follows from the triangular inequality and (3.17). The proof is completed.  $\square$

**Remark 3.6.** When  $\beta = 0$ , by the same argument as above, we obtain

$$\|u_p - \pi_p u\|_0 \leq \|u_p - \pi_p u\|_G \lesssim |u - \pi_p u|_G + \|u - \pi_p u\|_0.$$

In light of (3.16) and (3.18), we have

$$\|u - u_p\|_0 \lesssim |u - \pi_p u|_G + \|u - \pi_p u\|_0 \lesssim p^{-\frac{1}{2}} \left(\frac{heM}{4p}\right)^{p+1}. \quad (3.22)$$

#### 4. Superconvergence

In this section, we shall present superconvergence properties of the FV solution at some special points, that is, Gauss and Lobatto points. This motivates the following definition (see, [20]).



**Definition 4.1.** The error  $u - u_p$  is said to have convergence at a set of points  $\{\xi_{p,j}\}$  with order  $\lambda > 0$  if there exists a constant  $c > 0$  such that when  $p$  tends to infinity,

$$|(u - u_p)(\xi_{p,j})| \leq cp^{-\lambda} \|u - u_p\|_{0,\infty}.$$

We first present superconvergence properties of the derivative approximation at Gauss points.

**Theorem 4.2.** Suppose all the conditions of Theorem 3.5 hold. Then for all  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}$ ,

$$|(u' - u'_p)(g_{i,j})| \lesssim e^{-\sigma(p+1)}. \tag{4.1}$$

Moreover, if  $\beta = 0$ ,

$$|(u' - u'_p)(g_{i,j})| \lesssim p^{-\frac{1}{2}} e^{-\sigma(p+1)}, \tag{4.2}$$

if  $\beta = \gamma = 0$ ,

$$|(u' - u'_p)(g_{i,j})| \lesssim p^{-\frac{1}{2}} e^{-2(\sigma+\sigma_1)p}, \tag{4.3}$$

where  $\sigma$  is the same as in Theorem 3.5 and  $\sigma_1 = \ln 2$ .

**Proof.** For all  $x \in [a, b]$ , let

$$F(x) = \int_a^x (\gamma(s) - \beta'(s))(u - u_p)(s) ds,$$

$$\kappa(x) = \alpha(x)(u - u_p)'(x) - \beta(x)(u - u_p)(x) - F(x).$$

Note that both  $u$  and  $u_p$  satisfy (2.2), then for all  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i}$ ,

$$\kappa(g_{i,j}) - \kappa(g_{i,j+1}) = 0.$$

Namely,

$$\alpha(g_{i,j})(u - u_p)'(g_{i,j}) - \beta(g_{i,j})(u - u_p)(g_{i,j}) - F(g_{i,j}) = \kappa(g_{i,j}) = C_0, \tag{4.4}$$

where  $C_0$  is a constant independent of  $i, j$ . Multiplying (4.4) with  $A_{i,j}\alpha(g_{i,j})^{-1}$  and then summing up for all  $i, j$ , we obtain

$$\sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j}\alpha(g_{i,j})^{-1} \kappa(g_{i,j}) = C_0 \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j}\alpha(g_{i,j})^{-1}.$$

Then

$$|C_0| \lesssim \left| \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j}(u - u_p)'(g_{i,j}) \right| + \|u - u_p\|_{0,\infty} + \|F\|_{0,\infty}$$

$$\lesssim |E| + \|u - u_p\|_{0,\infty} + \|u - u_p\|_0,$$

where

$$E = \int_a^b (u - u_p)' dx - \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j}(u - u_p)'(g_{i,j})$$

is the error of Gauss quadrature. By [24, (p. 98, (2.7.12))], we have

$$|E| \lesssim \sum_{k=1}^N h_k^{2p_k+1} \frac{(p_k!)^4}{(2p_k + 1)[(2p_k)!]^3} |u|_{2p_k,\infty} \lesssim p^{-\frac{1}{2}} \left( \frac{heM}{8p} \right)^{2p}.$$

Here in the last step, we have used Stirling's formula

$$\sqrt{2\pi} p^{p+\frac{1}{2}} < p! e^p < \sqrt{2\pi} p^{p+\frac{1}{2}} \left( 1 + \frac{1}{4p} \right).$$

In light of (3.20)–(3.21), we obtain

$$|C_0| \lesssim \left( \frac{heM}{4p} \right)^{p+1}.$$

Then

$$|(u - u_p)'(g_{i,j})| \lesssim |(u - u_p)(g_{i,j})| + |F(g_{i,j})| + |C_0| \lesssim \left(\frac{heM}{4p}\right)^{p+1}, \quad \forall i, j.$$

Similarly, when  $\beta = 0$ , we have from (3.22) and (4.4)

$$|C_0| \lesssim |E| + \|u - u_p\|_0 \lesssim p^{-\frac{1}{2}} \left(\frac{heM}{4p}\right)^{p+1}.$$

Consequently,

$$|(u - u_p)'(g_{i,j})| \lesssim |C_0| + \|u - u_p\|_0 \lesssim p^{-\frac{1}{2}} \left(\frac{heM}{4p}\right)^{p+1}.$$

When  $\beta = \gamma = 0$ , we have

$$|C_0| \lesssim |E| \lesssim p^{-\frac{1}{2}} \left(\frac{heM}{8p}\right)^{2p},$$

which yields

$$|(u - u_p)'(g_{i,j})| \lesssim |C_0| \lesssim p^{-\frac{1}{2}} \left(\frac{heM}{8p}\right)^{2p}.$$

The proof is completed.  $\square$

As a direct consequence of the above theorem,

$$|\pi_p u - u_p|_1 = |\pi_p u - u_p|_G \lesssim \left(\frac{heM}{4p}\right)^{p+1}. \tag{4.5}$$

Apparently, the convergence rate of  $|\pi_p u - u_p|_1$  is  $\frac{1}{2}$  order higher than the optimal convergence rate  $p^{-\frac{1}{2}} e^{-\sigma p}$  under the  $H^1$ -norm, which indicates that the FV solution  $u_p$  is super-close to the function  $\pi_p u$ .

