

Superconvergence of Jacobi–Gauss-Type Spectral Interpolation

Li-Lian Wang · Xiaodan Zhao · Zhimin Zhang

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Abstract In this paper, we extend the study of superconvergence properties of Chebyshev–Gauss-type spectral interpolation in Zhang (SIAM J Numer Anal 50(5):2966–2985, 2012) to general Jacobi–Gauss-type interpolation. We follow the same principle as in Zhang (SIAM J Numer Anal 50(5):2966–2985, 2012) to identify superconvergence points from interpolating analytic functions, but rigorous error analysis turns out much more involved even for the Legendre case. We address the implication of this study to functions with limited regularity, that is, at superconvergence points of interpolating analytic functions, the leading term of the interpolation error vanishes, but there is no gain in order of convergence, which is in distinctive contrast with analytic functions. We provide a general framework for exponential convergence and superconvergence analysis. We also obtain interpolation error bounds for Jacobi–Gauss-type interpolation, and explicitly characterize the dependence of the underlying parameters and constants, whenever possible. Moreover, we provide illustrative numerical examples to show tightness of the bounds.

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L.-L. Wang (✉) · X. Zhao
Division of Mathematical Sciences, School of Physical and Mathematical Sciences,
Nanyang Technological University, Singapore 637371, Singapore
e-mail: LiLian@ntu.edu.sg

Z. Zhang
Beijing Computational Science Research Center, Beijing, China

Z. Zhang
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

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1 Introduction

The study of superconvergence phenomenon for h -version methods has had a great impact on scientific computing, especially on *a posteriori* error estimates and adaptive methods. With a belief that the scientific community would also benefit from the study of superconvergence phenomenon of spectral methods, the third author studied spectral collocation related to the Chebyshev polynomials under the analytic assumption, and the Legendre polynomials under even more restrictive assumption—polynomials of one degree higher than the approximating polynomial space [24]. Two issues remain open: *What happens to general Jacobi polynomials? Is the superconvergence phenomenon still valid for functions with limited regularity?*

This work is the second stage towards superconvergence phenomenon of orthogonal polynomial interpolation, in which we will study the aforementioned two issues. Answer to the first one is affirmative. The main effort here is devoted to identifying superconvergence points for interpolation by general Jacobi polynomials. We would like to emphasize that the proof for general case is very different from the Chebyshev case, in which a special closed form in terms of trigonometry functions can be utilized. In general case, analysis is much more involved and complicated. For functions with limited regularity, answer is partially positive in the sense that the convergence rate remains the same at all points. Nevertheless, the numerical error is significantly smaller at those superconvergence points obtained from interpolating analytic functions.

Rather than the conventional algebraic order (i.e., $O(N^{-r})$ -type) error estimates, we will establish error bounds in the form of $C(N)\rho^{-N}$ with $\rho > 1$ (where $N + 1$ is the number of interpolation points) under analytic assumption, and explicitly characterize the dependence of $C(N)$ on N . This kind of error bounds has been studied in a number of articles in literature. For example, Tadmor [16], and Reddy and Weideman [13] analyzed the Fourier and/or Chebyshev interpolation of analytic functions, while Xie, Wang and Zhao [21] conducted analysis of Gegenbauer–Gauss and Gegenbauer–Gauss–Lobatto interpolations. The exponential convergence of orthogonal polynomial expansions under analytic assumption was studied by e.g., Gottlieb and Shu et al. [5,6], Zhang [22–24], Wang and Xiang [18], Xiang [20], and Zhao, Wang and Xie [25]. The interested readers are also referred to some insightful discussions and applications in e.g., [7,11,17]. It is worthwhile to point out that the argument for Legendre–Gauss–Radau case in this paper appears similar to that in [21], but it needs much more delicate analysis, and the extension is nontrivial.

With the aforementioned issues in mind, we identify in Sect. 2 superconvergence points from interpolating analytic functions, and address the implication to functions with limited regularity. In Sect. 3, we present a general framework for analyzing exponential convergence and superconvergence of polynomial interpolation. In Sect. 4, we apply the general result to Jacobi–Gauss-type interpolation, and provide detailed estimates with explicit dependence of all involved parameters for Legendre–Gauss–Radau case. The last section is for numerical results and concluding remarks.

2 Superconvergence Points

In this section, we identify superconvergence points for derivatives of general Jacobi–Gauss-type interpolants under analytic assumption, and discuss the indication to functions with limited regularity.

2.1 Jacobi–Gauss-Type Interpolation of Analytic Functions

Throughout this paper, let $P_N^{(\alpha,\beta)}(x)$ ($x \in [-1, 1]$ and $\alpha, \beta > -1$) be the Jacobi polynomial of degree N , as normalized in [15]. In particular, we denote the Legendre polynomial by $P_N(x) (= P_N^{(0,0)}(x))$, and the Chebyshev polynomial by $T_N(x) (= \arccos(N \cos x))$. We collect in Appendix A some relevant properties of Jacobi polynomials.

Hereafter, let \mathbb{P}_N be the set of all real polynomials of degree at most N . Let $\{x_j\}_{j=0}^N$ be a set of generic distinct interpolation points on $[-1, 1]$, and let $I_N : C([-1, 1]) \rightarrow \mathbb{P}_N$ be the Lagrange interpolation operator such that $\{(I_N u)(x_j) = u(x_j)\}_{j=0}^N$ for any $u \in C([-1, 1])$.

Recall that Jacobi–Gauss-type interpolation is on one of the following sets of points:

- The Jacobi–Gauss (JG) points $\{x_j = \xi_{G,j}^{\alpha,\beta}\}_{j=0}^N$ are zeros of the Jacobi polynomial $P_{N+1}^{(\alpha,\beta)}(x)$.
- The (left) Jacobi–Gauss–Radau (JGR) points $\{x_j = \xi_{R,j}^{\alpha,\beta}\}_{j=0}^N$ (with fixed left endpoint $x_0 = -1$) are zeros of $(1+x)P_N^{(\alpha,\beta+1)}(x)$. With a change of variable $x \rightarrow -x$, JGR interpolation with fixed right endpoint $x_N = 1$, can be obtained.
- The Jacobi–Gauss–Lobatto (JGL) points $\{x_j = \xi_{L,j}^{\alpha,\beta}\}_{j=0}^N$ (with $x_0 = -1, x_N = 1$) are zeros of $(1-x^2)\partial_x P_N^{(\alpha,\beta)}(x)$.

It is known from the seminal work of Bernstein [2] that any analytic function $u(x)$ on $[-1, 1]$ can be continued analytically to an elliptic domain enclosed by the so-called *Bernstein ellipse* \mathcal{E}_ρ , with foci ± 1 and ρ being the sum of semi-axes. The Bernstein ellipse can be obtained from the circle $C_\rho := \{w = \rho e^{i\theta}\}$ through the Joukowski transformation (cf. [9]):

$$\mathcal{E}_\rho := \left\{ z \in \mathbb{C} : z = \frac{1}{2}(w + w^{-1}), w = \rho e^{i\theta}, \theta \in [0, 2\pi] \right\}, \quad \rho > 1, \tag{2.1}$$

where \mathbb{C} is the set of all complex numbers, and $i = \sqrt{-1}$ is the complex unit. The perimeter of \mathcal{E}_ρ can be approximated by (see Ramanujan [12]): $\pi(3\rho - \sqrt{4\rho^2 - \rho^{-2}})$. A tighter bound is given by [19, (2.15)]:

$$L(\mathcal{E}_\rho) \leq \frac{(24 - 5\sqrt{2}\pi)\rho - (8 - (1 + \sqrt{2})\pi)(\rho - \rho^{-3}) - (16 - 5\pi)\sqrt{\rho^2 + \rho^{-2}}}{2(3 - 2\sqrt{2})}. \tag{2.2}$$

The distance from \mathcal{E}_ρ to $[-1, 1]$ is

$$d_\rho = \frac{1}{2}(\rho + \rho^{-1}) - 1. \tag{2.3}$$

Define

$$\mathcal{A}_\rho := \{u : u \text{ is analytic on and within } \mathcal{E}_\rho\}, \quad 1 < \rho < \rho_{\max}, \tag{2.4}$$

where $\mathcal{A}_{\rho_{\max}}$ labels the largest ellipse within which u is analytic. In particular, if $\rho_{\max} = \infty$, u is an entire function.

