Superconvergence of Conforming Finite Element for Fourth-Order Singularly Perturbed Problems of Reaction Diffusion Type in 1D

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We consider conforming finite element approximation of fourth-order singularly perturbed problems of reaction diffusion type. We prove superconvergence of standard C^1 finite element method of degree p on a modified Shishkin mesh. In particular, a superconvergence error bound of $\left(N^{-1}\ln(N+1)\right)^p$ in a discrete energy norm is established. The error bound is uniformly valid with respect to the singular perturbation parameter ϵ. Numerical tests indicate that the error estimate is sharp. © 2013 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 30: 550–566, 2014

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I. INTRODUCTION

Singularly perturbed problems are important in both theoretical analysis and practice. Classical second-order singularly perturbed problems have been extensively studied, and many efficient numerical methods and techniques have been developed. For the literature, see monograph [1, 2] and references therein.

In this article, we consider numerical approximation of a class of fourth-order singularly perturbed problems of reaction diffusion type. In particular, we consider conforming finite element approximation of the following fourth order elliptic problem:

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$$\begin{cases} \epsilon^2 u^{(4)}(x) - (a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), \quad x \in \Omega = (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{1}$$

The functions $a(x), b(x), c(x)$, and $f(x)$ are assumed to be sufficiently smooth on [0, 1], with $a(x) \geq \alpha^2 > 0$ and $c(x) - \frac{1}{2}b'(x) \geq \beta^2 \geq 0$ for all $x \in [0,1]$. We are particularly interested in the case when the parameter ϵ is small, which is $0 < \epsilon \ll 1$.

Problem (1) is a variant of the Orr-Sommerfeld equation. This differential equation also governs the deflection of an elastic beam with small flexural rigidity under tension subject to a specific load f, according to the linearized Euler-Bernoulli beam theory, see [3] for details. A similar differential equation is obtained from the Green's function of the Navier Stokes equation. Many researchers have been attracted into this field. Here, we only review a few related to our work. Sempter [4] studied conforming finite element approximation of problem (1) on a quasi-equidistant mesh. Sun and Stynes [3, 5] considered general higher order singularly perturbed problems on the Shishkin mesh and gave uniform convergent results for both reaction diffusion and convection diffusion type. Shanthi and Ramanujam [6–8] transformed the fourth-order problem to a system of two ODEs subject to suitable boundary conditions and then used asymptote numerical methods to solve it. Recently, Chen and Huang used spline to solve higher order reaction-diffusion problems and employed an interpolation postprocessing technique to get uniform superconvergence [9]. However, their result is limited to cubic splines.

In this article, we use conforming finite element of degree p on a modified Shishkin mesh to solve (1). This modified Shishkin mesh was used in [10] for the second-order problem. Different from the results listed above, we obtain a uniform superconvergence rate $\left(N^{-1}\ln(N)\right)^p$ under a discrete energy norm for any C^1 polynomial finite element space of degree p $(p \geq 3)$ by extending the superconvergence argument in [10] for second-order singularly perturbed problems to four-order problems and using a generalized Gauss-Lobatto interpolation. The superconvergence result is uniformly valid with respect to the singular perturbation parameter ϵ. Note that for a fourth-order problem such as (1), the optimal rate of convergence in the energy norm is $\left(N^{-1}\ln(N)\right)^{p-1}$. Numerical tests illustrate that our theoretical analysis is sharp. In particular, our methods can be generalized to general higher order singular perturbed problems of reaction diffusion type as considered in [3] by using corresponding Generalized Gauss-Lobatto interpolation. The superconvergence of fourth-order singularly perturbed problems of convection diffusion type is under investigation.

The rest of this article is organized as follows. In Section 2, we introduce some preliminary knowledge on conforming finite element approximation for the model problem on a modified Shishkin mesh and its associated generalized Gauss-Lobatto interpolation. In Section 3, we prove our superconvergence results by choosing a special interpolation and superconvergence argument in finite element analysis. In Section 4, we present two numerical examples to demonstrate the sharpness of our theoretical results.

II. PRELIMINARIES

Throughout this article, we use standard notation for Sobolev spaces, and we use C as a generic constant independent of ϵ and N (or h). Also, we use subindex to indicate a particular norm or inner product that is only used for a subdomain. For example, $||v||_{\Omega_i}$, $(v, w)_{(0,\tau)}$ and so forth.

A. Conforming Finite Element Approximation

From the regularity result in [11] and [3], we can write the solution u of (1) in the form as

$$u = \bar{u} + u_{\epsilon,0} + u_{\epsilon,1}, \tag{2}$$

with $x \in [0,1]$ and

$$||\bar{u}^{(k)}||_{L^\infty(\Omega)} \le C, \quad |u_{\epsilon,0}^{(k)}(x)| \le C\epsilon^{1-k}e^{-\alpha x/\epsilon}, \quad |u_{\epsilon,1}^{(k)}(x)| \le C\epsilon^{1-k}e^{-\alpha(1-x)/\epsilon}. \tag{3}$$

Here k is a fixed nonnegative integer, and C is a constant independent of ϵ. Thus, for $x \in [0,1]$,

$$|u^{(k)}(x)| \le C, \quad k = 0, 1. \tag{4}$$

Furthermore, we may assume that

$$u_{\epsilon,0}(1) = u'_{\epsilon,0}(1) = 0, \quad u_{\epsilon,1}(0) = u'_{\epsilon,1}(0) = 0, \tag{5}$$

without destroying the regularity of the above decomposition. In fact, we may use

$$\bar{u}_{\epsilon,0}(x) = u_{\epsilon,0}(x) - (3-2x)\,x^2 u_{\epsilon,0}(1) - (x-1)\,x^2 u'_{\epsilon,0}(1),$$
$$\bar{u}_{\epsilon,1}(x) = u_{\epsilon,1}(x) - (2x+1)\,(x-1)^2 u_{\epsilon,1}(0) - x(x-1)^2 u'_{\epsilon,1}(0).$$

to replace $u_{\epsilon,0}$ and $u_{\epsilon,1}$, respectively, and absorb other terms in the above expression into $\bar{u}$. Boundedness condition (4) guarantees the same regularity result.