We next study superconvergence properties of  $u_p$  at Lobatto points. We denote by  $l_{i,0}, \dots, l_{i,p_i}$  the Lobatto points of degree  $p_i + 1$  on the interval  $\tau_i$ ,  $i \in \mathbb{Z}_N$ . That is,  $l_{i,0} = x_{i-1}$ ,  $l_{i,p_i} = x_i$  and  $l_{i,j}$ ,  $j \in \mathbb{Z}_{p_i-1}$  are zeros of the derivative of Legendre polynomial of degree  $p_i$ .

We start with Green function. Given  $y \in \Omega$ , let  $G_y(x)$  be the Green function associated with  $y$  for the problem (2.1). Then

$$v(y) = A(v, G_y), \quad \forall v \in H_0^1(\Omega), \tag{4.6}$$

where  $A(\cdot, \cdot)$  is defined for all  $v, w \in H_0^1(\Omega)$  by

$$A(v, w) = \int_a^b \alpha(x)v'(x)w'(x)dx + \int_a^b (\beta(x)v'(x) + \gamma(x)v(x))w(x)dx.$$

We have the following superconvergence result at nodal points  $x_i$ ,  $i \in \mathbb{Z}_N$ .

**Theorem 4.3.** Suppose  $\alpha$  is a constant and all the conditions of Theorem 3.5 hold. Then

$$|(u - u_p)(x_i)| \lesssim p^{-\frac{1}{2}} e^{-\sigma p} e^{-\sigma_2 p}, \quad \forall i \in \mathbb{Z}_N. \tag{4.7}$$

Moreover, if  $\beta = \gamma = 0$ ,

$$|(u - u_p)(x_i)| \lesssim p^{-\frac{1}{2}} e^{-2(\sigma + \sigma_1)p}. \tag{4.8}$$

Here  $\sigma$  and  $\sigma_1$  are defined in Theorems 3.5 and 4.2, respectively; and  $\sigma_2 = -\ln \frac{heM'}{4p}$  with  $M' = \max(1, M)$ .

**Proof.** Note that

$$(u - u_p)(x_i) = \int_0^{x_i} (u - u_p)' dx, \quad \forall i \in \mathbb{Z}_N,$$

then (4.8) follows from Gauss quadrature and (4.3).

We next consider (4.7). By choosing  $v = u - u_p$  in (4.6), we obtain

$$(u - u_p)(x_i) = A(u - u_p, G_{x_i} - \pi_p G_{x_i}) + A(u - u_p, \pi_p G_{x_i}) = I_1 + I_2.$$

For all  $v \in H_0^1(\Omega)$ , we have from integrating by parts

$$A(u - u_p, v) = \int_a^b \kappa(x)v'(x)dx,$$

where  $\kappa(x)$  is the same as in Theorem 4.2. Then

$$|I_1| \lesssim \|u - u_p\|_1 \|G_{x_i} - \pi_p G_{x_i}\|_1 \lesssim p^{-1} \left(\frac{heM}{4p}\right)^p \left(\frac{he}{4p}\right)^p.$$

Here in the last step, we have used (3.16) and (3.20) and the fact that the Green function  $G_{x_i}$  has bounded derivatives of any order on each  $\tau_k$ ,  $k \in \mathbb{Z}_N$ . As for  $I_2$ , a direct calculation from Gauss quadrature yields

$$I_2 = \sum_{k=1}^N \sum_{j=1}^{p_k} A_{k,j} \kappa(g_{k,j}) (\pi_p G_{x_i})'(g_{k,j}) + \sum_{k=1}^N \frac{h_k^{2p+1} (p_k!)^4}{(2p_k + 1) [(2p_k)!]^3} (\kappa (\pi_p G_{x_i})')^{2p_k}(\xi_k),$$

where  $\xi_k \in (x_{k-1}, x_k)$ . By (4.4), we have

$$\sum_{k=1}^N \sum_{j=1}^{p_k} A_{k,j} \kappa(g_{k,j}) (\pi_p G_{x_i})'(g_{k,j}) = C_0 \int_a^b (\pi_p G_{x_i})'(x) dx = 0.$$

Then by the Leibnitz formula for the derivative and Stirling's formula, we have

$$\begin{aligned} |I_2| &\lesssim \sum_{k=1}^N h_k p_k^{-\frac{1}{2}} \left(\frac{h_k}{4p_k}\right)^{2p_k} \max_{p_k+1 \leq j \leq 2p_k} |\kappa|_{j, \infty, \tau_k} \\ &\lesssim p^{-\frac{1}{2}} \left(\frac{he}{4p}\right)^{2p} M^p (M')^p. \end{aligned}$$