We now introduce the major analysis tool—*Hermite’s contour integral* (see e.g., [3, (3.6.5)–(3.6.6)]), that is, for any $u \in \mathcal{A}_\rho$,

$$(I_N u)(x) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{Q_{N+1}(z) - Q_{N+1}(x)}{z - x} \frac{u(z)}{Q_{N+1}(z)} dz, \quad \forall x \in [-1, 1], \tag{2.5}$$

and

$$(u - I_N u)(x) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{Q_{N+1}(x)}{z - x} \frac{u(z)}{Q_{N+1}(z)} dz, \quad \forall x \in [-1, 1]. \tag{2.6}$$

In particular, for Jacobi–Gauss-type interpolation, we have

$$Q_{N+1}(x) = c_N \left\{ P_{N+1}^{(\alpha, \beta)}(x), (1+x)P_N^{(\alpha, \beta+1)}(x) \text{ or } (1-x^2)\partial_x P_N^{(\alpha, \beta)}(x) \right\}, \tag{2.7}$$

where c_N is any nonzero factor.

Thus, by (2.6),

$$(u - I_N u)'(x) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \left(\frac{Q'_{N+1}(x)}{z - x} + \frac{Q_{N+1}(x)}{(z - x)^2} \right) \frac{u(z)}{Q_{N+1}(z)} dz, \tag{2.8}$$

and

$$(u - I_N u)''(x) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \left(\frac{Q''_{N+1}(x)}{z - x} + \frac{2Q'_{N+1}(x)}{(z - x)^2} + \frac{2Q_{N+1}(x)}{(z - x)^3} \right) \frac{u(z)}{Q_{N+1}(z)} dz. \tag{2.9}$$

Likewise, we can compute higher order derivatives by direct differentiation.

2.2 Superconvergence Points for Derivatives

As observed in [24] for the Chebyshev case, a differentiation of Q_{N+1} magnifies a factor of N or N^2 , so the highest derivative of Q_{N+1} dominates the error, and superconvergence can be attained at zeros of the highest derivative of $Q_{N+1}(x)$. Following the same principle, we have the following claim.

Proposition 2.1 *If we interpolate a function $u \in \mathcal{A}_\rho$ at the zeros of $Q_{N+1}(x)$ (e.g., (2.7)), then the superconvergence points for the first (resp. second) derivative are the zeros of $Q'_{N+1}(x)$ (resp. $Q''_{N+1}(x)$), and likewise for higher order derivatives.*

Remark 2.1 Extreme points of $Q_{N+1}(x)$ are superconvergence points of the first derivative. □

Applying the above rule, we can locate superconvergence points for Jacobi–Gauss-type interpolation as follows.

Proposition 2.2 (Jacobi–Gauss interpolation) *For Jacobi–Gauss interpolation at zeros of $Q_{N+1}(x) = P_{N+1}^{(\alpha, \beta)}(x)$, the first derivative superconverges at N interior Jacobi–Gauss–Lobatto points, i.e., zeros of $\partial_x P_{N+1}^{(\alpha, \beta)}(x)$; the second derivative superconverges at $N - 1$ Jacobi–Gauss points with the parameter $(\alpha + 2, \beta + 2)$, i.e., zeros of $P_{N-1}^{(\alpha+2, \beta+2)}(x)$ ($= \gamma_N^{\alpha, \beta} \partial_x^2 P_{N+1}^{(\alpha, \beta)}(x)$ by (6.1)).*

Proposition 2.3 (left) Jacobi–Gauss–Radau interpolation) For (left) Jacobi–Gauss–Radau interpolation at zeros of $(1 + x)P_N^{(\alpha, \beta+1)}(x)$, the first derivative superconverges at zeros of

$$Q'_{N+1}(x) = \partial_x P_{N+1}^{(\alpha, \beta)}(x) + \frac{N + \beta + 1}{N + 1} \partial_x P_N^{(\alpha, \beta)}(x); \tag{2.10}$$

the second derivative superconverges at zeros of $Q''_{N+1}(x)$.

Remark 2.2 Note that the identity (2.10) (up to a constant multiple) follows from (6.2). □

Proposition 2.4 (Jacobi–Gauss–Lobatto interpolation) For Jacobi–Gauss–Lobatto interpolation at zeros of $(1 - x^2)\partial_x P_N^{(\alpha, \beta)}(x)$, the first derivative superconverges at zeros of

$$Q'_{N+1}(x) = \partial_x P_{N+1}^{(\alpha, \beta)}(x) + (\beta - \alpha) \frac{2N + \alpha + \beta + 1}{(N + 1)(2N + \alpha + \beta)} \partial_x P_N^{(\alpha, \beta)}(x) - \frac{(N + \alpha)(N + \beta)(2N + \alpha + \beta + 2)}{N(N + 1)(2N + \alpha + \beta)} \partial_x P_{N-1}^{(\alpha, \beta)}(x); \tag{2.11}$$

the second derivative superconverges at zeros of $Q''_{N+1}(x)$.

Remark 2.3 For Legendre–Gauss–Lobatto (LGL) case, (2.11) reduces to

$$Q'_{N+1}(x) = P'_{N+1}(x) - P'_{N-1}(x) = (2N + 1)P_N(x),$$

where the last identity can be found in e.g., [15]. Therefore, superconvergence points of first derivative for LGL interpolation are Legendre–Gauss points (involving only N points); while superconvergence points for second derivative are $N - 1$ interior LGL points. In fact, only the Legendre case enjoys this property. □

2.3 Computation of Superconvergence Points

It is known that Jacobi–Gauss-type points $\{x_j\}$ can be computed by the eigen-method (see e.g., [4, 14]). It is known that between every two consecutive zeros of $Q_{N+1}(x)$, there exists exactly one extreme point of $Q_{N+1}(x)$, so we can set $\{y_j^0 = (x_{j-1} + x_j)/2\}_{j=1}^N$ as initial guesses, and then use the Newton’s iterative method to locate the superconvergence points $\{y_j\}_{j=1}^N$ for first derivative. Similarly, we can compute those for second derivative by using $\{(y_{j-1} + y_j)/2\}$ as initial guesses. In Fig. 1, we plot zeros of $Q'_{N+1}(x)$ with $Q_{N+1}(x) = (1 - x^2)\partial_x P_N^{(\alpha, \beta)}(x)$ for some α, β and N .

2.4 Implication to General Non-analytic Functions

We reiterate that the superconvergence points are at which the leading term of the error remainder vanishes (see (2.8)–(2.9) and Propositions 2.2–2.4). It is also based on this principle that Zhang [24] locates the superconvergence points for the function values, when one interpolates the first derivative values.

A natural question to ask is what would be the indication of superconvergence points (identified from interpolating analytic functions) to functions which are not analytic? We will examine this from two perspectives, and illustrate that the leading term of the interpolation error of functions with finite regularity vanishes at superconvergence points, but there is no gain in order of convergence, as opposite to the analytic case.

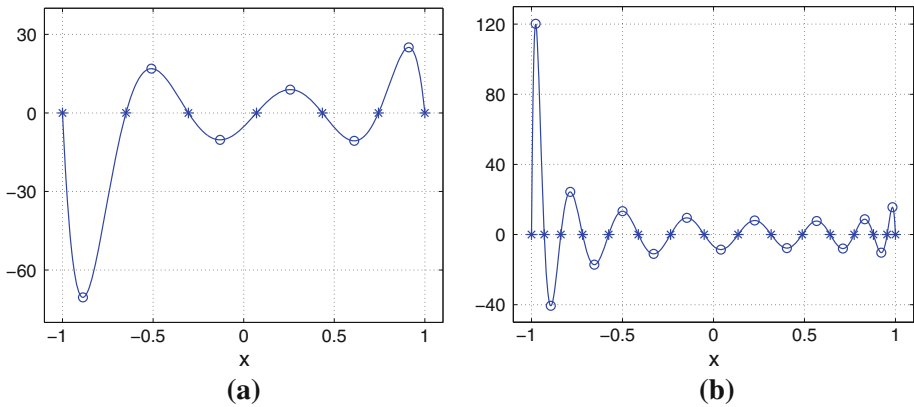


Fig. 1 Zeros (marked by “*”) and extreme points (marked by “o”) of $Q_{N+1}(x)$. **a** $\alpha = 2, \beta = 3, N = 6$. **b** $\alpha = 1, \beta = 2, N = 15$