Because there are boundary layers at $x = 0$ and $x = 1$, the Shishkin mesh [12] is widely used. Here we use the same Shishkin mesh as in [10]. We choose $\tau = \min\left(\frac{1}{4}, \frac{\epsilon}{\alpha}(p+1.5)\ln(N+1)\right)$ to locate two transition points at τ and $1-\tau$. Then, the intervals $(0,\tau)$ and $(1-\tau,1)$ are each divided into N equal subintervals, whereas the interval $(\tau, 1-\tau)$ is divided into $2N$ equal subintervals. Hence the interval length in $(0,\tau)$ and $(1-\tau,1)$ is $h_i = \underline{h} = \frac{\tau}{N}$ whereas in $(\tau,1-\tau)$ is $h_i = \bar{h} = \frac{(1-2\tau)}{2N}$. Therefore,

$$\frac{1}{4N} \le h = \max_i h_i = \frac{1-2\tau}{2N} \le \frac{1}{2N}. \tag{6}$$

In this article, we shall only consider the case when

$$\frac{\epsilon}{\alpha}(p+1.5)\ln(N+1) \le \frac{1}{4}, \tag{7}$$

since otherwise, the problem is either regular (when ϵ is not small) or p and N are large enough to catch the boundary layer [10] and hence traditional analysis can be employed.

For our model problem (1), its associated abstract variational problem reads as finding $u \in H_0^2(\Omega)$ such that

$$B_\epsilon(u,v) = (f,v) \quad \forall v \in H_0^2(\Omega), \tag{8}$$

where

$$B_\epsilon(u,v) = \epsilon^2\big(u'',v''\big) + \big(au',v'\big) + \big(bu',v\big) + (cu,v), \tag{9}$$

and $(\cdot,\cdot)$ is the standard L_2 inner product on Ω. Conforming finite element approximation of (1) is to find $u_N \in V_N^{\epsilon,p} \subset H_0^2(\Omega)$ such that

$$B_\epsilon(u_N,v) = (f,v) \quad v \in V_N^{\epsilon,p}, \tag{10}$$

where $V_N^{\epsilon,p}$ is the space of standard C^1 piecewise polynomials of degree p on the modified Shishkin mesh defined above.

Define the energy norm $||\cdot||_\epsilon$ by

$$||v||_\epsilon^2 = |v|_\epsilon^2 + \alpha^2|v|_1^2 + \beta^2||v||^2 \tag{11}$$

where

$$|v|_\epsilon^2 = \epsilon^2\left(v'', v''\right), \quad |v|_1^2 = \left(v', v'\right), \quad ||v||^2 = (v, v). \tag{12}$$

In light of homogeneous Dirichlet boundary condition, we can show that

$$\left(bv', v\right) = -\frac{1}{2}\left(b'v, v\right), \quad \forall v \in H_0^2(\Omega),$$

which, together with the assumption $a(x) \geq \alpha^2$ and $c(x) - \frac{1}{2}b'(x) \geq \beta^2$, etc follows that

$$B_\epsilon(v, v) \geq ||v||_\epsilon^2 \quad \forall v \in H_0^2(\Omega). \tag{13}$$

To proceed, we define a discrete energy norm $||\cdot||_{\epsilon,N}$ by

$$||v||_{\epsilon,N}^2 = |v|_{\epsilon,N}^2 + \alpha^2|v|_1^2 + \beta^2||v||^2, \tag{14}$$

with

$$|v|_{\epsilon,N}^2 = \epsilon^2\sum_{i=1}^{4N} h_i \sum_{j=1}^{p-1}\omega_j v''\left(x_{ij}\right)^2, \tag{15}$$

where x_{ij} are Gaussian points in element $\Omega_i = (x_{i-1}, x_i)$ and $\omega_j > 0$ are weights for the $(p-1)$-point Gaussian quadrature rule. Because the $(p-1)$-point Gaussian quadrature rule is exact for polynomials of degree less than or equal to $2p-3$, therefore $|v|_{\epsilon,N}^2 = |v|_\epsilon^2$ for $v \in V_N^{\epsilon,p}$ and hence $||v||_{\epsilon,N} = ||v||_\epsilon$ for all $v \in V_N^{\epsilon,p}$.

B. Generalized Gauss Lobatto Interpolant

In the following, we shall estimate the errors involving a generalized Gauss-Lobatto interpolation. Let $x_{i-1} = t_1^{(i)} = t_2^{(i)} < \cdots < t_p^{(i)} = t_{p+1}^{(i)} = x_i$ are zeros of generalized Jacobi polynomial $J_{p+1}^{-2,-2}(x)$ [13, 14] on $\Omega_i = (x_{i-1}, x_i)$. Let w_{I} be a polynomial of degree p such that

$$\begin{cases} w_{\mathrm{I}}\left(t_1^{(i)}\right) = w\left(t_1^{(i)}\right), \\ w_{\mathrm{I}}'\left(t_2^{(i)}\right) = w'\left(t_2^{(i)}\right), \\ w_{\mathrm{I}}\left(t_k^{(i)}\right) = w\left(t_k^{(i)}\right), \quad k = 3, \ldots, p-1, \\ w_{\mathrm{I}}\left(t_p^{(i)}\right) = w\left(t_p^{(i)}\right), \\ w_{\mathrm{I}}'\left(t_{p+1}^{(i)}\right) = w'\left(t_{p+1}^{(i)}\right). \end{cases} \tag{16}$$

Then the remainder of the above special interpolation can be expressed as

$$(w - w_{\rm I})(x) = \phi_{p+1}(x) w\left[t_1^{(i)}, t_2^{(2)}, \ldots, t_{p+1}^{(i)}, x\right], \tag{17}$$

where $w\left[t_1^{(i)}, t_2^{(i)}, \ldots, t_{p+1}^{(i)}, x\right]$ is $p+1$th-order Newton divided difference [15] and

$$\phi_{p+1}(x) = \left(x - t_1^{(i)}\right)\left(x - t_2^{(i)}\right)\cdots\left(x - t_p^{(i)}\right)\left(x - t_{p+1}^{(i)}\right). \tag{18}$$

The remainder (17) can be proved by using the same method used in [15]. Note that $\phi''_{p+1}(x)$ is a multiple of the Legendre polynomial of degree $p - 1$ on Ω_i [14]; therefore

$$\left(\phi''_{p+1}, v''\right)_{\Omega_i} = 0, \quad \forall v \in \mathrm{V}_N^{\epsilon,p} \subset H_0^2(\Omega). \tag{19}$$

When w is a polynomial of degree $p+1$ on Ω_i, the remainder $w - w_{\rm I} = c\phi_{p+1}$ with constant $c = w^{(p+1)}/(p+1)!$. Thus, $\left(w'' - w''_{\rm I}, v''\right)_{\Omega_i} = 0$ for any $v \in V_N^{\epsilon,p}$. Using the Bramble-Hilbert Lemma [16] and a scaling argument, we can derive

$$|\left(w'' - w''_{\rm I}, v''\right)|_{\Omega_i} \leq C h_i^p |w|_{p+2,\Omega_i} |v|_{2,\Omega_i}. \tag{20}$$

