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IS 2K-CONJECTURE VALID FOR FINITE VOLUME METHODS?*

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Abstract. This paper is concerned with superconvergence properties of a class of finite volume methods of arbitrary order over rectangular meshes. Our main result is to prove the $2k$-*conjecture*: at each vertex of the underlying rectangular mesh, the bi-k degree finite volume solution approximates the exact solution with an order $O(h^{2k})$, where h is the mesh size. As byproducts, superconvergence properties for finite volume discretization errors at Lobatto and Gauss points are also obtained. All theoretical findings are confirmed by numerical experiments.

Key words. superconvergence, finite volume method, $2k$-conjecture

AMS subject classifications. 65N15, 65N30

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1. Introduction.

As a popular numerical method for partial differential equations (PDEs), the finite volume (FV) method (FVM) has a wide range of applications and attracts intensive theoretical studies; see, e.g., [3, 4, 6, 7, 14, 15, 17, 18, 19, 21, 22, 23, 25, 28, 32] for an incomplete list of publications. However, most theoretical studies in the literature have been focused on linear or quadratic schemes. Recently, arbitrary order FV schemes have been constructed and analyzed for elliptic problems in [8] and [29]. The basic idea to design an FV scheme of any order k in [8, 29] is to choose standard finite element (FE) space as the trial space and construct control volumes with Gauss points in the primal partition. These FV schemes are shown to be convergent with optimal rates under both energy and L^2 norms.

In 1974 Douglas and Dupont proved that the kth order C^0 finite element method (FEM) to the two-point boundary value problem converges with rate h^{2k} at nodal points (see [13]). Since then, it has been conjectured (based on a large amount of numerical evidence) that the same is true for bi-k FE approximation under rectangular meshes for the Poisson equation. This conjecture was settled (see [12]) recently after almost 40 years. Our earlier study reveals that a class of FVMs of arbitrary degree have similar (and even better in some special cases) superconvergence properties as counterpart FEMs in the one dimensional setting [8, 9]. It is natural to ask whether the $2k$-*conjecture* is valid for FVMs? In this work, we will provide a confirmatory answer to this question. To be more precise, we shall investigate superconvergence properties of any order FV schemes studied in [29]. In particular, we show that the underlying FVM has all the superconvergence properties of the counterpart FEM.

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We begin with a model problem:

$$-\triangle u = f \text{ in } \Omega, \text{ and u} = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega = [a, b] \times [c, d]$ and f is a real-valued function defined on Ω.

Techniques used in [8, 9] are very difficult to apply to FV schemes in the two dimensional setting. Inspired by a recent work [12] for the FEM, our approach here is to construct a suitable function to correct the error between the exact solution u and its interpolation u_I. Due to the different nature of the FVM, the construction here is different from that of the FEM, some novel design has to be made to serve our purpose. In particular, we construct our correction function by designing some special operators, instead of a complicated iterative procedure used in the FEM case (see section 3). In addition, using a special mapping from the trial space to test space [29], the FV bilinear form can be regarded as a Gauss quadrature of its corresponding FE bilinear form. Then by taking special care for the residual term of the Gauss quadrature, we show that our correction function also has the desired properties. Once the correction function is constructed, superconvergence properties at some special points can be obtained with standard arguments. Our main results can be summarized as the following.

We first establish superconvergence at nodes: the bi-k degree FV solution u_h superconverges to u with order $2k$ at any nodal point P, i.e.,

$$(u - u_h)(P) = O(h^{2k}) \qquad \text{(compared to the optimal global rate } O(h^{k+1})), \tag{1.2}$$

which is termed by Zhou and Lin [30] as the $2k$*-conjecture* in the FE regime; see also, e.g., [5, 26] for literature along this line.

Our superconvergence results also include

$$(u - u_h)(L) = O(h^{k+2}) \qquad \text{(compared to } \|u - u_h\|_0 = O(h^{k+1})), \tag{1.3}$$

where L is an interior Lobatto point, and

$$\nabla(u - u_h)(G) = O(h^{k+1}) \qquad \text{(compared to } \|u - u_h\|_1 = O(h^k)), \tag{1.4}$$

where G is a Gauss point. As the reader may recall, these rates are the same as the counterpart FEM.

The rest of the paper is organized as follows. In section 2, we present our FV scheme for (1.1) and discuss the relationship between FV and FE bilinear forms. Section 3 is the most technical part, where we construct a correction function and study its properties. In section 4, we prove our main results (1.2)–(1.4). Finally, we provide some carefully designed numerical examples to support our theoretical findings in section 5.

Throughout this paper, we adopt standard notation for Sobolev spaces such as $W^{m,p}(D)$ on subdomain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and seminorm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. The notation $A \lesssim B$ implies that A can be bounded by B multiplied by a constant independent of the mesh size h. $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$.