The first perspective is from the remainder of the interpolation error for $u \in C^{k+1}[-1, 1]$ with finite integer k (see [10]):

$$R_N(x) = (u - I_N u)(x) = \frac{(-1)^k}{k!} \sum_{j=0}^N h_j(x) \int_{x_j}^x (t - x_j)^k u^{(k+1)}(t) dt, \quad (2.12)$$

where $\{h_j\}$ are the Lagrange basis polynomials given by

$$h_j(x) = \frac{Q_{N+1}(x)}{(x - x_j)Q'_{N+1}(x_j)}, \quad 0 \leq j \leq N.$$

A direct calculation leads to

$$R'_N(x) := \frac{(-1)^k}{k!} (R_1(x) + R_2(x)),$$

where

$$R_1(x) = \sum_{j=0}^N \frac{Q'_{N+1}(x)}{(x - x_j)Q'_{N+1}(x_j)} \int_{x_j}^x (t - x_j)^k u^{(k+1)}(t) dt;$$

$$R_2(x) = \sum_{j=0}^N \frac{Q_{N+1}(x)}{(x - x_j)Q'_{N+1}(x_j)} \left((x - x_j)^k u^{(k+1)}(x) - \frac{1}{x - x_j} \int_{x_j}^x (t - x_j)^k u^{(k+1)}(t) dt \right).$$

Observe that at interpolation points $\{x_i\}_{i=0}^N$, we have $R'_N(x_i) = (-1)^k R_1(x_i)/k!$, while at superconvergence points $\{y_i\}_{i=1}^N$, we have $R'_N(y_i) = (-1)^k R_2(y_i)/k!$. In fact, $R_1(x)$ dominates the error. For example, we consider the function:

$$u(x) = x^a \exp\left(\frac{1}{2} \sin(x/3)\right), \quad x \in (-1, 1), \quad (2.13)$$

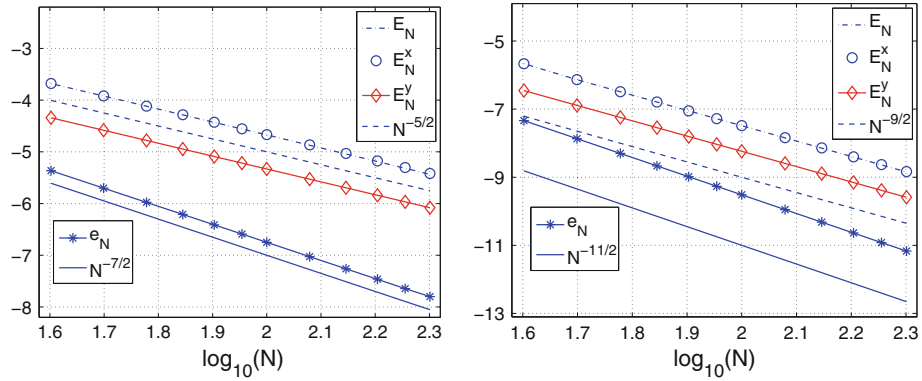


Fig. 2 Errors for $a = 7/2$ (left) and $a = 11/2$ (right) with $\alpha = \beta = 0$

where $a > 1$ is a non-integer number with integer part $[a]$. It is clear that $u \in C^{[a]}[-1, 1]$, but $u \notin C^{[a]+1}[-1, 1]$. Denote

$$e_N = \max_{|x| \leq 1} |R_N(x)|, \quad E_N = \max_{|x| \leq 1} |R'_N(x)|, \quad E_N^x = \max_{0 \leq j \leq N} |R'_N(x_j)|, \quad E_N^y = \max_{1 \leq i \leq N} |R'_N(y_i)|. \tag{2.14}$$

We compute e_N and E_N by a dense sampling of uniform points on $[-1, 1]$, and evaluate the derivatives and integrals in $R_1(x)$ and $R_2(x)$ exactly.

In Fig. 2, we plot e_N, E_N, E_N^x, E_N^y and the reference lines N^{-a}, N^{1-a} in log-log scale with $a = 7/2, 11/2$. We find that (i) the errors E_N, E_N^x, E_N^y decay at the same rate as N^{1-a} (expected order for E_N , see e.g., [14]); (ii) the convergence rate in the first derivative is one order lower than e_N ; and (iii) E_N^x dominates the error E_N . We also see that it is more accurate at superconvergence points, but there is no gain in convergence order. This should be in contrast with interpolating analytic functions, where an order of N^2 can be gained at the superconvergence points (see Theorem 4.1).

We next take a different viewpoint from the spectral expansion of a function with finite regularity. Let us consider the JGL case (see Proposition 2.4). Define

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_k(x) = (1 - x^2) \partial_x P_{k-1}^{(\alpha, \beta)}(x), \quad k \geq 2. \tag{2.15}$$

The so-defined $\{\phi_k\}_{k \geq 2}$ are mutually orthogonal in $L^2_{\omega^{\alpha-1, \beta-1}}(-1, 1)$ (where $\omega^{\alpha-1, \beta-1} = (1-x)^{\alpha-1}(1+x)^{\beta-1}$), which follows from (6.1) and the orthogonality of Jacobi polynomials. Moreover, $\{\phi_k\}_{k \geq 0}$ form a complete basis of $X := \mathbb{P}_1 \cup L^2_{\omega^{\alpha-1, \beta-1}}(-1, 1)$ (which reduces to $L^2_{\omega^{\alpha-1, \beta-1}}(-1, 1)$, if $\alpha, \beta > 0$). For any $u \in X$, we expand it as $u(x) = \sum_{k=0}^{\infty} \hat{u}_k \phi_k(x)$, where $\{\hat{u}_k\}$ can be uniquely determined. Then we have

$$(u - I_N u)(x) = \hat{u}_{N+1}(\phi_{N+1} - I_N \phi_{N+1})(x) + \sum_{k=N+2}^{\infty} \hat{u}_k(\phi_k - I_N \phi_k)(x). \tag{2.16}$$

We have $I_N \phi_{N+1} \equiv 0$ (which is actually aliased). Indeed, from the expansion: $(I_N \phi_{N+1})(x) = \sum_{k=0}^N \tilde{u}_k P_k^{(\alpha, \beta)}(x)$, we find that the pseudospectral coefficients $\{\tilde{u}_k\}$ vanish, as

$$\tilde{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^N \phi_{N+1}(x_j) P_k^{(\alpha, \beta)}(x_j) \omega_j = 0, \quad 0 \leq k \leq N,$$

where $\{\gamma_k\}$ are constants, $\{\omega_j\}$ are the JGL quadrature weights and we have used the fact $\phi_{N+1}(x_j) = 0$ for $0 \leq j \leq N$. This implies

$$(u - I_N u)'(x) = \hat{u}_{N+1} \phi'_{N+1}(x) + \sum_{k=N+2}^{\infty} \hat{u}_k (\phi_k - I_N \phi_k)'(x). \tag{2.17}$$

Consequently, at the zeros of ϕ'_{N+1} (i.e., the superconvergence points $\{y_j\}_{j=1}^N$ in Proposition 2.4), the leading term of the error remainder in (2.17) vanishes.

In summary, we infer from the previous discussions that at the superconvergence points, the leading term of the interpolation error remainder for functions with limited regularity, vanishes. This leads to better accuracy, but there is no gain in order of convergence, which should be in contrast with interpolating analytic functions.

3 Error Estimates: A General Framework

In this section, we analyze interpolation errors of analytical functions, and intend to estimate

$$\max_j |(u - I_N u)^{(k)}(x_j)|, \quad \max_i |(u - I_N u)^{(k)}(y_i)|, \quad k = 1, 2, \tag{3.1}$$

where we recall that $\{x_j\}$ are the Jacobi–Gauss-type points, and $\{y_i\}$ are the corresponding superconvergence points in Proposition 2.1.

From the reminders in (2.6) and (2.8)–(2.9), we derive the following general results. Note that at this stage, the estimates depend on several constants involving N , which will be estimated in Sect. 4 for specific sets of interpolation points.