Combining  $I_1$  with  $I_2$ , we obtain (4.7) directly.  $\square$

To investigate superconvergence properties of  $u_p$  at interior Lobatto points  $l_{i,j}$ ,  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i-1}$ , we first estimate  $\|D_i^{-1}v\|_{0, \infty}$  for all  $v(x) \in H_0^1(\Omega)$ , where the operator  $D_i^{-1}$  is defined as

$$(D_i^{-1}v)(x) := \int_{x_{i-1}}^x (v - \pi_p v)(s) ds, \quad \forall i \in \mathbb{Z}_N.$$

**Lemma 4.4.** Assume that  $v$  satisfies the regularity condition (3.14) and  $p$  satisfies (3.15). Then

$$\|D_i^{-1}v\|_{0, \infty} \lesssim \left(\frac{heM}{4p}\right)^{p+2}. \tag{4.9}$$

**Proof.** For all  $x \in \tau_i$ , recall the definition of  $\pi_p$ , we have

$$(v - \pi_p v)(x) = \sum_{j=p_i+1}^{\infty} v_j \phi_j(t) = \sum_{j=p_i+1}^{\infty} \frac{v_j}{2j-1} (P_j - P_{j-2})(t), \tag{4.10}$$

where  $v_j$  is the same as in (3.12)–(3.13) and  $t = (2x - x_{i-1} - x_i)/h_i \in [-1, 1]$ . Integrating (4.10) on the interval  $[x_{i-1}, x]$ , we derive

$$\begin{aligned} (D_i^{-1}v)(x) &= \sum_{j=p_i+1}^{\infty} \frac{h_i v_j}{2(2j-1)} \int_{-1}^t (P_j - P_{j-2})(t) dt \\ &= \frac{h_i}{2} \left( -\frac{v_{p_i+1}}{2p_i+1} \phi_{p_i} - \frac{v_{p_i+2}}{2p_i+3} \phi_{p_i+1} + \sum_{j=p_i+2}^{\infty} \left( \frac{v_j}{2j-1} - \frac{v_{j+2}}{2j+3} \right) \phi_{j+1} \right). \end{aligned}$$

From [20], we have

$$v_{j+1} \sim j \left(\frac{h_i eM}{4j}\right)^{j+1}, \quad \frac{v_j}{2j-1} - \frac{v_{j+2}}{2j+3} \lesssim \frac{h^j M^j}{2^j (2j-1)!!}.$$

Note that  $\frac{1}{(2j-1)!!} = \frac{2^j j!}{(2j)!} \sim \left(\frac{e}{2j}\right)^j$  and

$$|\phi_j| = \frac{1}{2j-1} |P_j - P_{j-2}| \lesssim \frac{1}{2j-1}.$$

Then  $-\frac{v_{p_i+1}}{2p_i+1}\phi_{p_i} - \frac{v_{p_i+2}}{2p_i+3}\phi_{p_i+1}$  is the dominant term in the formula of  $(D_i^{-1}v)(x)$ , which yields the estimate

$$\|D_i^{-1}v\|_{0,\infty} \lesssim \left| \frac{v_{p_i+1}h_i}{2p_i+1}\phi_{p_i} \right| + \left| \frac{v_{p_i+2}h_i}{2p_i+3}\phi_{p_i+1} \right| \lesssim \left(\frac{heM}{4p}\right)^{p+2}.$$

This completes our proof.  $\square$

We are now ready to present superconvergence properties of  $u_p$  at interior Lobatto points.

**Theorem 4.5.** *Suppose  $\alpha$  is a constant and all the conditions of Theorem 3.5 hold. Then*

$$|(u - u_p)(l_{i,j})| \lesssim e^{-\sigma(p+2)}, \quad \forall (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i-1}, \tag{4.11}$$

where  $\sigma$  is the same as in Theorem 3.5.

**Proof.** Given  $y \in \Omega$ , we have from (4.6)

$$(\pi_p u - u_p)(y) = A(\pi_p u - u_p, G_y) = J_1 + J_2 + J_3,$$

where

$$J_1 = A(\pi_p u - u_p, G_y - \pi_p G_y), \quad J_2 = A(\pi_p u - u, \pi_p G_y), \quad J_3 = A(u - u_p, \pi_p G_y).$$

We next estimate  $J_i$ ,  $i = 1, 2, 3$ , respectively. Noticing that

$$\int_a^b \alpha (G_y - \pi_p G_y)'(x) v_p'(x) dx = 0, \quad \forall v_p \in U_p,$$

we have

$$J_1 = \int_a^b (\beta(\pi_p u - u_p)' + \gamma(\pi_p u - u_p))(x)(G_y - \pi_p G_y)(x) dx.$$

For any  $x \in \tau_i$ ,  $i \in \mathbb{Z}_N$ , assume that

$$(\pi_p u - u_p)(x) = \sum_{j=0}^{p_i} u_j \phi_j(t), \quad G_y(x) = \sum_{j=0}^{\infty} a_j \phi_j(t),$$

where  $t = (2x - x_i - x_{i-1})/h_i$  and the coefficients  $u_j, a_j$  are determined by (3.12)–(3.13). By the quasi-orthogonal property of the Lobatto polynomials, we have

$$\int_{x_{i-1}}^{x_i} (\pi_p u - u_p)'(x)(G_y - \pi_p G_y)(x) dx = u_{p_i} a_{p_i+1} \int_{-1}^1 P_{p_i-1}(t) \phi_{p_i+1}(t) dt.$$