To end this introduction, we would like to emphasize that this work is a theoretical investigation. Our intention here is not to provide a practical method or anything similar, rather, we settle a conjecture on the convergence rate in the best possible case under a very limited special situation.

Compared with the rich literature on superconvergence of the FEM (see, e.g., [2, 5, 10, 11, 20, 26, 24, 27, 31]), the superconvergence study for the FVM is still in its infancy, especially for high order schemes.

2. FV schemes of arbitrary order. In this section, we first recall FV schemes introduced in [29], then we discuss briefly the relationship between the FV and its corresponding FE bilinear forms.

Let $\mathcal{T}_h$ be a rectangular partition of Ω, where h is the maximum length of all edges. For any $\tau \in \mathcal{T}_h$, we denote by h_τ^x, h_τ^y the lengths of x- and y- directional edges of τ, respectively. We assume that the mesh $\mathcal{T}_h$ is *quasi uniform* in the sense that there exist constants $c_1, c_2 > 0$ such that

$$h \le c_1 h_\tau^x, \quad h \le c_2 h_\tau^y \quad \forall \tau \in \mathcal{T}_h.$$

We denote by $\mathcal{E}_h$ and $\mathcal{N}_h$ the set of edges and vertices of $\mathcal{T}_h$, respectively.

We construct control volumes using Gauss points described below. Define reference element $\hat{\tau} = [-1,1] \times [-1,1]$ and $\mathbb{Z}_r = \{1, 2, \ldots, r\}, \mathbb{Z}_r^0 = \{0, 1, \ldots, r\}$ for any positive integer r. Let $G_j, j \in \mathbb{Z}_k$, be Gauss points of degree k (zeros of the Legendre polynomial P_k) in $[-1,1]$. Then $g_{i,j}^{\hat{\tau}} = (G_i, G_j), i, j \in \mathbb{Z}_k$, constitutes k^2 Gauss points in $\hat{\tau}$. Given $\tau \in \mathcal{T}_h$, let F_τ be the affine mapping from $\hat{\tau}$ to τ. Then Gauss points in τ are

$$\mathcal{G}_\tau = \{g_{i,j}^\tau : g_{i,j}^\tau = F_\tau(G_i, G_j),\ i, j \in \mathbb{Z}_k\}.$$

Similarly, let $L_i, i \in \mathbb{Z}_k^0$ be Lobatto points of degree $k+1$ on the interval $[-1,1]$, i.e., $L_0 = -1, L_k = 1$, and $L_i, i \in \mathbb{Z}_{k-1}$, are zeros of P_k'. Then

$$\mathcal{L}_\tau = \{l_{i,j}^\tau : l_{i,j}^\tau = F_\tau(L_i, L_j),\ i, j \in \mathbb{Z}_k^0\}$$

constitutes $(k+1)^2$ Lobatto points on τ. We denote by

$$\mathcal{G}_h = \bigcup_{\tau \in \mathcal{T}_h} \mathcal{G}_\tau, \qquad \mathcal{L}_h = \bigcup_{\tau \in \mathcal{T}_h} \mathcal{L}_\tau$$

the set of Gauss and Lobatto points on the whole domain, respectively, and $\mathcal{L}_h^0$ the set of interior Lobatto points by excluding Lobatto points on the boundary $\partial\Omega$. For any $P \in \mathcal{L}_h^0$, the control volume surrounding P is the rectangle K_P^* formed by four segments connecting the four Gauss points in $\mathcal{G}_h$ closest to P. Then

$$\mathcal{T}_h^* = \bigcup_{P \in \mathcal{L}_h} K_P^*$$

constitutes a dual partition of $\mathcal{T}_h$.

Next, we denote by $\mathbb{P}_k$ the space of polynomials with degree no more than k, and $\psi_{K_P^*}$ the characteristic function of K_P^*. Then the trial and test spaces are defined as

$$U_h = \{v \in C^0(\Omega) : v|_\tau \in \mathbb{P}_k \times \mathbb{P}_k, \tau \in \mathcal{T}_h, v|_{\partial\Omega} = 0\}$$

and

$$V_h = \mathrm{Span}\{\psi_{K_P^*} : P \in \mathcal{L}_h^0\},$$

respectively. We see that U_h is the bi-k degree FE space, and V_h is the piecewise constant space with respect to the partition $\mathcal{T}_h^*$. They both vanish on the boundary of Ω.