Theorem 3.1 *Let $\{x_j\}_{j=0}^N$ be the zeros of $Q_{N+1}(x)$ defined in (2.7), and let $I_N u$ be the associated Lagrange interpolant. Denote*

$$m_Q = \min_{z \in \mathcal{E}_\rho} |Q_{N+1}(z)|, \quad M_Q = \max_{|x| \leq 1} |Q_{N+1}(x)|, \quad M_{Q'} = \max_{|x| \leq 1} |Q'_{N+1}(x)|, \tag{3.2}$$

and likewise $M_{Q''}$. For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, define $M = \max_{z \in \mathcal{E}_\rho} |u(z)|$. Then

(i) for the first derivative, we have

$$\max_{0 \leq j \leq N} |(u - I_N u)'(x_j)| \leq \frac{ML(\mathcal{E}_\rho) M_{Q'}}{2\pi d_\rho m_Q}, \tag{3.3}$$

and at the corresponding superconvergence points $\{y_j\}_{j=1}^N$ (i.e., zeros of $Q'_{N+1}(x)$), we have

$$\max_{1 \leq j \leq N} |(u - I_N u)'(y_j)| \leq \frac{ML(\mathcal{E}_\rho) M_Q}{2\pi d_\rho^2 m_Q}; \tag{3.4}$$

(ii) for the second derivative, we have

$$\max_{0 \leq j \leq N} |(u - I_N u)''(x_j)| \leq \frac{ML(\mathcal{E}_\rho)}{2\pi d_\rho} \left(\frac{M_{Q''}}{m_Q} + \frac{2}{d_\rho} \frac{M_{Q'}}{m_Q} \right), \tag{3.5}$$

and at the corresponding superconvergence points $\{z_j\}_{j=1}^{N-1}$ (i.e., zeros of $Q''_{N+1}(x)$), we have

$$\max_{1 \leq j \leq N-1} |(u - I_N u)''(z_j)| \leq \frac{ML(\mathcal{E}_\rho)}{\pi d_\rho^2} \left(\frac{M_{Q''}}{m_Q} + \frac{1}{d_\rho} \frac{M_Q}{m_Q} \right). \tag{3.6}$$

Here, the constants $L(\mathcal{E}_\rho)$ and d_ρ are defined in (2.2) and (2.3), respectively.

Proof As $Q_{N+1}(x_j) = 0$, we derive from (2.8) that

$$\max_{0 \leq j \leq N} |(u - I_N u)'(x_j)| \leq \frac{M_{Q'} M}{2\pi m_Q} \max_{0 \leq j \leq N} \oint_{\mathcal{E}_\rho} \frac{|dz|}{|z - x_j|} \leq \frac{ML(\mathcal{E}_\rho)}{2\pi d_\rho} \frac{M_{Q'}}{m_Q}, \tag{3.7}$$

which leads to (3.3). Similarly, (3.5) follows from (2.9).

Using the fact $Q'_{N+1}(y_j) = 0$ (resp. $Q''_{N+1}(z_j) = 0$, see Proposition 2.1), we can obtain (3.4) (resp. (3.6)) in a fashion very similar to (3.7). \square

The following property is a direct consequence of Theorem 3.1.

Corollary 3.1 *At superconvergence points, the maximum of the first derivative of the interpolation error converges at the same rate as the maximum interpolation error. Similarly, the maximum of the second derivative of the interpolation error at superconvergence points enjoys the same convergence rate as that of the first derivative.*

Proof We find from (2.6) that

$$\max_{|x| \leq 1} |(u - I_N u)(x)| \leq \frac{ML(\mathcal{E}_\rho)}{2\pi d_\rho} \frac{M_Q}{m_Q}. \tag{3.8}$$

Hence, we claim the first statement from (3.4). Similarly, by (2.8),

$$\max_{|x| \leq 1} |(u - I_N u)'(x)| \leq \frac{ML(\mathcal{E}_\rho)}{2\pi d_\rho} \left(\frac{M_{Q'}}{m_Q} + \frac{1}{d_\rho} \frac{M_Q}{m_Q} \right). \tag{3.9}$$

Thus, the second claim follows from (3.6). \square

Remark 3.1 Since the above error bounds share the common factor $1/m_Q$, it suffices to show that $M_{Q'} \sim N^\gamma M_Q$ and $M_{Q''} \sim N^\delta M_{Q'}$ (for $\gamma, \delta > 0$) to obtain the gain in order of N at superconvergence points. \square

4 Error Estimates for Jacobi–Gauss-Type Interpolation

To derive the error bounds, it remains to estimate the constants $M_Q, M_{Q'}, M_{Q''}$ and m_Q . For a specific set of interpolation points, the former three constants are not difficult to obtain from the relevant properties of orthogonal polynomials (which only contribute to some powers of N), but much care has to be taken to estimate m_Q (which actually leads to the exponential convergence). Note that for the Chebyshev case, $Q_{N+1}(z)$ with $z \in \mathcal{E}_\rho$ has a simple closed form, so the estimation of m_Q becomes straightforward (see e.g., [3, 8, 13, 24]). The recent work [21] derived the lower bound of m_Q for the Gegenbauer–Gauss (GG) and Gegenbauer–Gauss–Lobatto (GGL) points. Typically, they are symmetric cases, as the interior interpolation points are zeros of certain Jacobi polynomial with parameters $\alpha = \beta$. However, the results for the non-symmetric cases with $\alpha \neq \beta$, for example, the Gauss–Radau interpolation and Jacobi–Gauss-type interpolation with $\alpha \neq \beta$, are lacking. The rest of this paper is devoted to the analysis of these missing cases. We first consider the general Jacobi case, and then refine the results for some special cases including the Gegenbauer–Gauss–Radau and Legendre–Gauss–Radau interpolation.

4.1 Jacobi–Gauss-Type Interpolation

To fix idea, we focus on Jacobi–Gauss interpolation. In this case, we have $Q_{N+1}(x) = P_{N+1}^{(\alpha,\beta)}(x)$, and need to estimate the corresponding $M_Q, M_{Q'}, M_{Q''}$ and m_Q in Theorem 3.1.

We start with a property of the Gamma function. Recall that (see [1, (6.1.38)]):

$$\Gamma(x + 1) = \sqrt{2\pi}x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right), \quad \forall x > 0, \quad 0 < \theta < 1. \tag{4.1}$$

Using this formula, we can show that (see [25, Lemma 2.1]) for any constants a, b , independent of n , and for $n \geq 1, n + a > 1$ and $n + b > 1$,

$$\exp(-\Upsilon_n^{b,a})n^{a-b} \leq \frac{\Gamma(n+a)}{\Gamma(n+b)} \leq \exp(\Upsilon_n^{a,b})n^{a-b}, \tag{4.2}$$

where

$$\Upsilon_n^{a,b} = \frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-1)(a-b)}{n}. \tag{4.3}$$

Note that $\exp(\Upsilon_n^{a,b}) \approx 1$ for large n .

Let $q := \max(\alpha, \beta)$ with $\alpha, \beta > -1$. Then we obtain from Szegő [15, Theorem 7.32.1] that

$$M_Q = \max_{|x| \leq 1} |P_{N+1}^{(\alpha,\beta)}(x)| = \begin{cases} |P_{N+1}^{(\alpha,\beta)}(x')| \leq c_N(\alpha, \beta)N^{-1/2}, & \text{if } q < -1/2; \\ \frac{\Gamma(N+q+2)}{\Gamma(N+2)\Gamma(q+1)} \leq \frac{\exp(\Upsilon_N^{q+2,2})}{\Gamma(q+1)}N^q, & \text{if } q \geq -1/2, \end{cases} \tag{4.4}$$

where x' is one of two maximum points nearest $(\beta - \alpha)/(\alpha + \beta + 1)$, and $c_N(\alpha, \beta) \approx 1$ for large N . Hence, by (6.1) and (4.4),

$$\begin{aligned} M_{Q'} &= \max_{|x| \leq 1} |\partial_x P_{N+1}^{(\alpha,\beta)}(x)| = \frac{1}{2}(N + \alpha + \beta + 2) \max_{|x| \leq 1} |P_N^{(\alpha+1,\beta+1)}(x)| \\ &\leq \frac{1}{2}(N + \alpha + \beta + 2) \frac{\exp(\Upsilon_N^{q+2,1})}{\Gamma(q+2)}N^{q+1}; \\ M_{Q''} &\leq \frac{1}{4}(N + \alpha + \beta + 2)(N + \alpha + \beta + 3) \frac{\exp(\Upsilon_N^{q+2,0})}{\Gamma(q+3)}N^{q+2}. \end{aligned} \tag{4.5}$$

To estimate m_Q , we resort to the asymptotic representation (see Szegő [15, Theorem 8.21.9]):

$$P_N^{(\alpha,\beta)}(z) = \phi_0(w; \alpha, \beta)N^{-1/2}w^N + O(N^{-1}), \quad \forall z = \frac{1}{2}(w + w^{-1}) \in \mathcal{E}_\rho, \quad |w| = \rho, \tag{4.6}$$

where $\phi_0(w; \alpha, \beta)$ is regular for $|w| = \rho > 1$, and $|w| = 1$ but $w \neq \pm 1$. Thus, we have

$$m_Q = \min_{z \in \mathcal{E}_\rho} |P_{N+1}^{(\alpha,\beta)}(z)| \geq C(\rho; \alpha, \beta)N^{-\frac{1}{2}}\rho^{N+1}(1 + O(N^{-1})), \tag{4.7}$$

where $C(\rho; \alpha, \beta) = \min_{|w|=\rho} |\phi_0(w; \alpha, \beta)|$, independent of N .