By (3.13) and integrating by parts, we obtain

$$|a_{p_i}| = \frac{2p_i - 1}{2} \left| \int_{x_{i-1}}^{x_i} G_y'(x) \phi_{p_i}'(t) dx \right| \lesssim h_i \|G_y\|_{2,1,\tau_i}.$$

Similarly, by (3.13) and (4.5), we have

$$|u_{p_i}| \lesssim p_i^{\frac{1}{2}} \|\pi_p - u_p\|_{1,\tau_i} \lesssim p_i^{\frac{1}{2}} \left(\frac{heM}{4p}\right)^{p+1}.$$

Then

$$\left| \int_{x_{i-1}}^{x_i} \beta(\pi_p u - u_p)'(G_y - \pi_p G_y) \right| \lesssim \left(\frac{heM}{4p}\right)^{p+2} \|G_y\|_{2,1,\tau_i},$$

which yields

$$\left| \int_a^b \beta(\pi_p u - u_p)'(G_y - \pi_p G_y) \right| \lesssim \left(\frac{heM}{4p}\right)^{p+2} \|G_y\|_{2,1}.$$

A similar result can be obtained for the term  $\int_a^b \gamma(\pi_p u - u_p)(G_y - \pi_p G_y)$ . Consequently,

$$|J_1| \lesssim \left(\frac{heM}{4p}\right)^{p+2} \|G_y\|_{2,1}.$$

As for  $J_2$ , Note that

$$\int_a^b \alpha(u - \pi_p u)'(\pi_p G_y)' = 0.$$

Then we have from integrating by parts and (4.9)

$$\begin{aligned} |J_2| &= \left| \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (D_i^{-1} u)((\beta \pi_p G_y)' - \gamma \pi_p G_y)' \right| \\ &\lesssim \sum_{i=1}^N \|D_i^{-1} u\|_{0,\infty} \|\pi_p G_y\|_{2,1,\tau_i} \lesssim \left(\frac{heM}{4p}\right)^{p+2} \|G_y\|_{2,1}. \end{aligned}$$

Now we turn to estimating  $J_3$ . Let  $\zeta(x) = \int_a^x \kappa(s) ds$  with  $\kappa$  defined in Theorem 4.2. A direct calculation yields

$$J_3 = \int_a^b \zeta'(\pi_p G_y)' = \int_a^b (\pi_p \zeta)'(\pi_p G_y)'.$$

By (4.4), we know that

$$\zeta'(g_{i,j}) = \kappa(g_{i,j}) = C_0$$

is a constant independent of  $i, j$ . Then we have from (3.18)

$$|((\pi_p \zeta)' - C_0)(g_{i,j})| = |(\pi_p \zeta - \zeta)'(g_{i,j})| \lesssim \left(\frac{heM}{4p}\right)^{p+2}, \quad \forall i, j.$$

Note that  $(\pi_p \zeta)'(\pi_p G_y)'|_{\tau_i} \in \mathbb{P}_{2p_i-1}$ ,  $i \in \mathbb{Z}_N$ , by Gauss quadrature,

$$\begin{aligned} |J_3| &= \left| \sum_{i=1}^N \sum_{j=1}^{p_i} A_{i,j} ((\pi_p \zeta)' - C_0)(g_{i,j})(\pi_p G_y)'(g_{i,j}) \right| \\ &\lesssim \left(\frac{heM}{4p}\right)^{p+2} \|G_y\|_{2,1}. \end{aligned}$$

Combining  $J_1, J_2$  with  $J_3$ , we obtain

$$|(\pi_p u - u_p)(y)| \lesssim \left(\frac{heM}{4p}\right)^{p+2} \|G_y\|_{2,1} \lesssim \left(\frac{heM}{4p}\right)^{p+2}. \tag{4.12}$$

Here we have used the regularity result (see, [27])

$$\|G_y\|_{2,1} = \sum_{i=1}^N \|G_y\|_{2,1,\tau_i} \lesssim 1.$$

Especially, by choosing  $y = l_{i,j}$ ,  $\forall (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_{p_i-1}$ , we have

$$|(\pi_p u - u_p)(l_{i,j})| \lesssim \left(\frac{heM}{4p}\right)^{p+2}.$$

A similar argument as in [20] yields

$$|(u - \pi_p u)(l_{i,j})| \lesssim \left(\frac{heM}{4p}\right)^{p+2} \quad \forall i, j.$$

Then the desired result (4.11) follows.  $\square$

**Remark 4.6.** The theoretical results in Theorems 4.3 and 4.5 are obtained under the condition that all the coefficients  $\alpha, \beta, \gamma$  are constants. However, it seems from numerical experiments that the same results still hold true for variable coefficients.

As a direct consequence of Theorem 4.5, we have the following estimate for  $\|u - u_p\|_{1,\infty}$ .