To end this section, we list some basic facts, which will be frequently used in the sequel:

$$||e^{-\alpha x/\epsilon}||^2_{(0,\tau)} = \int_0^\tau e^{-2\alpha x/\epsilon} dx = \frac{\epsilon}{2\alpha}\left(1 - N^{-2p-3}\right) < \frac{\epsilon}{2\alpha}, \tag{21a}$$

$$||e^{-\alpha x/\epsilon}||^2_{(\tau,1)} = \int_\tau^1 e^{-2\alpha x/\epsilon} dx = \frac{\epsilon}{2\alpha}\left(N^{-2p-3} - e^{-2\alpha/\epsilon}\right) < \frac{\epsilon}{2\alpha N^{2p+3}}, \tag{21b}$$

$$||e^{-\alpha(1-x)/\epsilon}||^2_{(0,1-\tau)} = \int_0^{1-\tau} e^{-\alpha(1-x)/\epsilon} dx = \frac{\epsilon}{2\alpha}\left(\frac{1}{N^{2p+3}} - e^{-2\alpha/\epsilon}\right) < \frac{\epsilon}{2\alpha N^{2p+3}}, \tag{21c}$$

$$||e^{-\alpha(1-x)/\epsilon}||^2_{(1-\tau,1)} = \int_{1-\tau}^1 e^{-\alpha(1-x)/\epsilon} dx = \frac{\epsilon}{2\alpha}\left(1 - \frac{1}{N^{2p+3}}\right) < \frac{\epsilon}{2\alpha}. \tag{21d}$$

3. SUPERCONVERGENCE ANALYSIS

Before giving our main superconvergence result, we introduce some lemmas.

Lemma 3.1. *Let u be the solution of model problem (1) with $u = \bar{u} + u_{\epsilon,0} + u_{\epsilon,1}$ satisfying regularity (3). Then for any $v \in H_0^2(\Omega)$ we have*

$$|\left(u''_{\epsilon,0} - u''_{\epsilon,0,\rm I}, v''\right)_{(0,\tau)}| \leq \frac{C}{\sqrt{\epsilon}}\left(\frac{\ln(N+1)}{N}\right)^p |v|_2,$$

$$||u''_{\epsilon,0}||_{(\tau,1)} \leq \frac{C}{\sqrt{\epsilon} N^{p+1.5}};$$

$$|\left(u''_{\epsilon,1} - u''_{\epsilon,1,\rm I}, v''\right)_{(1-\tau,1)}| \leq \frac{C}{\sqrt{\epsilon}}\left(\frac{\ln(N+1)}{N}\right)^p |v|_2,$$

$$||u''_{\epsilon,1}||_{(0,1-\tau)} \leq \frac{C}{\sqrt{\epsilon} N^{p+1.5}}.$$

Proof. Using inequality (20), regularity (3), and (21a), we derive

$$\begin{aligned}|(u''_{\epsilon,0}-u''_{\epsilon,0,\mathrm{I}},v'')_{(0,\tau)}| &\le C\underline{h}^p|u_{\epsilon,0}|_{p+2,(0,\tau)}|v|_{2,(0,\tau)}\\ &\le C\underline{h}^p\epsilon^{-p-1}\left(\int_0^\tau e^{-2\alpha x/\epsilon}\right)^{1/2}|v|_{2,(0,\tau)}\\ &\le \frac{C}{\sqrt{\epsilon}}(\underline{h}/\epsilon)^p\left(\epsilon^{-1}\int_0^\tau e^{-2\alpha x/\epsilon}dx\right)^{1/2}|v|_{2,(0,\tau)}\\ &\le \frac{C}{\sqrt{\epsilon}}\left(\frac{\ln(N+1)}{N}\right)^p|v|_{2,(0,\tau)}.\end{aligned}$$

Using regularity (3) and (21b), it follows

$$||u''_{\epsilon,0}||_{(\tau,1)} \le C\epsilon^{-1}||e^{-\alpha x/\epsilon}||_{(\tau,1)} \le \frac{C}{\sqrt{\epsilon}N^{p+1.5}}.$$

Similarly, we can prove the other two inequalities for $u_{\epsilon,1}$. ∎

Lemma 3.2. *Let u be the solution of model problem (1) with $u=\bar{u}+u_{\epsilon,0}+u_{\epsilon,1}$ satisfying regularity (3). Then*

$$||u_{\epsilon,0}-u_{\epsilon,0,\mathrm{I}}||^2_{(0,\tau)} \le C\epsilon^3\left(\frac{\ln(N+1)}{N}\right)^{2(p+1)}, \quad ||u_{\epsilon,0}||^2_{(\tau,1)} \le \frac{C\epsilon^3}{N^{2p+3}};$$

$$||u'_{\epsilon,0}-u'_{\epsilon,0,\mathrm{I}}||^2_{(0,\tau)} \le C\epsilon\left(\frac{\ln(N+1)}{N}\right)^{2p}, \quad ||u'_{\epsilon,0}||^2_{(\tau,1)} \le \frac{C\epsilon}{N^{2p+3}};$$

$$||u_{\epsilon,1}-u_{\epsilon,1,\mathrm{I}}||^2_{(1-\tau,1)} \le C\epsilon^3\left(\frac{\ln(N+1)}{N}\right)^{2(p+1)}, \quad ||u_{\epsilon,1}||^2_{(0,1-\tau)} \le \frac{C\epsilon^3}{N^{2p+3}};$$

$$||u'_{\epsilon,1}-u'_{\epsilon,1,\mathrm{I}}||^2_{(1-\tau,1)} \le C\epsilon\left(\frac{\ln(N+1)}{N}\right)^{2p}, \quad ||u'_{\epsilon,1}||^2_{(0,1-\tau)} \le \frac{C\epsilon}{N^{2p+3}}.$$

Proof. Consider $\Omega_i=(x_{i-1},x_i)\in(0,\tau)$, we have, from the standard approximation theory, regularity (3), and inequality (21a)

$$\begin{aligned}||u_{\epsilon,0}-u_{\epsilon,0,\mathrm{I}}||^2_{\Omega_i} &\le Ch_i^{2(p+1)}||u_{\epsilon,0}||^2_{p+1,\Omega_i} \le C\epsilon^2\left(\frac{h_i}{\epsilon}\right)^{2(p+1)}h_ie^{-2\alpha x_{i-1}/\epsilon}\\ &= C\epsilon^2(N+1)^{(2p+3)/N}\left(\frac{h_i}{\epsilon}\right)^{2(p+1)}h_ie^{-2\alpha x_i/\epsilon}.\end{aligned}$$