With a little abuse of notation, we use $c_N(\alpha, \beta)$ to denote the generic positive constant such that $c_N(\alpha, \beta) \approx 1$ for large N . For example, for $q = \max(\alpha, \beta) \geq -1/2$, we denote

$$c_N(\alpha, \beta) = \frac{\exp\left(\Upsilon_N^{q+2,1}\right)}{1 + O(N^{-1})},$$

which is extracted from the ratio $M_{Q'}/m_{Q'}$, and likewise for M_Q/m_Q and $M_{Q''}/m_{Q''}$. Then the error bounds for Jacobi–Gauss interpolation follow from Theorem 3.1.

Theorem 4.1 *For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, let $(I_N u)(x)$ be the interpolant of $u(x)$ at the Jacobi–Gauss points $\{x_j\}_{j=0}^N$, and let $q = \max(\alpha, \beta)$ for $\alpha, \beta > -1$. Then*

(i) *we have*

$$\max_{0 \leq j \leq N} |(u - I_N u)'(x_j)| \leq \frac{c_N(\alpha, \beta)}{4\pi\Gamma(q + 2)} \frac{ML(\mathcal{E}_\rho)}{d_\rho C(\rho; \alpha, \beta)} \frac{N^{q+5/2}}{\rho^{N+1}}, \tag{4.8}$$

and at the corresponding superconvergence points $\{y_j\}_{j=1}^N$, we have

$$\begin{aligned} &\max_{1 \leq j \leq N} |(u - I_N u)'(y_j)| \\ &\leq \frac{c_N(\alpha, \beta)ML(\mathcal{E}_\rho)}{2\pi d_\rho^2 C(\rho; \alpha, \beta)} \frac{1}{\rho^{N+1}} \begin{cases} 1, & \text{if } q < -1/2, \\ N^{q+1/2}/\Gamma(q + 1), & \text{if } q \geq -1/2; \end{cases} \end{aligned} \tag{4.9}$$

(ii) *for the second derivative, we have*

$$\max_{0 \leq j \leq N} |(u - I_N u)''(x_j)| \leq \frac{c_N(\alpha, \beta)}{8\pi\Gamma(q + 3)} \frac{ML(\mathcal{E}_\rho)}{d_\rho C(\rho; \alpha, \beta)} \frac{N^{q+9/2}}{\rho^{N+1}}, \tag{4.10}$$

and at the corresponding superconvergence points $\{z_j\}_{j=1}^{N-1}$, we have

$$\max_{1 \leq j \leq N-1} |(u - I_N u)''(z_j)| \leq \frac{c_N(\alpha, \beta)}{2\pi\Gamma(q + 2)} \frac{ML(\mathcal{E}_\rho)}{d_\rho^2 C(\rho; \alpha, \beta)} \frac{N^{q+5/2}}{\rho^{N+1}}. \tag{4.11}$$

Remark 4.1 We see that we gain a factor of N^2 at superconvergence points. □

Remark 4.2 In both the Jacobi–Gauss–Radau and Jacobi–Gauss–Lobatto cases, Q_{N+1} is a linear combination of $\{P_{N+k}^{(\alpha, \beta)} : k = 0, \pm 1\}$ (see (6.2)–(6.3)), so we can estimate the interpolation errors in a very similar fashion. □

It is seen from (4.7) that the dependence of $C(\rho; \alpha, \beta)$ in the asymptotic estimate (4.7) on ρ is implicit. However, for the Chebyshev polynomial, we can explicitly characterize this, and also the estimate of m_Q is valid for all $N \geq 1$. Indeed, we have the closed form (see e.g., [3]):

$$T_{N+1}(z) = \frac{1}{2}(w^{N+1} + w^{-(N+1)}), \quad z \in \mathcal{E}_\rho, \quad |w| = \rho,$$

which implies

$$\min_{z \in \mathcal{E}_\rho} |T_{N+1}(z)| \geq \frac{1}{2}(\rho^{N+1} - \rho^{-N-1}). \tag{4.12}$$

With a re-normalization of T_{N+1} , we find from (4.2) that

$$\begin{aligned}
 m_Q &= \min_{z \in \mathcal{E}_\rho} |P_{N+1}^{(-1/2, -1/2)}(z)| = \frac{\Gamma(N + 3/2)}{\sqrt{\pi}(N + 1)!} \min_{z \in \mathcal{E}_\rho} |T_{N+1}(z)| \\
 &\geq \frac{\exp(-\Upsilon_{N+1}^{1,1/2})}{2\sqrt{\pi}} \left(1 - \frac{1}{\rho^{2N+2}}\right) N^{-1/2} \rho^{N+1}, \quad N \geq 1.
 \end{aligned}
 \tag{4.13}$$

In fact, such an estimate is also available for the Gegenbauer polynomial (i.e., $\alpha = \beta$), but the analysis is much more involved (see [21]). However, the analysis in [21] appears nontrivial to be extended to the case with $\alpha \neq \beta$. For example, m_Q in the Gegenbauer–Gauss–Radau interpolation is associated with the Jacobi polynomials with the parameter $(\alpha, \alpha + 1)$. For clarity of presentation, we just give the details for estimating m_Q of the Legendre–Gauss–Radau (LGR) interpolation (i.e., $\alpha = 0$), which can be extended to the Gegenbauer case (i.e., $\alpha \neq 0$) straightforwardly.

4.2 LGR Interpolation

In this case, the interpolation points are zeros of

$$Q_{N+1}(x) = P_N(x) + P_{N+1}(x) = (1 + x)P_N^{(0,1)}(x),
 \tag{4.14}$$

where the last identity follows from (6.2).

We state the main result on the estimate for m_Q as follows.

Theorem 4.2 *Let Q_{N+1} be the polynomial defined in (4.14). Then for any integer $1 < K < N$, we have*

$$m_Q = \min_{z \in \mathcal{E}_\rho} |Q_{N+1}(z)| \geq A(K, N; \rho) \frac{e^{-\frac{5}{6N}} \rho^{N+1}}{\sqrt{\pi} \sqrt{N}},
 \tag{4.15}$$

where

$$A(K, N; \rho) = \frac{\rho - 1}{\sqrt{1 + \rho^2}} - \left\{ \left(\sqrt{\frac{\rho + 1}{\rho - 1}} - 1 \right) D_{K,N} + \frac{1}{\rho^{2K}(\rho - 1)} \right\},
 \tag{4.16}$$

with

$$D_{K,N} = \frac{K}{2(N - K)} + \frac{2N - K}{2(N - K)} \left(e^{\frac{5}{12(N-K)}} - 1 \right).
 \tag{4.17}$$

In particular, if $N \gg 1$, we can choose $K = o(N)$ such that

$$m_Q \geq \frac{c_N}{\sqrt{\pi}} \frac{\rho - 1}{\sqrt{1 + \rho^2}} \frac{\rho^{N+1}}{\sqrt{N}},
 \tag{4.18}$$

where the constant $c_N \approx 1$ for large N .

The proof of this theorem is rather involved. To prevent distracting from the main results, we postpone the proof to Appendix B.

Next, we provide some numerical illustrations to the tightness of the lower bound of m_Q . For fixed ρ , we compute m_Q by sampling very dense points on \mathcal{E}_ρ , and scale m_Q and the corresponding lower bound by the factor ρ^{N+1}/\sqrt{N} . More precisely, we look at the exact value $m_Q \sqrt{N}/\rho^{N+1}$ and the scaled lower bound $A(K, N; \rho) e^{-\frac{5}{6N}}/\sqrt{\pi}$. We choose $K = \lceil N^\varepsilon \rceil$ with $\varepsilon = 0.5$, and plot in Fig. 3 the exact value and lower bound with $\rho = 1.2$

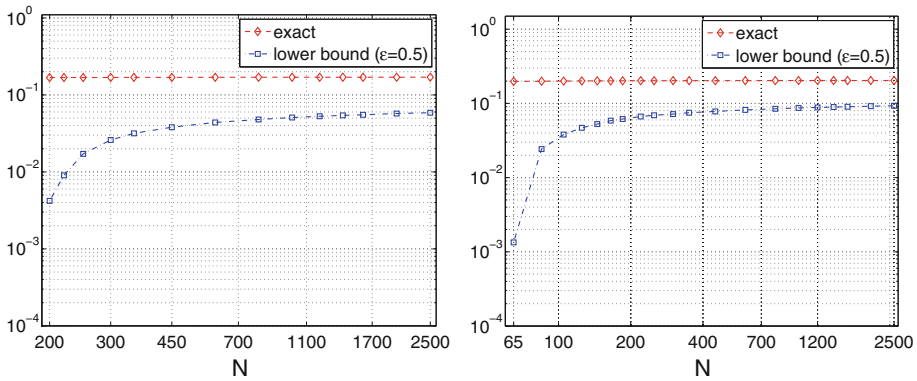


Fig. 3 Comparison of m_Q and the lower bound with $\rho = 1.2$ (left) and $\rho = 1.3$ (right)

(left) and $\rho = 1.3$ (right), respectively. Observe that when N increases, the lower bound tends to m_Q gradually and the rate is $N^{\epsilon-1}$ (see (7.19)).