**Table 1**  
Approximate errors in various norms.

$r$	$ u - u_p _1$	$\ u - u_p\ _0$	$\ u - u_p\ _{0,\infty}$	$ u - u_p _L$
15	1.0783e-4	1.8193e-6	1.7380e-6	2.3354e-7
16	7.5982e-5	1.6780e-6	1.3749e-6	7.4393e-8
17	3.6946e-6	5.4672e-8	5.3504e-8	6.2421e-9
18	2.6039e-6	5.0957e-8	4.1595e-8	2.0090e-9
19	1.0076e-7	1.3288e-9	1.3072e-9	1.3553e-10
20	7.1032e-8	1.2486e-9	1.0133e-9	4.4267e-11
21	2.2411e-9	2.6669e-11	2.5307e-11	2.4706e-12
22	1.5802e-9	2.5221e-11	2.0046e-11	8.1435e-13
23	4.1452e-11	4.5006e-13	4.4320e-13	-
24	2.9233e-11	4.2767e-13	3.5466e-13	-

**Corollary 4.7.** Assume that  $\alpha$  is a constant and all the conditions of Theorem 3.5 hold. Suppose  $p_i \lesssim p$  and  $h \lesssim h_i$ ,  $i \in \mathbb{Z}_N$ . Then

$$\|u - u_p\|_{1,\infty} \lesssim e^{-\sigma p}.$$

**Proof.** From (4.12), we have

$$\|u_p - \pi_p u\|_{0,\infty} \lesssim \left(\frac{heM}{4p}\right)^{p+2}.$$

By the inverse inequality of  $hp$ -version,

$$\|u_p - \pi_p u\|_{1,\infty} \lesssim \frac{p^2}{h} \|u_p - \pi_p u\|_{0,\infty} \lesssim p \left(\frac{heM}{4p}\right)^{p+1}.$$

On the other hand, note that

$$\|u - \pi_p u\|_{1,\infty} \lesssim \left(\frac{heM}{4p}\right)^p.$$

The desired result then follows.  $\square$

**Remark 4.8.** Since the convergence rate of  $\|u' - u'_p\|_{0,\infty}$  is  $e^{-\sigma p}$ , we observe from (4.1)–(4.3) that the convergence rate of the derivative error at Gauss points is one order higher than that of  $\|u' - u'_p\|_{0,\infty}$  in case  $\beta \neq 0$  and 3/2 order higher in case  $\beta = 0$ ,  $\gamma \neq 0$ . While, when  $\beta = 0$ ,  $\gamma = 0$ , the derivative error at Gauss points converges faster than  $\|u' - u'_p\|_{0,\infty}$  by an exponential rate. This phenomenon is similar to that of the  $h$ -version FVM in [21]. In this sense, the derivative error is superconvergent at all Gauss points.

Similarly, recall the convergence rate of  $\|u - u_p\|_{0,\infty}$ , we know that the error in function values approximation at interior Lobatto points is one order higher than that in the  $L^\infty$ -norm. Note that this error bound  $e^{-\sigma(p+2)}$  for function values approximation at Lobatto points is similar to that of the spectral collocation method in [20]. While at nodal points, the convergence rate  $p^{-\frac{1}{2}} e^{-\sigma p} e^{-\sigma_2 p}$  almost doubles the global optimal rate  $e^{-\sigma(p+1)}$ , which is the same as the  $h$ -version finite volume method.

### 5. Numerical examples

In this section, we present numerical examples to support the theoretical analysis in the previous sections.

In our experiments, we obtain the mesh by dividing  $[-1, 1]$  into two elements of length  $h_1 = h_2 = 1$ . The exact solution is always  $u(x) = \sin(4\pi x)$  and the right-hand function  $f$  changes according to the coefficients in different cases. Note that  $\|u\|_{k,\infty} \leq (4\pi)^k$ . Therefore,  $M = 4\pi$ . The asymptotic condition  $(2p + 1)(2p + 3) > 2M^2$  in (3.15) suggests  $p > 8$ . We solve (2.1) by the  $p$ -version FV scheme (2.4) with polynomial degree  $p_1 = p_2 = p$ ,  $p = 9, \dots, 24$ . We define by

$$|u - u_p|_N = \max_i |(u - u_p)(x_i)|, \quad |u - u_p|_L = \max_{i,j} |(u - u_p)(l_{i,j})|, \quad |u - u_p|_g = \max_{i,j} |(u - u_p)'(g_{i,j})|$$

the maximum errors at nodes, Lobatto points and Gauss points, respectively.

**Example 1.** We consider (2.1) with

$$\alpha(x) = e^x, \quad \beta(x) = 1, \quad \gamma(x) = 7, \quad \forall x \in (-1, 1).$$

We list in Table 1 the approximate errors in various norms for different values of  $p = 15, \dots, 24$ .

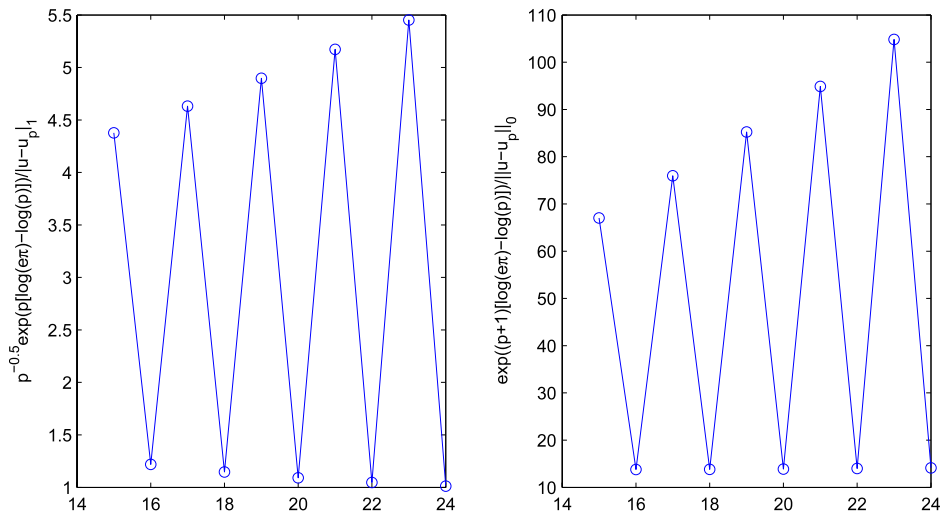


Fig. 1. Ratios of estimated errors and computed errors for  $\|u - u_p\|_1$  (left) and  $\|u - u_p\|_0$  (right).