The FVM for solving (1.1) is to find $u_h \in U_h$ satisfying the following local conservative property,

$$-\int_{\partial\tau^*} \frac{\partial u_h}{\partial \mathbf{n}} ds = \int_{\tau^*} f dxdy \quad \forall \tau^* \in \mathcal{T}_h^*,$$

or equivalently,

$$a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \tag{2.1}$$

where the bilinear form is defined for all $w \in H_0^1(\Omega), v_h \in V_h$ by

$$a_h(w, v_h) = -\sum_{E \in \mathcal{E}_{\mathcal{T}_h^*}} [v_h]_E \int_E \frac{\partial w}{\partial \mathbf{n}} ds. \tag{2.2}$$

Here $\mathcal{E}_{\mathcal{T}_h^*}$ is the set of interior edges of the dual partition $\mathcal{T}_h^*$, $[v_h]_E = v_h|_{\tau_2} - v_h|_{\tau_1}$ denotes the jump of v_h across the common edge $E = \tau_1 \cap \tau_2$ of two rectangles $\tau_1, \tau_2 \in \mathcal{T}_h^*$, and $\mathbf{n}$ denotes the normal vector on E pointing from τ_2 to τ_1.

The inf-sup property and continuity of the bilinear form $a_h(\cdot,\cdot)$ have been established in [29]. Moreover, we have the following convergence and superconvergence properties.

LEMMA 2.1 (cf. [29]). *Let $u \in H_0^1(\Omega) \cap H^{k+2}(\Omega)$ be the solution of* (1.1)*, and u_h be the solution of* (2.1)*. Then*

$$|u - u_h|_1 \lesssim h^k |u|_{k+1}, \qquad |u_h - \tilde{u}_I|_1 \lesssim h^{k+1}|u|_{k+2}, \tag{2.3}$$

where $\tilde{u}_I \in U_h$ is the function interpolating u at Lobatto points.

We next discuss the relationship between $a_h(\cdot,\cdot)$ and the FE bilinear form $a_e(\cdot,\cdot)$, which is defined for all $v, w \in H_0^1(\Omega)$ by

$$a_e(v, w) = \int_\Omega \nabla v \cdot \nabla w.$$

We begin with some necessary notation. Let $A_j, j \in \mathbb{Z}_k$ denote the weights of the Gauss quadrature $Q_k(F) = \sum_{j=1}^k A_j F(G_j)$ for computing the integral $I(F) = \int_{-1}^1 F(x)dx$. For all $v_1, v_2 \in L^2(\Omega)$, we define a discrete inner product on Ω by

$$\langle v_1, v_2 \rangle = \sum_{\tau \in \mathcal{T}_h} \langle v_1, v_2 \rangle_\tau, \quad \langle v_1, v_2 \rangle_\tau = \sum_{i,j=1}^k A_{\tau,i}^x A_{\tau,j}^y v_1(g_{i,j}^\tau) v_2(g_{i,j}^\tau),$$

where for all $\tau \in \mathcal{T}_h$,

$$A_{\tau,j}^x = \frac{1}{2} h_\tau^x A_j, \; A_{\tau,j}^y = \frac{1}{2} h_\tau^y A_j, \; j \in \mathbb{Z}_k,$$

are Gauss weights associated with τ. Writing $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}$ for simplicity, we denote, for all $w \in H_0^1(\Omega)$,

$$\partial_x^{-1} w(x,y) = \int_a^x w(x', y)dx', \quad \partial_y^{-1} w(x,y) = \int_c^y w(x, y')dy'.$$

A function $v_h \in V_h$ can be represented as

$$v_h = \sum_{P \in \mathcal{L}_h^0} (v_h)_P \psi_{K_P^*} = \sum_{P \in \mathcal{L}_h} (v_h)_P \psi_{K_P^*},$$

where $(v_h)_P$ is a constant on the control volume K_P^* for $P \in \mathcal{L}_h$. Here we use the fact $(v_h)_P = 0, P \in \partial\Omega$.

Furthermore, we denote the (double layer) jump of v_h at the Gauss point $g_{i,j}^\tau, \tau \in \mathcal{T}_h, i, j \in \mathbb{Z}_k$ as

$$\lfloor v_h \rfloor_{g_{i,j}^\tau} = (v_h)_{l_{i,j}^\tau} + (v_h)_{l_{i-1,j-1}^\tau} - (v_h)_{l_{i-1,j}^\tau} - (v_h)_{l_{i,j-1}^\tau}.$$

Note that when v_h is bilinear in τ (i.e., $k = 1$), the double layer jump $\lfloor v_h \rfloor_{g_{i,j}^\tau}$ is actually the mixed derivative $h_\tau^x h_\tau^y \partial_x \partial_y v_h$.