Now, we continue to estimate M_Q etc in Theorem 3.1. We find from (4.4), (4.5) and (4.14) that

$$\begin{aligned}
 M_Q &\leq 2, \quad M_{Q'} \leq \frac{1}{2} \left((N^2 + 2N) \exp(\Upsilon_N^{2,1}) + (N^2 - 1) \exp(\Upsilon_{N-1}^{2,1}) \right), \\
 M_{Q''} &\leq \frac{N+2}{8} \left((N+3)N^2 \exp(\Upsilon_N^{2,0}) + (N+1)(N-1)^2 \exp(\Upsilon_{N-1}^{2,0}) \right). \quad (4.19)
 \end{aligned}$$

From Theorem 3.1, (4.18) and (4.19), we obtain the following estimates.

Theorem 4.3 For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, let $(I_{Nu})(x)$ be the interpolant of $u(x)$ at the set of $(N + 1)$ Legendre–Gauss–Radau points $\{x_j\}_{j=0}^N$. Then

(i) we have

$$\max_{0 \leq j \leq N} |(u - I_{Nu})'(x_j)| \leq c_N \frac{ML(\mathcal{E}_\rho)}{2\sqrt{\pi}d_\rho} \frac{\sqrt{1 + \rho^2}}{\rho - 1} \frac{N^{5/2}}{\rho^{N+1}}, \quad (4.20)$$

and at the corresponding superconvergence points $\{y_j\}_{j=1}^N$, we have

$$\max_{1 \leq j \leq N} |(u - I_{Nu})'(y_j)| \leq c_N \frac{ML(\mathcal{E}_\rho)}{\sqrt{\pi}d_\rho^2} \frac{\sqrt{1 + \rho^2}}{\rho - 1} \frac{\sqrt{N}}{\rho^{N+1}}; \quad (4.21)$$

(ii) for the second derivative, we have

$$\max_{0 \leq j \leq N} |(u - I_{Nu})''(x_j)| \leq c_N \frac{ML(\mathcal{E}_\rho)}{8\sqrt{\pi}d_\rho} \frac{\sqrt{1 + \rho^2}}{\rho - 1} \frac{N^{9/2}}{\rho^{N+1}}, \quad (4.22)$$

and at the corresponding superconvergence points $\{z_j\}_{j=1}^{N-1}$, we have

$$\max_{1 \leq j \leq N-1} |(u - I_{Nu})''(z_j)| \leq c_N \frac{ML(\mathcal{E}_\rho)}{\sqrt{\pi}d_\rho^2} \frac{\sqrt{1 + \rho^2}}{\rho - 1} \frac{N^{5/2}}{\rho^{N+1}}. \quad (4.23)$$

Here, $M = \max_{z \in \mathcal{E}_\rho} |u(z)|$, and $c_N \approx 1$ for $N \gg 1$.

Remark 4.3 We can apply the same argument to estimate the Gegenbauer–Gauss–Radau interpolation errors, and have to resort to the formula in [21, Lemma 3.1] as a generalization of (7.1)–(7.2). Here, we just state the result (for the first derivative) and leave the proof to the interested readers. For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, let $(I_N u)(x)$ be the interpolant of $u(x)$ at the GGR points $\{x_j\}_{j=0}^N$. Then we have

$$\begin{aligned} & \max_{0 \leq j \leq N} |(u - I_N u)'(x_j)| \\ & \leq \frac{2c_N ML(\mathcal{E}_\rho)}{\pi d_\rho(\rho - 1)} \frac{\Gamma(\alpha + 3/2)}{\Gamma(2\alpha + 3)} \frac{N^{\alpha+5/2}}{\rho^N} \begin{cases} (1 - \rho^{-2})^{\alpha+1/2}, & \alpha < -1/2, \\ (1 + \rho^{-2})^{\alpha+1/2}, & \alpha > -1/2; \end{cases} \end{aligned} \tag{4.24}$$

and at the corresponding superconvergence points $\{y_j\}_{j=1}^N$, we have

$$\begin{aligned} & \max_{1 \leq j \leq N} |(u - I_N u)'(y_j)| \\ & \leq \frac{c_N ML(\mathcal{E}_\rho)}{\pi d_\rho^2(\rho - 1)} \frac{1}{\rho^N} \begin{cases} 2^{1/2-\alpha}(1 - \rho^{-2})^{\alpha+1/2}, & \alpha < -1/2, \\ \frac{2^{-2\alpha}\sqrt{\pi}}{\Gamma(\alpha + 1)}(1 + \rho^{-2})^{\alpha+1/2}N^{\alpha+1/2}, & \alpha > -1/2. \end{cases} \end{aligned} \tag{4.25}$$

Note that we exclude the Chebyshev case (with $\alpha = -1/2$), as it can be derived from (4.13) (see e.g., [13, 24]). □

5 Numerical Results and Concluding Remarks

We present in this section some numerical examples to illustrate the superconvergence phenomenon by comparing the convergence behavior between the superconvergence points and the interpolation points. We then conclude the paper with some remarks.

5.1 Numerical Examples

We first consider the function:

$$u(x) = \frac{1}{1 + 25x^2}, \tag{5.1}$$

which has two simple poles at $\pm i/5$ in the complex plane. Hence, it is analytic within the Bernstein ellipse satisfying $(\rho - \rho^{-1})/2 < 1/5$, so we have $1 < \rho < (1 + \sqrt{5^2 + 1})/5 \approx 1.2198$.

Let us look at the Legendre–Gauss–Lobatto case. We know from Remark 2.3 that the superconvergence points for the first derivative are the N Legendre–Gauss points $\{y_j\}_{j=1}^N$. Let $r_N(x) = (u - I_N u)(x)$. In Fig. 4a, we plot the error curve $r'_N(x)$ and the samples $\{r'_N(x_j)\}_{j=0}^N$ and $\{r'_N(y_j)\}_{j=1}^N$ with $N = 55$. Observe that the interior LGL points are the extreme points of $r'_N(x)$, while the magnitude of the errors at the superconvergence points is significantly smaller. We now numerically verify the claims in Corollary 3.1:

$$\max_{1 \leq j \leq N} |r'_N(y_j)| \sim \max_{|x| \leq 1} |r_N(x)| \sim N^{-2} \max_{0 \leq j \leq N} |r'_N(x_j)|.$$

For this purpose, we eliminate the exponential order ρ^{-N} with $\rho = 1.2198$, and plot them with various N in Fig. 4b. The numerical results agree well with what we have claimed. Very

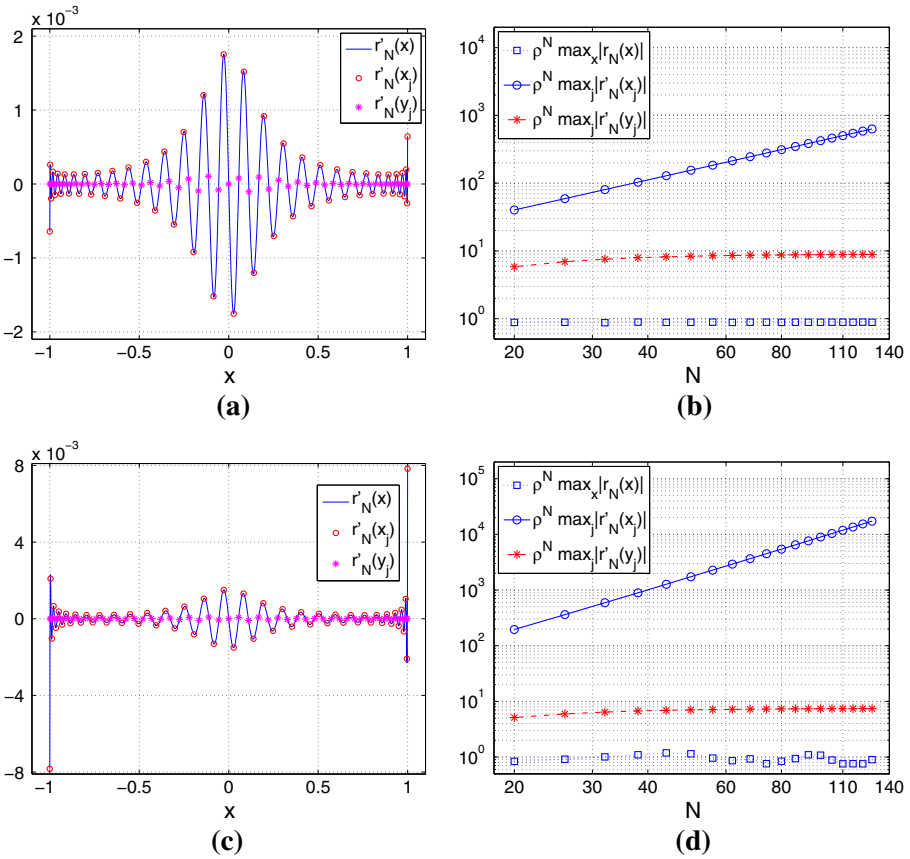


Fig. 4 Superconvergence of Gauss-Lobatto interpolation. **a** LGL interpolation error curve. **b** ρ^N multiple of the errors (LGL). **c** JGL interpolation error curve ($\alpha = \beta = 1$). **d** ρ^N multiple of the errors (JGL $\alpha = \beta = 1$)