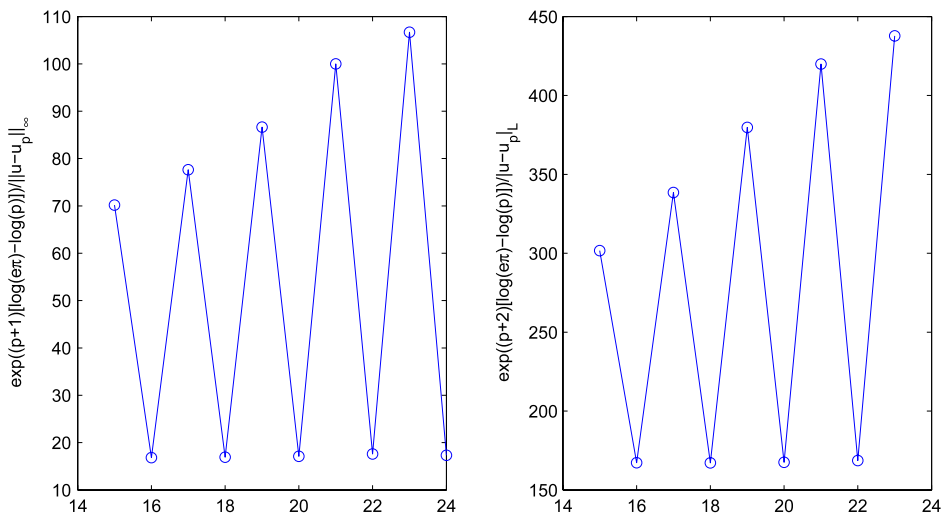


Fig. 2. Ratios of estimated errors and computed errors for  $\|u - u_p\|_{0,\infty}$  (left) and  $\|u - u_p\|_L$  (right).

To verify the error bounds we have established for  $H^1, L^2$  and  $L^\infty$ -norms in Section 3, we plot in Fig. 1 the ratios  $p^{-\frac{1}{2}} \left(\frac{heM}{4p}\right)^p : \|u - u_p\|_1$  (left) and  $\left(\frac{heM}{4p}\right)^{p+1} : \|u - u_p\|_0$  (right), and in Fig. 2 the ratio  $\left(\frac{heM}{4p}\right)^{p+1} : \|u - u_p\|_{0,\infty}$  (left). We observe that all the ratios are constants for even  $p$  and increase linearly for odd  $p$ , indicating better convergence rates for odd  $p$ . These results are consistent with (3.20) and (3.21). Therefore, the error bounds given by Theorem 3.5 are sharp.

To observe the superconvergence phenomenon of FV approximation at Lobatto points, we plot in Fig. 2 the ratio  $\left(\frac{heM}{4p}\right)^{p+2} : \|u - u_p\|_L$  (right). Just as predicted in (4.11), we observe that the ratio is a constant for even  $p$ , indicating an order-one superconvergence rate. Once again, we observe a better convergence rate for odd  $p$ . Hence, the superconvergence phenomenon of FV approximation at Lobatto points exists.

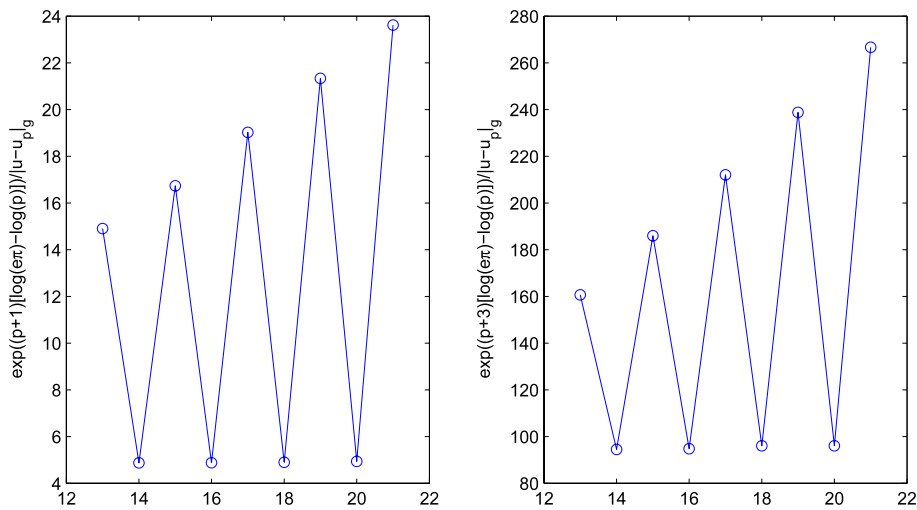
**Example 2.** In this example, we test the convergence behavior of the derivative error at Gauss points and the function values error at nodes in three different cases. They are

- Case 1 :  $\alpha(x) = e^x, \beta(x) = 1 + \cos(x), \gamma(x) = 12 + 4x$ ;
- Case 2 :  $\alpha(x) = e^x, \beta(x) = 0, \gamma(x) = 12 + 4x$ ;
- Case 3 :  $\alpha(x) = e^x, \beta(x) = 0, \gamma(x) = 0$ .