Denote by $\mathcal{E}_y$ and $\mathcal{E}_x$ the sets of y- and x-directional interior edges of the dual partition $\mathcal{T}_h^*$, respectively, then $\mathcal{E}_{\mathcal{T}_h^*} = \mathcal{E}_y \cup \mathcal{E}_x$. Next we only consider the edges in $\mathcal{E}_y$, since the argument for those in $\mathcal{E}_x$ is similar. For all $\tau \in \mathcal{T}_h$, let

$$\mathcal{E}_{y,\tau} = \mathcal{E}_y \cap \tau = \{\overline{g_{i,j}^\tau g_{i,j+1}^\tau} : i \in \mathbb{Z}_k, j \in \mathbb{Z}_k^0\}.$$

Then $\mathcal{E}_y = \bigcup_{\tau \in \mathcal{T}_h} \mathcal{E}_{y,\tau}$. For all $E = \overline{g_{i,j}^\tau g_{i,j+1}^\tau}, i \in \mathbb{Z}_k, j \in \mathbb{Z}_k^0$,

$$[v_h]_E = [v_h]_{i,j}^{y,\tau} = (v_h)_{l_{i-1,j}^\tau} - (v_h)_{l_{i,j}^\tau},$$

and

$$\int_E \frac{\partial w}{\partial n} ds = \int_E \frac{\partial w}{\partial x} dy = \partial_y^{-1} \partial_x w(g_{i,j+1}^\tau) - \partial_y^{-1} \partial_x w(g_{i,j}^\tau).$$

Therefore,

$$\begin{aligned}
\sum_{E \in \mathcal{E}_y} [v_h]_E \int_E \frac{\partial w}{\partial n} ds &= \sum_{\tau \in \mathcal{T}_h} \sum_{E \in \mathcal{E}_{y,\tau}} [v_h]_E \int_E \frac{\partial w}{\partial x} dy \\
&= \sum_{\tau \in \mathcal{T}_h} \sum_{i=1}^k \sum_{j=0}^k [v_h]_{i,j}^{y,\tau} \left(\partial_y^{-1} \partial_x w(g_{i,j+1}^\tau) - \partial_y^{-1} \partial_x w(g_{i,j}^\tau) \right) \\
&= \sum_{\tau \in \mathcal{T}_h} \left(\sum_{i,j=1}^k \left([v_h]_{i,j-1}^{y,\tau} - [v_h]_{i,j}^{y,\tau} \right) \partial_y^{-1} \partial_x w(g_{i,j}^\tau) + bdy_\tau \right) \\
&= \sum_{\tau \in \mathcal{T}_h} \left(\sum_{i,j=1}^k \lfloor v_h \rfloor_{g_{i,j}^\tau} \partial_y^{-1} \partial_x w(g_{i,j}^\tau) + bdy_\tau \right),
\end{aligned}$$

where the elementwise boundary term

$$bdy_\tau = \sum_{i=1}^k \left([v_h]_{i,k}^{y,\tau} \partial_y^{-1} \partial_x w(g_{i,k+1}^\tau) - [v_h]_{i,0}^{y,\tau} \partial_y^{-1} \partial_x w(g_{i,0}^\tau) \right).$$

Noticing that $v_h = 0$ on the boundary of Ω and the function $v_h \partial_y^{-1} \partial_x w$ is continuous with respect to y across the horizontal edges in $\mathcal{T}_h$, we have

$$\sum_{\tau \in \mathcal{T}_h} bdy_\tau = 0.$$

Consequently,

$$\sum_{E\in\mathcal{E}_y}[v_h]_E\int_E\frac{\partial w}{\partial n}ds=\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k}\lfloor v_h\rfloor_{g_{i,j}^\tau}\partial_y^{-1}\partial_x w(g_{i,j}^\tau).$$

Similarly, we have

$$\sum_{E\in\mathcal{E}_x}[v_h]_E\int_E\frac{\partial w}{\partial n}ds=\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k}\lfloor v_h\rfloor_{g_{i,j}^\tau}\partial_x^{-1}\partial_y w(g_{i,j}^\tau).$$

Then

$$\begin{aligned}a_h(w,v_h)&=-\sum_{E\in\mathcal{E}_y}[v_h]_E\int_E\frac{\partial w}{\partial \mathbf{n}}ds-\sum_{E\in\mathcal{E}_x}[v_h]_E\int_E\frac{\partial w}{\partial \mathbf{n}}ds\\ &=-\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k}\left(\partial_x^{-1}\partial_y w+\partial_y^{-1}\partial_x w\right)(g_{i,j}^\tau)\lfloor v_h\rfloor_{g_{i,j}^\tau}.\end{aligned}\tag{2.4}$$