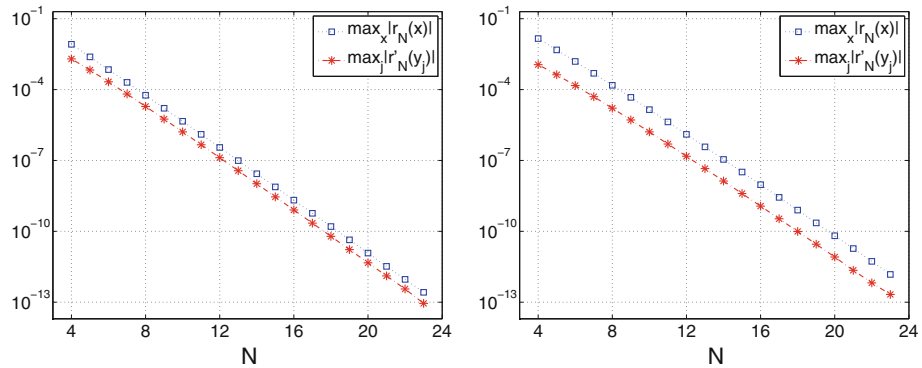


Fig. 5 Superconvergence of GGR interpolation: $\alpha = 0$ (left) and $\alpha = 1$ (right)

similar behavior is observed from Fig. 4c–d for the Jacobi–Gauss–Lobatto interpolation with $\alpha = \beta = 1$.

Next, we consider the interpolation of $u(x) = \frac{1}{x-2}$, which has a simple pole $x = 2$ on the x -axis. It is analytic on and within the Bernstein ellipse with $1 < \rho < 2 + \sqrt{3}$. In Fig. 5, we plot the maximum interpolation error and the error at the superconvergence points for the first derivative of the Gauss–Radau interpolation. Once again, we observe the same convergence behavior as claimed in Corollary 3.1.

5.2 Concluding Remarks

In this paper, we studied the superconvergence of Jacobi–Gauss-type spectral interpolation. Following [24], we identified superconvergence points from the interpolation error remainder represented by Hermite’s contour integral under analytic assumption. The main contributions of this paper resided in that (i) for the first time, we had useful insights into the superconvergence of functions with limited regularity, and (ii) we provided a general framework for exponential convergence and superconvergence analysis, and obtained the error bounds of the type $C(N)\rho^{-N}$ (with explicit dependence of $C(N)$ on N) for general Jacobi–Gauss-type interpolations.

6 Appendix A: Jacobi Polynomials

The Jacobi polynomials satisfy the derivative relation (see [15, (4.21.7)]):

$$\partial_x P_N^{(\alpha,\beta)}(x) = \frac{1}{2}(N + \alpha + \beta + 1)P_{N-1}^{(\alpha+1,\beta+1)}(x), \tag{6.1}$$

and there holds (see [15, (4.5.4)]):

$$P_N^{(\alpha,\beta+1)}(x) = \frac{2}{2N + \alpha + \beta + 2} \frac{(N + \beta + 1)P_N^{(\alpha,\beta)}(x) + (N + 1)P_{N+1}^{(\alpha,\beta)}(x)}{1 + x}. \tag{6.2}$$

Another recurrence relation reads (see [15, (4.5.5)–(4.5.6)]):

$$(1 - x^2)\partial_x P_N^{(\alpha,\beta)}(x) = AP_{N-1}^{(\alpha,\beta)}(x) + BP_N^{(\alpha,\beta)}(x) + CP_{N+1}^{(\alpha,\beta)}(x), \tag{6.3}$$

where

$$A = \frac{2(N + \alpha)(N + \beta)(N + \alpha + \beta + 1)}{(2N + \alpha + \beta)(2N + \alpha + \beta + 1)}, \quad B = \frac{2(\alpha - \beta)N(N + \alpha + \beta + 1)}{(2N + \alpha + \beta)(2N + \alpha + \beta + 2)},$$

$$C = -\frac{2N(N + 1)(N + \alpha + \beta + 1)}{(2N + \alpha + \beta + 1)(2N + \alpha + \beta + 2)}.$$

7 Appendix B: Proof of Theorem 4.2

We first present some necessary lemmas for its proof.

The following formula (see [3, Lemma 12.4.1]) is of paramount importance.

Lemma 7.1 *Let $z = (w + w^{-1})/2$. Then*

$$P_N(z) = \sum_{k=0}^N g_k g_{N-k} w^{N-2k}, \tag{7.1}$$

where

$$g_k = \frac{(2k)!}{(k!)^2 2^{2k}} = \frac{\Gamma(k + 1/2)}{\sqrt{\pi} k!}, \quad k \geq 0. \tag{7.2}$$

In fact, the coefficients $\{g_k\}$ relate to the following Laurent series expansion.

Lemma 7.2 *We have*

$$F(w) := (1 - w^{-2})^{-1/2}(1 + w^{-1}) = \sum_{k=0}^{\infty} \frac{g_k}{w^{2k}} + \sum_{k=0}^{\infty} \frac{g_k}{w^{2k+1}}, \tag{7.3}$$

which converges uniformly for all complex-valued w such that $|w| > 1$.

Proof Recall the binomial expansion:

$$(1 - w^{-2})^{-1/2} = \sum_{k=0}^{\infty} \frac{g_k}{w^{2k}}, \quad \forall |w| > 1. \tag{7.4}$$

This implies (7.3). □

The key idea of estimating m_Q is to show that for $z \in \mathcal{E}_\rho$ with $|w| = \rho > 1$,

$$\left| F(w) - \frac{Q_{N+1}(z)}{g_{N+1}w^{N+1}} \right| \rightarrow 0 \text{ as } N \rightarrow \infty, \tag{7.5}$$

and more importantly, we care about the rate it decays. For this purpose, let us split the error term into two parts:

$$\begin{aligned} \left| F(w) - \frac{Q_{N+1}(z)}{g_{N+1}w^{N+1}} \right| &= \left| \left(\sum_{k=0}^{\infty} \frac{g_k}{w^{2k}} - \sum_{k=0}^{N+1} \frac{g_k g_{N+1-k}}{g_{N+1}w^{2k}} \right) \right. \\ &\quad \left. + \left(\sum_{k=0}^{\infty} \frac{g_k}{w^{2k+1}} - \sum_{k=0}^N \frac{g_k g_{N-k}}{g_{N+1}w^{2k+1}} \right) \right| \\ &\leq R_N^e(\rho) + R_N^o(\rho), \end{aligned} \tag{7.6}$$

where

$$R_N^e(\rho) = \sum_{k=0}^{N+1} \frac{|q_k||g_k|}{\rho^{2k}} + \sum_{k=N+2}^{\infty} \frac{|g_k|}{\rho^{2k}}; \quad R_N^o(\rho) = \sum_{k=0}^N \frac{|q_{k+1}||g_k|}{\rho^{2k+1}} + \sum_{k=N+1}^{\infty} \frac{|g_k|}{\rho^{2k+1}}, \tag{7.7}$$

with

$$q_k := q_k(N) := \frac{g_{N+1-k}}{g_{N+1}} - 1, \quad 0 \leq k \leq N + 1. \tag{7.8}$$

We deduce from (7.2) and (7.8) the following useful properties of $\{g_k\}$ and $\{q_k\}$.