Summing up, we derive

$$||u_{\epsilon,0}-u_{\epsilon,0,\mathrm{I}}||^2_{(0,\tau)} \le C\epsilon^2\left(\frac{\ln(N+1)}{N}\right)^{2(p+1)}\int_0^\tau e^{-2\alpha x/\epsilon}dx \le C\epsilon^3\left(\frac{\ln(N+1)}{N}\right)^{2(p+1)}.$$

Similarly, we have

$$||u'_{\epsilon,0} - u'_{\epsilon,0,\mathrm{I}}||^2_{\Omega_i} \le Ch_i^{2p}||u_{\epsilon,0}||^2_{p+1,\Omega_i} \le C\left(\frac{h_i}{\epsilon}\right)^{2p} h_i e^{-2\alpha x_{i-1}/\epsilon}$$

$$= C(N+1)^{(2p+3)/N}\left(\frac{h_i}{\epsilon}\right)^{2p} h_i e^{-2\alpha x_i/\epsilon}.$$

Summing up yields

$$||u'_{\epsilon,0} - u'_{\epsilon,0,\mathrm{I}}||^2_{(0,\tau)} \le C\left(\frac{\ln(N+1)}{N}\right)^{2p}\int_0^\tau e^{-2\alpha x/\epsilon} \le C\epsilon\left(\frac{\ln(N+1)}{N}\right)^{2p}.$$

From the regularity (3) and inequality (21b), we have

$$||u_{\epsilon,0}||^2_{(\tau,1)} \le C\epsilon^2\int_\tau^1 e^{-2\alpha x/\epsilon}dx \le \frac{C\epsilon^3}{N^{2p+3}},$$

$$||u'_{\epsilon,0}||^2_{(\tau,1)} \le C\int_\tau^1 e^{-2\alpha x/\epsilon}dx \le \frac{C\epsilon}{N^{2p+3}}.$$

The result follows by taking the square root.

We can prove the other four inequalities virtually by the same argument. ∎

Lemma 3.3. *Let u be the solution of model problem (1) with $u = \bar{u} + u_{\epsilon,0} + u_{\epsilon,1}$ satisfying regularity (3). Then*

$$|u_{\epsilon,0} - u_{\epsilon,0,\mathrm{I}}|_{\epsilon,N,(0,\tau)} \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p, \quad |u_{\epsilon,0}|_{\epsilon,N,(\tau,1)} \le \frac{C\sqrt{\epsilon}}{N^{p+1.5}};$$

$$|u_{\epsilon,1} - u_{\epsilon,1,\mathrm{I}}|_{\epsilon,N,(1-\tau,1)} \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p, \quad |u_{\epsilon,1}|_{\epsilon,N,(0,1-\tau)} \le \frac{C\sqrt{\epsilon}}{N^{p+1.5}}.$$

Proof. Since ϕ''_{p+1} is a multiple of the Legendre polynomial of degree $p-1$ on $\Omega_i = (x_{i-1}, x_i)$, it vanishes at Gaussian points x_{ij}, $j = 1, \dots, p-1$ on Ω_i. Therefore,

$$|u_{\epsilon,0} - u_{\epsilon,0,\mathrm{I}}|^2_{\epsilon,N,\Omega_i} = |u_{\epsilon,0} - u_{\epsilon,0,\mathrm{I}} - c\phi_{p+1}|^2_{\epsilon,N,\Omega_i}$$

for any $c \in \mathbb{R}$. Note that $\omega_j > 0$ and $\sum_j \omega_j = 1$, we have

$$|u_\epsilon - u_{\epsilon,0,\mathrm{I}}|^2_{\epsilon,N,\Omega_i} \le \epsilon^2 h_i \inf_{c\in\mathbb{R}}||\left(u_{\epsilon,0} - u_{\epsilon,0,\mathrm{I}} - c\phi_{p+1}\right)''||^2_{L^\infty(\Omega_i)} \le C\epsilon^2 h_i h_i^{2p}||u^{(p+2)}_{\epsilon,0}||^2_{L^\infty(\Omega_i)},$$

since by choosing

$$c = \frac{\left(u''_{\epsilon,0}, \phi''_{p+1}\right)}{||\phi''_{p+1}||^2_{L^2(\Omega_i)}}$$

the operator $w'' \to w_{\mathrm{I}}'' + c\phi_{p+1}''$ reproduces w'' if w is a polynomial of degree $p+1$. Using the regularity (3), it follows that

$$|u_{\epsilon,0} - u_{\epsilon,0,\mathrm{I}}|^2_{\epsilon,N,(0,\tau)} \le C \sum_{i=1}^{N} \left(\frac{h_i}{\epsilon}\right)^{2p} h_i e^{-2\alpha x_{i-1}/\epsilon}$$
$$\le C\left(\frac{\ln(N+1)}{N}\right)^{2p} \int_0^\tau e^{-2\alpha x/\epsilon} dx$$
$$\le C\epsilon\left(\frac{\ln(N+1)}{N}\right)^{2p}.$$

Here we have used the estimate

$$h_i e^{-2\alpha x_{i-1}/\epsilon} < (N+1)^{(2p+3)/N} \int_{x_{i-1}}^{x_i} e^{-2\alpha x/\epsilon} dx.$$

Also using regularity (3), it follows that

$$|u_{\epsilon,0}|^2_{\epsilon,N,(\tau,1)} = \epsilon^2 \sum_{i=N+1}^{4N} h_i \sum_{j=1}^{p-1} \omega_j u''_{\epsilon,0}(x_{ij})^2$$
$$\le \sum_{i=N+1}^{4N} h_i \sum_{j=1}^{p-1} e^{-2\alpha x_{ij}/\epsilon}$$
$$\le C \int_\tau^1 e^{-2\alpha x/\epsilon} dx$$
$$\le \frac{C\epsilon}{N^{2p+3}}.$$

Here we have used the remainder for the Gauss-Legendre quadrature [17]

$$\int_{x_{i-1}}^{x_i} e^{-2\alpha x/\epsilon} dx - h_i \sum_{j=1}^{p-1} \omega_j e^{-2\alpha x_{ij}/\epsilon} = \frac{h_i^{2p-1}[(p-1)!]^4}{(2p+1)\,[(2\,(p-1))!]^3} \left(\frac{2\alpha}{\epsilon}\right)^{2p-2} e^{-2\alpha\eta_i/\epsilon} > 0$$

with $\eta_i \in (x_{i-1}, x_i)$ for $i = N+1, \ldots, 4N$. Similar argument shows results for $u_{\epsilon,1}$. Then, our conclusions follow by taking the square roots. ■