In [29], a linear mapping $\Pi:U_h\to V_h$,

$$\Pi v=v_h=:\sum_{P\in\mathcal{L}_h^0}(v_h)_P\psi_{K_P^*}\in V_h,\quad v\in U_h,\tag{2.5}$$

is defined by letting

$$\lfloor v_h\rfloor_{g_{i,j}^\tau}=A_{\tau,i}^xA_{\tau,j}^y\partial_{x,y}^2v(g_{i,j}^\tau)\ \ \forall g_{i,j}^\tau\in\mathcal{G}_h.\tag{2.6}$$

Note that although the number of constraints in (2.6) (which equals the cardinality of $\mathcal{G}_h$) is different from the dimensionality of the test space (which equals the cardinality of $\mathcal{L}_h^0$), it has been rigourously shown in [29] that Π is well defined. With this mapping, we have

$$a_h(w,\Pi v)=-\langle\partial_x^{-1}\partial_y w,\partial_{x,y}^2v\rangle-\langle\partial_y^{-1}\partial_x w,\partial_{x,y}^2v\rangle.\tag{2.7}$$

On the other hand, for all $v\in U_h$, since $v(x,c)=v(x,d)=0\ \forall x\in[a,b]$, we have

$$\partial_x v(x,c)=\partial_x v(x,d)=0.$$

Note that $\partial_y^{-1}\partial_x w\partial_x v$ is continuous with respect to y across the horizontal edges in $\mathcal{T}_h$; then an elementwise integration by parts for variable y yields

$$\sum_{\tau\in\mathcal{T}_h}\int_\tau\partial_x w\partial_x v dxdy=-\sum_{\tau\in\mathcal{T}_h}\int_\tau(\partial_y^{-1}\partial_x w)\partial_{x,y}^2v dxdy.$$

Following the same line, we obtain

$$\sum_{\tau\in\mathcal{T}_h}\int_\tau\partial_y w\partial_y v dxdy=-\sum_{\tau\in\mathcal{T}_h}\int_\tau(\partial_x^{-1}\partial_y w)\partial_{x,y}^2v dxdy.$$

Then

$$a_e(w,v)=\sum_{\tau\in\mathcal{T}_h}\int_\tau\nabla w\nabla v dxdy=-\sum_{\tau\in\mathcal{T}_h}\int_\tau\left(\partial_x^{-1}\partial_y w+\partial_y^{-1}\partial_x w\right)\partial_{x,y}^2v dxdy.\tag{2.8}$$

Note that (2.8) is a novel representation of the classic FEM bilinear form. This representation might have potential applications when the dual arguments are adopted. To show (2.8), we have used a combination of cancelation techniques which have been first used in the proof of the FVM bilinear form (2.7).

From (2.7) and (2.8), we observe that the FV bilinear form $a_h(\cdot, \Pi\cdot)$ is a discretization (Gauss quadrature) of its corresponding FE bilinear form $a_e(\cdot,\cdot)$. Moreover, since the k-point Gauss quadrature is exact for polynomials of degree $2k-1$ for all $w, v \in \mathbb{P}_{k-1}(x) \times \mathbb{P}_{k-1}(y)$,

$$a_h(w, \Pi v) = a_e(w, v).$$

Thanks to the important equivalence between the FV and FE bilinear forms, orthogonality techniques which are used to analyze the superconvergence of the FEM can be applied to the FVM.

3. Correction function. Superconvergence analysis at a special point can usually be reduced to estimating

$$a_h(u - u_I, \Pi v) \quad \forall v \in U_h,$$

where $u_I \in U_h$ is an interpolant of u which will be defined in (3.11). A straightforward analysis using the continuity of $a_h(\cdot,\cdot)$ results in

$$|a_h(u - u_I, \Pi v)| \lesssim h^k$$

due to the restriction of optimal error bound

$$|u - u_I|_1 \lesssim h^k.$$

Further analysis based on the standard superconvengence argument may lead to

$$|a_h(u - u_I, \Pi v)| \lesssim h^{k+1},$$

an improvement by order one, but is still far from our need. To obtain desired superconvergence results, more delicate analysis is necessary. In this section, we shall construct a correction function w_h with the following properties.

PROPOSITION 3.1. *Assume that $u \in H^{\alpha+1}(\Omega), \alpha = k+2$ (or $2k$). Then there exists a function $w_h \in U_h$ such that $w_h = 0$ at all nodes and*

$$\|w_h\|_{0,\infty} \lesssim h^{k+2}|\ln h|^{\frac{1}{2}}\|u\|_{\alpha+1}. \tag{3.1}$$

Furthermore,

$$|a_h(u - u_I - w_h, \Pi v)| \lesssim h^{\alpha}\|u\|_{\alpha+1}\|v\|_1 \ \forall v \in U_h. \tag{3.2}$$

In the rest of this section, we will first construct w_h and then verify that w_h satisfies Proposition 3.1.