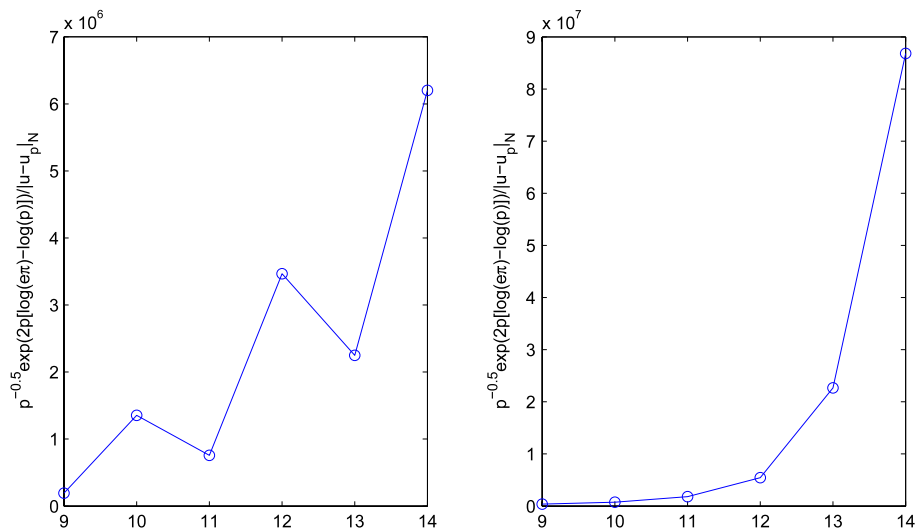
We list in Table 2 the derivative error at Gauss points and function values error at nodes in three different cases for  $p = 9, \dots, 14$ .

**Table 2**  
The derivative error at Gauss points and the function values error at nodes in three cases.

$r$	Case 1		Case 2		Case 3	
	$ u - u_p _g$	$ u - u_p _N$	$ u - u_p _g$	$ u - u_p _N$	$ u - u_p _g$	$ u - u_p _N$
9	5.4417e-2	6.6848e-7	4.8124e-3	3.3608e-7	3.6918e-6	7.4941e-7
10	3.5008e-2	9.9343e-9	1.3731e-3	1.8226e-8	1.0094e-7	2.0431e-8
11	3.7644e-3	1.5200e-9	2.1287e-4	6.3543e-10	2.2497e-9	4.5438e-10
12	2.4266e-3	2.3711e-11	6.3362e-5	1.5094e-11	4.1695e-11	8.4236e-12
13	1.8692e-4	2.2182e-12	7.4840e-6	2.1982e-13	6.2425e-13	1.2610e-13
14	1.2363e-4	4.1526e-14	2.3713e-6	-	-	-



**Fig. 3.** Ratios of estimated errors and computed errors at Gauss points for Case 1 (left) and Case 2 (right).



**Fig. 4.** Ratios of estimated errors and computed errors at nodes for Case 1 (left) and Case 2 (right).

Fig. 3 plots the ratios  $\left(\frac{heM}{4p}\right)^{p+1} : |u - u_p|_g$  (left) for Case 1 and  $\left(\frac{heM}{4p}\right)^{p+3} : \|u - u_p\|_g$  (right) for Case 2, respectively, from which we observe that the derivative approximation at Gauss points is superconvergent with order one for Case 1 and order three for Case 2. Note that the superconvergence result for Case 1 is consistent with (4.1), while for Case 2, the convergence rate is better than the one given in (4.2). Fig. 4 plots the ratio  $p^{-0.5} \left(\frac{heM}{4p}\right)^{2p} : |u - u_p|_N$  for Case 1 (left) and Case 2 (right). We see that in both cases the ratio increases rapidly as  $p$  increases, indicating a better convergence rate than the one given in (4.7). Fig. 5 depicts the error curves of  $|u - u_p|_g$  and  $|u - u_p|_N$  (left) and the ratios  $p^{-0.5} \left(\frac{heM}{8p}\right)^{2p} : |u - u_p|_N$  and  $p^{-\frac{1}{2}} \left(\frac{heM}{8p}\right)^{2p} : |u - u_p|_g$  (right) for Case 3. We see that both of the ratios are constants, which verifies (4.3) and (4.8).



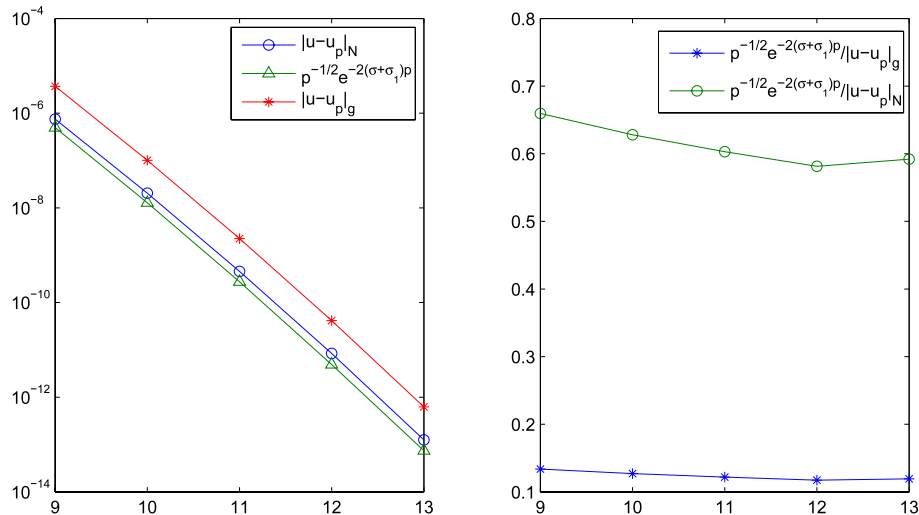


Fig. 5. Error curves (left) and ratios of estimated errors and computed errors at nodes and Gauss points for Case 3 (right).