Now we choose a special interpolation of the solution u as

$$\mathrm{I}_N^\epsilon u = \bar{u}_{\mathrm{I}} + u_{\epsilon,0,\mathrm{I}_\epsilon} + u_{\epsilon,1,\mathrm{I}_\epsilon}, \tag{22}$$

where w_{I} is the generalized Gauss-Lobatto interpolation of w which is defined on (16). Here $u_{\epsilon,0,\mathrm{I}_\epsilon}$ is defined as following

$$u_{\epsilon,0,\mathrm{I}_\epsilon} = \begin{cases} u_{\epsilon,0,\mathrm{I}} - \ell_{0,\tau} & 0 \le x \le \tau, \\ 0 & \tau \le x \le 1; \end{cases} \tag{23}$$

with

$$\ell_{0,\tau} = \begin{cases} 0 & 0 \le x \le \tau - \underline{h}, \\ u_{\epsilon,0}(\tau) \frac{\left(\underline{h}-2(x-\tau)\right)\left(x-\tau+\underline{h}\right)^2}{\underline{h}^3} + u'_{\epsilon,0}(\tau) \frac{(x-\tau)\left(x-\tau+\underline{h}\right)^2}{\underline{h}^2} & \tau - \underline{h} \le x \le \tau; \end{cases}$$

and $u_{\epsilon,1,\mathrm{I}_\epsilon}$ is defined as

$$u_{\epsilon,1,\mathrm{I}_\epsilon} = \begin{cases} 0 & 0 \le x \le 1-\tau, \\ u_{\epsilon,1,\mathrm{I}} - \ell_{1,\tau} & 1-\tau \le x \le 1; \end{cases} \tag{24}$$

with

$$\ell_{1,\tau} = \begin{cases} u_{\epsilon,1}(1-\tau) \frac{\left(2(x-1+\tau)+\underline{h}\right)\left(1-\tau-x+\underline{h}\right)^2}{\underline{h}^3} & \\ \quad +u'_{\epsilon,1}(1-\tau) \frac{(x-1+\tau)\left(1-\tau-x+\underline{h}\right)^2}{\underline{h}^2} & 1-\tau \le x \le 1-\tau+\underline{h}, \\ 0 & 1-\tau+\underline{h} \le x \le 1. \end{cases}$$

We can see that $\ell_{0,\tau}$ and $\ell_{1,\tau}$ are both only defined in the boundary layer region. Notice that

$$\ell_{0,\tau}\left(\tau-\underline{h}\right) = 0, \quad \ell'_{0,\tau}\left(\tau-\underline{h}\right) = 0, \quad \ell_{0,\tau}(\tau) = u_{0,\epsilon}(\tau), \quad \ell'_{0,\tau}(\tau) = u'_{0,\epsilon}(\tau);$$

and

$$\ell_{1,\tau}(1-\tau) = u_{1,\epsilon}(1-\tau), \quad \ell'_{1,\tau}(1-\tau) = u'_{1,\epsilon}(1-\tau), \quad \ell_{1,\tau}\left(1-\tau+\underline{h}\right) = 0,$$
$$\ell'_{1,\tau}\left(1-\tau+\underline{h}\right) = 0.$$

The definition (23) and (24) guarantee that the interpolation $u_{\epsilon,0,\mathrm{I}_\epsilon}$ and $u_{\epsilon,1,\mathrm{I}_\epsilon}$ are C^1 continuous at transient points τ and $1-\tau$, respectively. Then (5) implies that both $u_{\epsilon,0,\mathrm{I}_\epsilon}$ and $u_{\epsilon,1,\mathrm{I}_\epsilon}$ belong to $V_N^{\epsilon,p}$ and hence belong to $H_0^2(\Omega)$.

From a direct calculation, we have bounds for $i = 0, 1$,

$$||\ell_{i,\tau}||^2 \le \frac{C\epsilon^3 \ln(N+1)}{N^{2(p+2)}}, \quad ||\ell'_{i,\tau}||^2 \le \frac{C\epsilon}{N^{2(p+1)}\ln(N+1)}, \quad ||\ell''_{i,\tau}||^2 \le \frac{C}{\epsilon N^{2p}\ln^3(N+1)};$$

and

$$|\ell_{i,\tau}|^2_{\epsilon,N} = \epsilon^2 h_i \sum_{j=1}^{p-1} \omega_j \left(\ell''_{i,\tau}\right)^2 \le \frac{C\epsilon}{N^{2p}\ln^3(N+1)}.$$

Now, we are ready to prove our main theorems.

Theorem 3.1. *Let u be the solution of solve problem (1) and satisfy the regularity (3) with the boundary condition (5). Let $V_N^{\epsilon,p}$ be the C^1 finite element space with piecewise polynomials of degree p on the modified Shishkin mesh satisfying (7). Then for any $v \in H_0^2(\Omega)$ the special interpolation I_N^ϵ defined in (22) satisfies*

$$\left|B_\epsilon\left(u - \mathrm{I}_N^\epsilon u, v\right)\right| \le C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right)||v||_\epsilon,$$

$$||u - \mathrm{I}_N^\epsilon u||_{\epsilon,N} \le C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right);$$

if $\bar{u} \in V_N^{\epsilon,p}$ then

$$\left|B_\epsilon\left(u - \mathrm{I}_N^\epsilon u, v\right)\right| \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p ||v||_\epsilon,$$

$$||u - \mathrm{I}_N^\epsilon u||_{\epsilon,N} \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p.$$

Proof. By the Cauchy-Schwarz inequality and the estimates for $\ell_{i,\tau}$, we have

$$\begin{aligned}\left|B_\epsilon\left(\ell_{i,\tau}, v\right)\right| &\le \epsilon^2||\ell''_{i,\tau}||\,||v''|| + C_1||\ell'_{i,\tau}||\,||v'|| + C_2||\ell_{i,\tau}||\,||v|| \\ &\le \frac{C\epsilon\sqrt{\epsilon}}{N^p \ln^{\frac{3}{3}}(N+1)}||v''|| + \frac{C_1\sqrt{\epsilon}}{N^{(p+1)}\sqrt{\ln(N+1)}}||v'|| + \frac{C_2\epsilon\sqrt{\epsilon}\sqrt{\ln(N+1)}}{N^{(p+2)}}||v|| \\ &\le \frac{C\sqrt{\epsilon}}{N^p}||v||_\epsilon.\end{aligned}$$

for $i = 0, 1$.