3.1. Construction. In this subsection, we construct a suitable correction function w_h by introducing some special operators. Our device is very transparent and simpler than that in [12] for the FEM, where a complex iterative procedure is used.

We begin with notation and preliminaries. Since $\mathcal{T}_h$ is a partition of rectangles, there exist $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$ such that

$$\mathcal{T}_h = \{\tau_{i,j} : \tau_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}.$$

We denote by $B_i^x = [x_{i-1}, x_i] \times [c, d], i \in \mathbb{Z}_m$, the element band along the x-direction, and $B_j^y = [a, b] \times [y_{j-1}, y_j], j \in \mathbb{Z}_n$, the element band along the y-direction, respectively. For any rectangle $B \subset \Omega$, we define

$$U_h(B) = \{v \in C(\Omega) : v|_B \in \mathbb{P}_k(x) \times \mathbb{P}_k(y), v|_{\partial B} = 0\}.$$

Note that when $k = 1$, $U_h(B) = \{0\}$.

For all $i \in \mathbb{Z}_m$, let $\mathcal{L}_{B_i^x} : H_0^1(\Omega) \to U_h(B_i^x)$ be the operator which maps $w \in H_0^1(\Omega)$ to $\mathcal{L}_{B_i^x}(w)$ defined by

$$a_h(\mathcal{L}_{B_i^x}(w), \Pi v) = -\langle \partial_y^{-1} \partial_x w, \partial_{x,y}^2 v \rangle_{B_i^x} \quad \forall v \in U_h(B_i^x). \tag{3.3}$$

Note that on one hand, given $w \in H_0^1(\Omega)$,

$$-\langle \partial_y^{-1} \partial_x w, \partial_{x,y}^2 v \rangle_{B_i^x} \quad \forall v \in U_h(B_i^x)$$

is a bounded linear functional on $U_h(B_i^x)$. On the other hand, the coercivity and continuity of the bilinear form $a_h(\cdot, \Pi\cdot)$ have been established in [29]. Then by the Lax–Milgram lemma, (3.3) has a unique solution and thus the operator $\mathcal{L}_{B_i^x}$ is well defined.

We define a global operator $\mathcal{L}^x : H_0^1(\Omega) \to U_h$ by

$$\mathcal{L}^x(w)|_{B_i^x} := \mathcal{L}_{B_i^x}(w), \quad i \in \mathbb{Z}_m.$$

Since $\mathcal{L}_{B_i^x}(w) = 0$ on the boundary ∂B_i^x, $\mathcal{L}^x(w) = 0$ on all $\partial B_i^x, i \in \mathbb{Z}_m$. Consequently, $\mathcal{L}^x(w) = 0$ at all vertices.

By a slight modification, we can define another operator $\tilde{\mathcal{L}}^x : H_0^1(\Omega) \to U_h$ by letting

$$\tilde{\mathcal{L}}^x(w)|_{B_i^x} := \tilde{\mathcal{L}}_{B_i^x}(w), \quad i \in \mathbb{Z}_m,$$

where the local operator $\tilde{\mathcal{L}}_{B_i^x} : H_0^1(\Omega) \to U_h(B_i^x)$ is defined by

$$a_h(\tilde{\mathcal{L}}_{B_i^x}(w), \Pi v) = -\langle \partial_x^{-1} \partial_y w, \partial_{x,y}^2 v \rangle_{B_i^x} \quad \forall v \in U_h(B_i^x). \tag{3.4}$$

By the same token, we define $\mathcal{L}_{B_j^y}$, $\tilde{\mathcal{L}}_{B_j^y}$, $\mathcal{L}^y$, and $\tilde{\mathcal{L}}^y$.

Next we define some operators. Let $P_r, r \geq 0$, be the Legendre polynomial of degree r and denote by

$$\phi_0(s) = \frac{1-s}{2}, \quad \phi_1(s) = \frac{1+s}{2}, \quad \phi_{r+1}(s) = \int_{-1}^{s} P_r(t)dt, \ r \geq 1,$$

the series of Lobatto polynomials on the interval $[-1, 1]$. For all $\hat{v} \in H^1([-1, 1])$, $\hat{v}$ has the following Lobatto expansion

$$\hat{v}(s) = \sum_{j=0}^{\infty} v_j \phi_j(s),$$

where

$$v_0 = \hat{v}(-1), \ v_1 = \hat{v}(1), \ v_j = \frac{2j-1}{2} \int_{-1}^{1} \hat{v}'(s)\phi_j'(s)ds, \quad j \geq 2. \tag{3.5}$$