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## References

- [1] R.E. Bank, D.J. Rose, Some error estimates for the box scheme, *SIAM J. Numer. Anal.* 24 (1987) 777–787.
- [2] T. Barth, M. Oehlberger, Finite Volume Methods: Foundation and Analysis, in: *Encyclopedia of computational Mechanics*, vol. 1, John Wiley & Sons, 2004 (Chapter 15).
- [3] Z. Cai, On the finite volume element method, *Numer. Math.* 58 (1991) 713–735.
- [4] Z. Cai, J. Douglas, M. Park, Development and analysis of higher order finite volume methods over rectangles for elliptic equations, *Adv. Comput. Math.* 19 (2003) 3–33.
- [5] Z. Chen, J. Wu, Y. Xu, Higher-order finite volume methods for elliptic boundary value problems, *Adv. Comput. Math.* 37 (2012) 191–253.
- [6] R. Ewing, T. Lin, Y. Lin, On the accuracy of the finite volume element based on piecewise linear polynomials, *SIAM J. Numer. Anal.* 39 (2002) 1865–1888.
- [7] R. Eymard, T. Gallouet, R. Herbin, Finite volume methods, in: P.G. Ciarlet, J.L. Lions (Eds.), *Handbook of Numerical Analysis*, Vol. VII, North-Holland, Amsterdam, 2000, pp. 713–1020.
- [8] R. Li, Z. Chen, W. Wu, *The Generalized Difference Methods for Partial differential Equations*, Marcel Dekker, New York, 2000.
- [9] C. Ollivier-Gooch, M. Altena, A high-order-accurate unconstructed mesh finite-volume scheme for the advection–diffusion equation, *J. Comput. Phys.* 181 (2002) 729–752.
- [10] M. Plexousakis, G. Zouraris, On the construction and analysis of high order locally conservative finite volume type methods for one dimensional elliptic problems, *SIAM J. Numer. Anal.* 42 (2004) 1226–1260.
- [11] E. Süli, Convergence of finite volume schemes for Poisson's equation on nonuniform meshes, *SIAM J. Numer. Anal.* 28 (1991) 1419–1430.
- [12] J. Xu, Q. Zou, Analysis of linear and quadratic simplicial finite volume methods for elliptic equations, *Numer. Math.* 111 (2009) 469–492.
- [13] Q. Zou, Hierarchical error estimates for finite volume approximation solution of elliptic equations, *Appl. Numer. Math.* 60 (2010) 142–153.
- [14] I. Babuška, M. Suri, The  $p$  and  $hp$  version of the finite element method, basic principles and properties, *SIAM Rev.* 36 (1994) 578–632.
- [15] B. Guo, The  $h-p$  version of the finite element method for elliptic equations of order  $2m$ , *Numer. Math.* 53 (1998) 199–224.
- [16] J.M. Melenk, C. Schwab, An  $hp$  finite element method for convection–diffusion problems in one dimension, *IMA J. Numer. Anal.* 19 (1999) 425–453.
- [17] J.M. Melenk, C. Schwab, An  $hp$  streamline diffusion finite element method for convection dominated problems in one dimension, *East–West J. Numer. Math.* 7 (1999) 31–60.
- [18] C. Schwab, M. Suri, The  $p$  and  $hp$  versions of the finite element for problems with boundary layers, *Math. Comp.* 65 (1996) 1403–1429.
- [19] Z. Zhang, On the  $hp$  finite element method for the one dimensional singularly perturbed convection–diffusion problems, *J. Comput. Math.* 20 (2002) 599–610.
- [20] Z. Zhang, Superconvergence of spectral collocation and  $p$ -version methods in one dimensional problems, *Math. Comp.* 74 (2005) 1621–1636.
- [21] W. Cao, Z. Zhang, Q. Zou, Superconvergence of any order finite volume schemes for 1D general elliptic equations, *J. Sci. Comput.* 56 (2013) 566–590.
- [22] W. Cao, Z. Zhang, Q. Zou, Finite volume superconvergence approximation for 1D singular perturbed problems, *J. Comput. Math.* 31 (2013) 488–508.
- [23] L. Yi, B. Guo, Superconvergence of the  $h-p$  version of the finite element method in one dimension, *J. Comput. Appl. Math.* 233 (2009) 150–164.
- [24] P.J. Davis, P. Rabinowitz, *Methods of Numerical Integration*, second ed., Academic Press, Boston, 1984.
- [25] K. Forster, K. Petras, On estimates for the weights in Gaussian quadrature in the ultraspherical case, *Math. Comp.* 55 (1990) 243–264.
- [26] J. Xu, L. Zikatanov, Some observations on Babuška–Brezzi conditions, *Numer. Math.* 94 (2003) 195–202.
- [27] C. Chen, *Structure Theory of Superconvergence of Finite Elements*, Hunan Science and Technology Press, Hunan, China, 2001 (in Chinese).