Let $\mathcal{T}_b = (0, \tau)$ and $\mathcal{T}_r = (\tau, 1)$ if $i = 0$; $\mathcal{T}_r = (0, 1-\tau)$ and $\mathcal{T}_b = (1-\tau, 1)$ if $i = 1$. Then by Lemmas 3.1 and 3.2, we have

$$\begin{aligned}&\left|B_\epsilon(u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}}, v)_{\mathcal{T}_b}\right| \le \epsilon^2\left|(u''_{\epsilon,i} - u''_{\epsilon,i,\mathrm{I}}, v'')_{\mathcal{T}_b}\right| + C_1||u'_{\epsilon,i} - u'_{\epsilon,i,\mathrm{I}}||_{\mathcal{T}_b}||v'|| + C_2||u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}}||_{\mathcal{T}_b}||v|| \\ &\le C\epsilon\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p|v|_2 + C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p|v|_1 + C\epsilon\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p||v||_\epsilon \\ &\le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p||v||_\epsilon;\end{aligned}$$

and

$$\begin{aligned}\left|B_\epsilon\left(u_{\epsilon,i}, v\right)_{\mathcal{T}_r}\right| &\le \epsilon^2||u''_{\epsilon,0}||_{\mathcal{T}_r}|v|_2 + C_1||u'_{\epsilon,i}||_{\mathcal{T}_r}|v|_1 + C_2||u_{\epsilon,i}||_{\mathcal{T}_r}||v|| \\ &\le \frac{C\epsilon\sqrt{\epsilon}}{N^{p+1.5}}|v|_2 + \frac{C_1\sqrt{\epsilon}}{N^{p+1.5}}|v|_1 + \frac{C_2\epsilon\sqrt{\epsilon}}{N^{p+1.5}}||v|| \\ &\le \frac{C\sqrt{\epsilon}}{N^{p+1.5}}||v||_\epsilon.\end{aligned}$$

where $i = 0, 1$. Note that

$$\sum_{i=0}^{1} B_\epsilon(u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}_\epsilon}, v) = \sum_{i=0}^{1}[B_\epsilon(u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}}, v)_{\mathcal{T}_b} + B_\epsilon(\ell_{i,\tau}, v)_{\mathcal{T}_b} + B_\epsilon(u_{i,\tau}, v)_{\mathcal{T}_r}].$$

Thus

$$\sum_{i=0}^{1} B_\epsilon\left(u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}_\epsilon}, v\right) \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p||v||_\epsilon.$$

When the regular part $\bar{u}$ is in the finite element space $V_N^{\epsilon,p}$, we have

$$B_\epsilon\left(u - \mathrm{I}_N^\epsilon u, v\right) \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p ||v||_\epsilon.$$

When the regular part $\bar{u}$ of the solution is not in the finite element space $V_N^{\epsilon,p}$, we have from (3), (6), and (20)

$$\epsilon^2\left|\left(\bar{u}'' - \bar{u}_\mathrm{I}'', v''\right)\right| \le \frac{C\epsilon}{N^p}|v|_\epsilon.$$

From standard estimate, we have

$$|\left(\bar{u}' - \bar{u}_\mathrm{I}', v'\right)| \le \frac{C}{N^p}|v|_1,$$

$$|\left(\bar{u} - \bar{u}_\mathrm{I}, v\right)| \le \frac{C}{N^{(p+1)}}||v||.$$

The above three estimates imply that

$$|B_\epsilon\left(\bar{u} - \bar{u}_\mathrm{I}, v\right)| \le \frac{C}{N^p}\left(\epsilon|v|_\epsilon + |v|_1 + ||v||\right) \le \frac{C}{N^p}||v||_\epsilon.$$

Note that

$$B_\epsilon\left(u - \mathrm{I}_N^\epsilon u, v\right) = B_\epsilon\left(u_{\epsilon,0} - u_{\epsilon,0,\mathrm{I}_\epsilon}, v\right) + B_\epsilon\left(u_{\epsilon,1} - u_{\epsilon,1,\mathrm{I}_\epsilon}, v\right) + B_\epsilon\left(\bar{u} - \bar{u}_\mathrm{I}, v\right).$$

Therefore

$$\left|B_\epsilon\left(u - \mathrm{I}_N^\epsilon u, v\right)\right| \le C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right)||v||_\epsilon.$$

Recall Lemmas 3.2 and 3.3 and the estimates for $\ell_{i,\tau}, i = 0, 1$, and use the triangle inequality, we have

$$\begin{aligned}
||u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}}||_{\epsilon,N}^2 &= ||u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}} + \ell_{i,\tau}||_{\epsilon,N,\mathcal{T}_b}^2 + ||u_{\epsilon,i}||_{\epsilon,N,\mathcal{T}_r}^2 \\
&\le ||u_{\epsilon,i} - u_{\epsilon,i,\mathrm{I}}||_{\epsilon,N,\mathcal{T}_b}^2 + ||\ell_{i,\tau}||_{\epsilon,N,\mathcal{T}_b}^2 + ||u_{\epsilon,i}||_{\epsilon,N,\mathcal{T}_r}^2 \\
&\le C\epsilon\left(\frac{\ln(N+1)}{N}\right)^{2p},
\end{aligned}$$

which implies

$$||u - \mathrm{I}_N^\epsilon u||_{\epsilon,N} \le C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p,$$

when the regular part $\bar{u}$ belongs to the finite element space $V_N^{\epsilon,p}$; otherwise we can use the similar argument as in the proof of Lemma 3.3 to drive

$$|\bar{u} - \bar{u}_\mathrm{I}|_{\epsilon,N}^2 \le C\epsilon^2\sum_{i=1}^{4N} h_i h_i^{2p}||\bar{u}^{(p+2)}||_{L^\infty(\Omega_i)}^2 \le C\epsilon^2 h^{2p} \le Ch^{2p}.$$

Then (6) implies that

$$|\bar{u} - \bar{u}_{\mathrm{I}}|^2_{\epsilon,N} \leq \frac{C}{N^p}.$$

In addition, we have

$$||\bar{u} - \bar{u}_{\mathrm{I}}|| \leq \frac{C}{N^{p+1}}, \quad ||\bar{u}' - \bar{u}'_{\mathrm{I}}|| \leq \frac{C}{N^p},$$

from the standard interpolation theory. Combining those with the above estimates for the boundary layer term, we derive

$$||u - \mathrm{I}^{\epsilon}_N u||_{\epsilon,N} \leq C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right).$$

This completes our proof. ■

Theorem 3.2. *Assume the same hypothesis as in Theorem 3.1. Let* $u_N \in V_N^{\epsilon,p}$ *be the conforming finite element approximation of u. Then*

$$||u - u_N||_{\epsilon,N} \leq C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right);$$

if $\bar{u} \in V_N^{\epsilon,p}$ *, then*

$$||u - u_N||_{\epsilon,N} \leq C\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p.$$