We define the truncated operator $Q_p^s, p \geq 1$, as

$$(Q_p^s \hat{v})(s) := \sum_{j=0}^{p} v_j \phi_j(s).$$

The operator Q_p^s has many important properties. First, for any function $\hat{v} \in W^{r,q}([-1,1]), r \geq 0, q \geq 1$, since $Q_p^x \hat{v}$ is the truncated Lobatto expansion of $\hat{v}$, we have

$$|Q_p^s \hat{v}|_{r,q,[-1,1]} \lesssim |\hat{v}|_{r,q,[-1,1]}.$$

Namely, the operator Q_p^s is a continuous linear mapping on the space $W^{r,q}([-1,1])$. Second, $Q_p^s \hat{v} = \hat{v}$ for all $\hat{v} \in \mathbb{P}_p$. Consequently, by the Bramble–Hilbert lemma, there holds for all $\hat{v} \in W^{p+1,q}([-1,1])$

$$|(\hat{v} - Q_p^s \hat{v})|_{r,\infty,[-1,1]} \lesssim |\hat{v}|_{p+1,q,[-1,1]}, \quad r = 0, 1. \tag{3.6}$$

Third, by the orthogonalities of Legendre and Lobatto polynomials,

$$(\hat{v} - Q_p^s \hat{v})' \perp \mathbb{P}_{p-1}, \quad (\hat{v} - Q_p^s \hat{v}) \perp \mathbb{P}_{p-2}, \tag{3.7}$$

where $\mathbb{P}_{-1} = \emptyset$. Finally, the fact $\phi_r(\pm 1) = 0, r \geq 2$, yields that

$$(Q_p^s \hat{v})(-1) = \hat{v}(-1), \quad (Q_p^s \hat{v})(1) = \hat{v}(1). \tag{3.8}$$

For any $(x, y) \in B_i^x, i \in \mathbb{Z}_m$, denoting

$$v(x, y) = \hat{v}(s, y), \quad s = (2x - x_i - x_{i-1})/(x_i - x_{i-1}) \in [-1, 1],$$

we define a operator Q_p^x for $v(x, y)$ by

$$Q_p^x v(x, y) = Q_p^s \hat{v}(s, y).$$

Desired properties of Q_p^x can be obtained by using the linear transform from $[-1, 1]$ to $[x_{i-1}, x_i]$. In fact, in each element band $B_i^x, i \in \mathbb{Z}_m$, we have the remainder

$$(v - Q_p^x v)(x, y) = \sum_{j=p+1}^{\infty} v_j(y) \phi_j(s),$$

which means that the orthogonal property (3.7) is still valid for Q_p^x. Moreover, by (3.6) and (3.8), we obtain

$$|\partial_x^r (v - Q_p^x v)(x, y)| \lesssim h^{p-r} \int_{x_{i-1}}^{x_i} \left|\partial_x^{p+1} v(x, y)\right| dx, \quad r = 0, 1, \tag{3.9}$$

and

$$(Q_p^x v)(x_i, y) = v(x_i, y), \quad (Q_p^x v)(x_{i-1}, y) = v(x_{i-1}, y). \tag{3.10}$$

The operator $Q_p^y, p \geq 1$, can be defined similarly.

Now we define, for all v, an interpolation

$$v_I = Q_k^x Q_k^y v, \tag{3.11}$$

and the residuals

$$E^x v = v - Q_k^x v, \quad E^y v = v - Q_k^y v.$$

Then

$$v - v_I = E^x v + E^y v - E^y E^x v \tag{3.12}$$

and

$$(v - v_I)(P) = 0 \quad \forall P \in \mathcal{N}_h.$$

We are now in a perfect position to construct our correction function w_h. Let

$$w_h = w_1 + w_2, \quad w_1 = \mathcal{L}^x(E^x u) + \tilde{\mathcal{L}}^x(E^x u), \quad w_2 = \mathcal{L}^y(E^y u) + \tilde{\mathcal{L}}^y(E^y u). \tag{3.13}$$

Obviously, $w_h \in U_h$ and $w_h(P) = 0$ for all $P \in \mathcal{N}_h$.

3.2. Analysis. In this subsection, we shall prove that w_h defined in (3.13) satisfies all properties listed in Proposition 3.1. For simplicity, we assume that

$$h = h_\tau^x = h_\tau^y \quad \forall \tau \in \mathcal{T}_h.$$

Consider $\mathcal{L}^x(E^x u)$, the first term of w_1. For this purpose, we need to present (3.3) in its linear algebraic form. We begin with a presentation of a basis of $U_h(B_i^x), i \in \mathbb{Z}_m$. For all $(x, y) \in B_i^x$ and $0 \le p, q \le k$, let