Lemma 7.3 (i) *For $k \geq 0$, $g_k > 0$, and $\{g_k\}$ is strictly decreasing, namely,*

$$1 = g_0 > g_1 > \dots > g_k > g_{k+1} > \dots. \tag{7.9}$$

(ii) *We have*

$$0 = q_0 < q_1 < \dots < q_N < q_{N+1}. \tag{7.10}$$

Moreover, $(q_k + 1)g_k < 1$, for $1 \leq k \leq N$, and $(q_{k+1} + 1)g_k < 1$ for $1 \leq k \leq N - 1$. In addition, $(q_{N+1} + 1)g_{N+1} = 1$, and $q_{N+1}g_N < 1$ for $N \geq 2$.

Proof (i) It is clear that by (7.2), $g_0 = 1$ and $0 < g_k < 1$ for all $k \geq 1$. Since

$$\frac{g_{k+1}}{g_k} = \frac{k + 1/2}{k + 1} < 1, \tag{7.11}$$

$\{g_k\}$ is strictly decreasing with respect to k .

(ii) Observe from (7.8) and (i) that $q_0 = 0$ and $\{q_k\}$ is strictly increasing. A direct calculation leads to that for $1 \leq k \leq N - 1$,

$$\begin{aligned} (q_{k+1} + 1)g_k &= \frac{g_{N-k}}{g_{N+1}}g_k = \left(\prod_{j=0}^{k-2} \frac{1 - \frac{1/2}{k-j}}{1 - \frac{1/2}{N+1-j}} \right) \frac{(N - k + 2)(N - k + 1)}{2(N - k + 3/2)(N - k + 1/2)} \\ &\leq \frac{1}{2} \left(1 + \frac{1/2}{N - k + 3/2} \right) \left(1 + \frac{1/2}{N - k + 1/2} \right) \leq \frac{4}{5} < 1, \end{aligned}$$

where we used the fact $1 + \frac{1/2}{N-k+3/2}$ and $1 + \frac{1/2}{N-k+1/2}$ are strictly increasing with respect to k . Note that if $k = 1$, the first term in the second identity equals 1.

By (7.9)–(7.10), we have

$$(q_{k+1} + 1)g_{k+1} < (q_{k+1} + 1)g_k; \quad (q_k + 1)g_k < (q_{k+1} + 1)g_k,$$

which implies $(q_k + 1)g_k < 1$ for $1 \leq k \leq N$.

Next, by (7.2) and (7.8), we have $(q_{N+1} + 1)g_{N+1} = 1$ and

$$q_{N+1}g_N = \frac{g_N}{g_{N+1}} - g_N = 1 + \frac{1}{2N + 1} - g_N < 1,$$

since $g_N = \frac{\Gamma(N+1/2)}{\sqrt{\pi}N\Gamma(N)} > \frac{1}{\sqrt{\pi}N} > \frac{1}{2N+1}$ for $N \geq 2$. This ends the proof. □

Lemma 7.4 *We have*

$$0 < q_k \leq D_{k,N}, \quad 1 \leq k \leq N - 1, \tag{7.12}$$

where $D_{k,N}$ is defined in (4.17).

Proof By (7.2) and (7.8), we obtain from (4.2)–(4.3) that

$$\begin{aligned} q_k + 1 &= \frac{g_{N+1-k}}{g_{N+1}} = \frac{\Gamma(N - k + 3/2)}{\Gamma(N - k + 2)} \frac{\Gamma(N + 2)}{\Gamma(N + 3/2)} \\ &\leq \sqrt{\frac{N + 1}{N - k + 1}} \exp(\Upsilon_{N-k+1}^{1/2,1} + \Upsilon_{N+1}^{1,1/2}). \end{aligned}$$

A direct calculation leads to

$$\begin{aligned} \Upsilon_{N-k+1}^{1/2,1} + \Upsilon_{N+1}^{1,1/2} &= \frac{1}{6(2N - 2k + 1)} + \left(\frac{1}{2(2N + 1)} + \frac{1}{12(N + 1)} \right) \\ &\leq \frac{1}{12(N - k)} + \frac{1}{3N} \leq \frac{5}{12(N - k)}, \quad 1 \leq k \leq N - 1. \end{aligned}$$

Using the fact $\sqrt{1 + x} \leq 1 + \frac{x}{2}$ (with $x \geq 0$) yields

$$\sqrt{\frac{N + 1}{N - k + 1}} \leq 1 + \frac{k}{2(N - k + 1)} \leq 1 + \frac{k}{2(N - k)}.$$

Consequently, we obtain

$$q_k + 1 \leq \left(1 + \frac{k}{2(N - k)}\right) \exp\left(\frac{5}{12(N - k)}\right), \tag{7.13}$$

which gives the desired upper bound. \square

Proof of Theorem 4.2 By (7.6),

$$|Q_{N+1}(z)| \geq g_{N+1} \rho^{N+1} \{|F(w)| - (R_N^e(\rho) + R_N^o(\rho))\}. \tag{7.14}$$

As $|w| = \rho$, we find

$$|F(w)| = |1 - w^{-2}|^{-1/2} |1 + w^{-1}| \geq \frac{1 - \rho^{-1}}{\sqrt{1 + \rho^{-2}}} = \frac{\rho - 1}{\sqrt{1 + \rho^2}}. \tag{7.15}$$

By (7.2) and (4.2),

$$g_{N+1} \geq \frac{1}{\sqrt{\pi N}} \exp(-\Upsilon_N^{2,3/2}) \geq \frac{1}{\sqrt{\pi N}} e^{-\frac{5}{6N}}. \tag{7.16}$$

We now work on the upper bound of $R_N^e(\rho) + R_N^o(\rho)$ defined in (7.7). Using the properties in Lemma 7.3 and (7.3)–(7.4), we obtain that for some $1 < K < N$,

$$\begin{aligned} R_N^e(\rho) &= \sum_{k=1}^K \frac{q_k g_k}{\rho^{2k}} + \sum_{k=K+1}^{N+1} \frac{q_k g_k}{\rho^{2k}} + \sum_{k=N+2}^{\infty} \frac{g_k}{\rho^{2k}} \\ &\leq q_K \sum_{k=1}^K \frac{g_k}{\rho^{2k}} + \sum_{k=K+1}^{N+1} \frac{1}{\rho^{2k}} + \sum_{k=N+2}^{\infty} \frac{1}{\rho^{2k}} \\ &\leq q_K \left((1 - \rho^{-2})^{-1/2} - 1 \right) + \frac{1}{\rho^{2K}(\rho^2 - 1)}, \end{aligned}$$

and similarly,

$$\begin{aligned} R_N^o(\rho) &= \sum_{k=0}^{K-1} \frac{q_{k+1} g_k}{\rho^{2k+1}} + \sum_{k=K}^N \frac{q_{k+1} g_k}{\rho^{2k+1}} + \sum_{k=N+1}^{\infty} \frac{g_k}{\rho^{2k+1}} \\ &\leq q_K \sum_{k=0}^{K-1} \frac{g_k}{\rho^{2k+1}} + \sum_{k=K}^N \frac{1}{\rho^{2k+1}} + \sum_{k=N+1}^{\infty} \frac{1}{\rho^{2k+1}} \\ &\leq q_K \rho^{-1} (1 - \rho^{-2})^{-1/2} + \frac{1}{\rho^{2K-1}(\rho^2 - 1)}. \end{aligned}$$

Collecting the terms leads to the upper bound

$$R_N^e(\rho) + R_N^o(\rho) \leq \left(\sqrt{\frac{\rho + 1}{\rho - 1}} - 1 \right) q_K + \frac{1}{\rho^{2K}(\rho - 1)}. \tag{7.17}$$

Thus, a combination of (7.14)–(7.17) yields (4.15).

It remains to show the asymptotic estimate (4.18). Observe that for $N \gg 1$, if we choose $K = \lceil N^\varepsilon \rceil$ with $0 < \varepsilon < 1$, then

$$D_{K,N} = \frac{N^{\varepsilon-1}}{2} + O(N^{-1}). \tag{7.18}$$

Therefore,

$$A(K, N; \rho) = \frac{\rho - 1}{\sqrt{1 + \rho^2}} - \left(\sqrt{\frac{\rho + 1}{\rho - 1}} - 1 \right) \left(\frac{N^{\varepsilon-1}}{2} + O(N^{-1}) \right) - O(\rho^{-(2N^\varepsilon+1)}). \quad (7.19)$$

Thus, for $N \gg 1$,

$$A(K, N; \rho) \approx \frac{\rho - 1}{\sqrt{1 + \rho^2}}.$$

Hence, the conclusion follows from (4.15). \square

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