Proof. We only prove the case when the regular part $\bar{u}$ does not belong to the finite element space $V_N^{\epsilon,p}$, since the other case is the same. By (13) and the fact $V_N^{\epsilon,p} \subset H_0^2(\Omega)$ which is guaranteed by conforming finite element methods, we have

$$\begin{aligned}
||u_N - \mathrm{I}^{\epsilon}_N u||^2_{\epsilon} &\leq B_{\epsilon}\left(u_N - \mathrm{I}^{\epsilon}_N u, u_N - \mathrm{I}^{\epsilon}_N u\right) \\
&= B_{\epsilon}\left(u - \mathrm{I}^{\epsilon}_N u, u_N - \mathrm{I}^{\epsilon}_N u\right) \\
&\leq C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right)||u_N - \mathrm{I}^{\epsilon}_N u||_{\epsilon}.
\end{aligned}$$

Cancelling $||u_N - \mathrm{I}^{\epsilon}_I u||_{\epsilon}$ on both ends, we have

$$||u_N - \mathrm{I}^{\epsilon}_N u||_{\epsilon} \leq C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right),$$

which combined with the triangle inequality and Theorem 3.1, yields

$$\begin{aligned}
||u - u_N||_{\epsilon,N} &\leq ||u - \mathrm{I}^{\epsilon}_N u||_{\epsilon,N} + ||\mathrm{I}^{\epsilon}_N u - u_N||_{\epsilon,N} \\
&= ||u - \mathrm{I}^{\epsilon}_N u||_{\epsilon,N} + +||\mathrm{I}^{\epsilon}_N u - u_N||_{\epsilon} \\
&\leq C\left(\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p + \frac{1}{N^p}\right).
\end{aligned}$$

■

TABLE I. Error in the semienergy norm, **Example 4.1**, $p = 3$.

N	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$
	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$
4	1.199642837361798e-03	1.220228993598402e-04	1.220731614791486e-05
8	4.040247563514831e-04	4.048804067182617e-05	4.049091961157767e-06
16	1.115839816841199e-04	1.116341688574735e-05	1.116359187350829e-06
32	2.656446892867752e-05	2.656822763296837e-06	2.656834664073347e-07
64	5.679895456628676e-06	5.680169455975019e-07	5.680171139978733e-08
128	1.122416167701913e-06	1.122432866360375e-07	1.122417255982434e-08
256	2.090032200109513e-07	2.090226920424709e-08	2.107940865455582e-09

TABLE II. Error in the semienergy norm, **Example 4.1**, $p = 4$.

N	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$
	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$
1	3.976605993562015e-03	6.608984448376749e-04	6.645571174116623e-05
2	5.776168978088796e-04	1.277259891488829e-04	1.291691557334470e-05
4	1.163409610254992e-04	1.928130418450377e-05	1.959805654377102e-06
8	2.680631740784765e-05	3.047524647072133e-06	3.083126974922051e-07
16	4.853697936493047e-06	4.973889479534159e-07	4.999537731228687e-08
32	7.188778312204927e-07	7.222446658639692e-08	7.238362852330126e-09
64	9.206352347946794e-08	9.214558839074637e-09	9.232357945806096e-10
128	1.060333689831790e-08	1.060500613471576e-09	1.334225886595813e-10

Remark 3.1. Comparing with the literature, the superconvergence result in the discrete energy norm is new. As a by-product, the uniform convergence in the H^1-norm also takes the form of the right-hand sides of error bounds in Theorem 3.2, which is better than the error bounds in [3] by a factor $\sqrt{\epsilon}$.

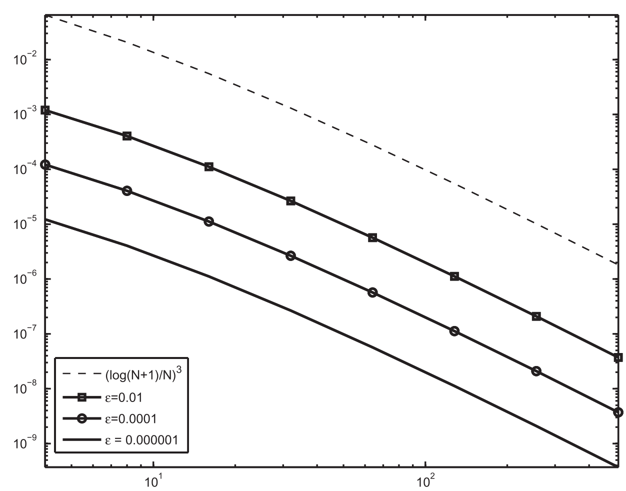

FIG. 1. Error curve, **Example 4.1**, $p = 3$.

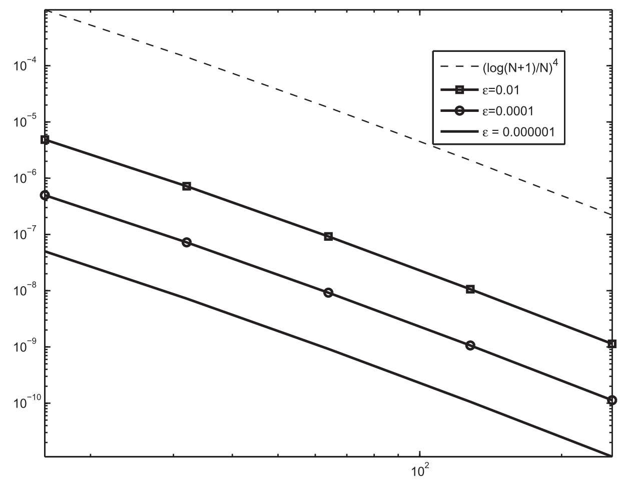

FIG. 2. Error curve, **Example 4.1**, $p = 4$.

TABLE III. Error in the semienergy norm, **Example 4.2**, $p = 3$.

	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$
N	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$
4	1.039986389674419e-03	1.044724341881863e-04	1.044723962085517e-05
8	3.562378909606350e-04	3.563698171877963e-05	3.563695437450857e-06
16	9.955953030594041e-05	9.955962399561371e-06	9.955959417185815e-07
32	2.384741173402196e-05	2.384715855652849e-06	2.384715431984535e-07
64	5.112204440580189e-06	5.112180998150958e-07	5.112177407074458e-08
128	1.011220828952090e-06	1.011218594978394e-07	1.011214328448602e-08
256	1.883615187724756e-07	1.883664646768138e-08	1.888637233868672e-09

4. NUMERICAL EXAMPLES

In this section, we present two numerical examples to support our theoretical results. Our numerical simulation was implemented under the frame of C++ Finite Element Package AFEPack [18]. We use Herimite element of degree $p = 3$ and $p = 4$. Since our model problem (1) has two

TABLE IV. Error in the semienergy norm, **Example 4.2**, $p = 4$.

	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$
N	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$	$\|u - u_N\|_{\epsilon,N}$
1	1.948487195493097e-03	3.303998550373655e-04	3.322780652300214e-05
2	2.885577963113786e-04	6.386058524878483e-05	6.458455472816753e-06
4	5.823551824614348e-05	9.641328735220617e-06	9.799035606310579e-07
8	1.340724166088775e-05	1.523852612676438e-06	1.541564528134647e-07
16	2.427179155328638e-06	2.486984175502978e-07	2.499769531321942e-08
32	3.594582724486571e-07	3.611222178574523e-08	3.619181741939649e-09
64	4.603265872771751e-08	4.607273885824092e-09	4.616179107581453e-10
128	5.301712126903481e-09	5.302495759323248e-10	6.671045669350635e-11

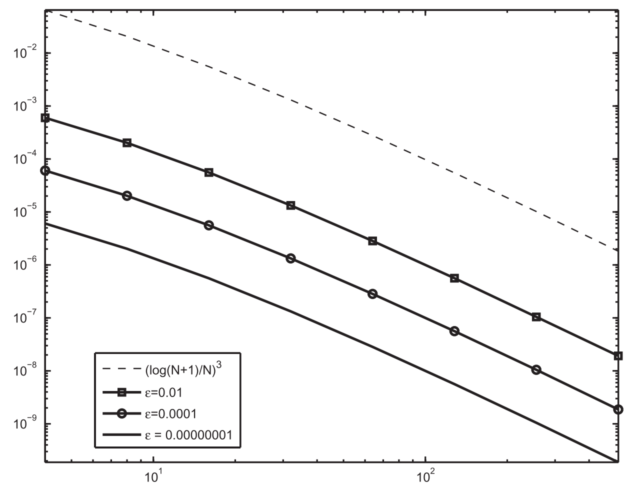

FIG. 3. Error curve, **Example 4.2**, $p = 3$.

boundary layers at $x = 0$ and $x = 1$, we use the modified Shishkin mesh. The transition points are τ and $1 - \tau$ where $\tau = \epsilon\,(p + 1.5)\ln(N + 1)$. Each interval $(0, \tau)$ and $(1 - \tau, 1)$ is equally divided into $N = 2^j$ subintervals and interval $(\tau, 1 - \tau)$ is equally divided into $2\,N$ subintervals, $j = 2, 3, \ldots, 9$ when $p = 3$ and $j = 0, 1, \ldots, 8$ when $p = 4$.

Example 4.1. *Here we use the same example as in [3]. Let*

$$a(x) = 1 + x\,(1 - x)\,, \quad b(x) = c(x) = 0,$$

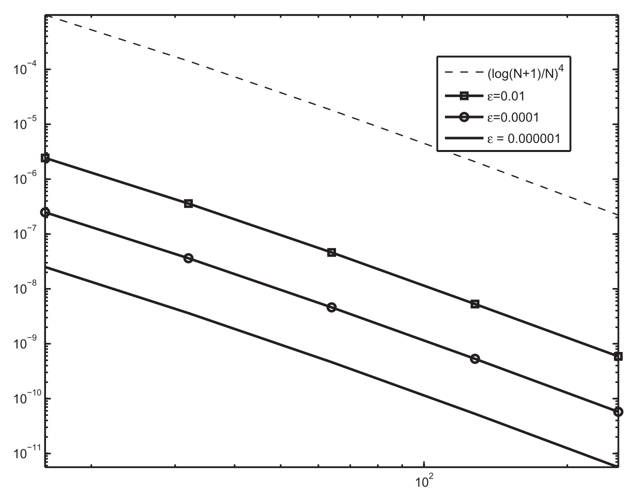

FIG. 4. Error curve, **Example 4.2**, $p = 4$.

in problem (1) and f(x) is chosen such that the solution of (1) is

$$u(x) = \epsilon \left\{ \frac{e^{-\frac{x}{\epsilon}} + e^{-\frac{1-x}{\epsilon}}}{1 + e^{-\frac{1}{\epsilon}}} - 1 \right\} + \frac{1 - e^{-\frac{1}{\epsilon}}}{1 + e^{-\frac{1}{\epsilon}}} x\,(1-x) + x^2(1-x)^2. \tag{25}$$

As in [3], u(x) exhibits typical boundary layer behavior. Note the regular part the solution is a polynomial of degree 4 which is not contained in the finite element space $V_N^{\epsilon,p}$ *when* $p = 3$ *but is contained in the finite element space* $V_N^{\epsilon,p}$ *when* $p = 4$.

Tables I and II list errors in the discrete semienergy norm $|u - u_N|_{\epsilon,N}$ *for three different values of* $\epsilon = 0.01, 0.0001, 0.000001$ *for cases* $p = 3, 4$, *respectively. The related convergent curves are depicted in Figs. 1 and 2, respectively. We see that the convergent curves are parallel to the reference convergent curve. Note that the gap between the curves is* $\sqrt{\epsilon}$ *which clearly indicate a rate of* $\sqrt{\epsilon}\left(\frac{\ln(N+1)}{N}\right)^p$ *as predicted by Theorem 2 and* $\sqrt{\epsilon}$ *cannot be ignored. The convergent curves clearly indicate that our error bound in Theorem 3.2 is sharp.*

Example 4.2. *Let us consider the following problem*

$$\begin{cases} \epsilon^2 u^{(4)}(x) - u''(x) = 1, & x \in (0,1) \\ u(0) = u'(0) = 0 = u(1) = u'(1). \end{cases} \tag{26}$$

The exact solution is

$$u(x) = \epsilon \frac{e^{-\frac{1-x}{\epsilon}} + e^{-\frac{x}{\epsilon}} - 1 - e^{-\frac{1}{\epsilon}}}{2 - 2e^{-\frac{1}{\epsilon}}} + \frac{1}{2} x\,(1-x)\,. \tag{27}$$

Note that the regular part $\bar{u}$ *is a polynomial of degree 2 and hence belongs to the finite element spaces* $V_N^{\epsilon,p}$ *for both* $p = 3$ *and* $p = 4$.

Tables III and IV list errors in the discrete semienergy norm $|u - u_N|_{\epsilon,N}$ *for three different values of* $\epsilon = 0.01, 0.0001, 0.000001$ *and for cases* $p = 3, 4$, *respectively. The related convergent curves are depicted in Figs. 3 and 4, respectively. We observe the similar behavior as in Example 4.1.*

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