$$\Psi_{p,q}(x, y) = \phi_p(s)\phi_q(t), \tag{3.14}$$

where $s = (2x - x_i - x_{i-1})/h, \quad t = (2y - d - c)/(d - c)$. Then the function system $\{\Psi_{p,q}, 2 \le p, q \le k\}$ constitutes a basis of $U_h(B_i^x)$. Since $\mathcal{L}_{B_i^x}(E^x u) \in U_h(B_i^x)$, we have the representation

$$\mathcal{L}_{B_i^x}(E^x u) = \sum_{p,q=2}^{k} w_{p,q} \Psi_{p,q}.$$

Let

$$D = (d_{p,q})_{(k-1)\times(k-1)}, \quad K = (m_{p,q})_{(k-1)\times(k-1)},$$

where

$$d_{p,q} = \langle \phi'_p, \phi'_q \rangle_{[-1,1]}, \qquad m_{p,q} = -\langle \partial^{-1}\phi_p, \phi'_q \rangle_{[-1,1]}, \ 2 \le p, q \le k,$$

with the discrete inner product defined by

$$\langle v_1, v_2 \rangle_{[-1,-1]} = \sum_{r=1}^{k} A_r v_1(G_r) v_2(G_r).$$

By [16, p. 98, (2.7.12)],

$$\langle v_1, v_2 \rangle_{[-1,-1]} = \int_{-1}^{1} (v_1 v_2)(s) ds - c_k (v_1 v_2)^{(2k)}(\xi), \tag{3.15}$$

where $c_k = \frac{2^{2k+1}(k!)^4}{(2k+1)[(2k)!]^3}$ and $\xi \in (-1,1)$. Taking $v = \Psi_{r,l}, r,l = 2,\dots,k$, in (3.3), we derive

$$\sum_{p,q=2}^{k} \Big((d-c)^2 d_{p,r} m_{q,l} + h^2 d_{q,l} m_{p,r}\Big) w_{p,q} = f_{r,l}, \tag{3.16}$$

where

$$f_{r,l} = -(d-c)h\langle \partial_y^{-1}\partial_x E^x u, \partial^2_{x,y}\Psi_{r,l}\rangle_{B_i^x}. \tag{3.17}$$

Denote the unknowns $X = (X_2,\dots X_k)^T$ and the right-hand side $F = (F_2,\dots,F_k)^T$ with vectors

$$X_r = (w_{r,2},\dots,w_{r,k})^T,\ F_r = (f_{r,2},\dots,f_{r,k})^T,\ r = 2,\dots,k.$$

Then (3.16) can be rewritten as

$$\Big((d-c)^2(D\otimes K) + h^2(K\otimes D)\Big)X = F, \tag{3.18}$$

where for two matrices $B_1 = (b^1_{p,q})_{k\times k}$ and $B_2 = (b^2_{p,q})_{k\times k}$, the tensor product $B_1\otimes B_2$ is a matrix of $k^2\times k^2$ defined by

$$B_1\otimes B_2 = (B_{p,q})_{k\times k},\quad B_{p,q} = b^1_{p,q}B_2,\quad p,q\le k.$$

With the linear system (3.18), the study of $\mathcal{L}^x(E^x u)$ is reduced to estimates of the vector F and the matrix $A = (d-c)^2(D\otimes K) + h^2(K\otimes D)$.

For any vector $\nu = (\nu_1,\dots,\nu_r)^T$, we denote by $\|\nu\|_\infty = \max_{1\le q\le r}|\nu_q|$ the maximal norm of ν. We have the following estimate for vector F.

Lemma 3.2. *If $u\in H^{\alpha+1}(\Omega), \alpha\ge k+1$, then*

$$\|F_p\|_\infty \lesssim h^{\min(\alpha,2k+2-p)}|\ln h|^{\frac12}\|u\|_{\alpha+1,B_i^x},\quad p = 2,\dots,k. \tag{3.19}$$

Proof. To show (3.19), we only need to prove

$$|f_{p,q}| \lesssim h^{\min(\alpha,2k+2-p)}|\ln h|^{\frac12}\|u\|_{\alpha+1,B_i^x},\quad p,q = 2,\dots,k. \tag{3.20}$$

By (3.17), we have

$$-\frac{f_{p,q}}{h(d-c)} = \langle \partial_y^{-1}\partial_x(E^x u - Q^x_{\alpha-1}E^x u), \partial^2_{x,y}\Psi_{p,q}\rangle_{B_i^x} + \langle \Theta, \partial^2_{x,y}\Psi_{p,q}\rangle_{B_i^x} = I_1 + I_2,$$

where $\Theta = \partial_y^{-1}\partial_x(Q^x_{\alpha-1}E^x u)$. We next estimate I_1 and I_2, respectively.

A direct calculation from (3.9) yields

$$|E^x u - Q^x_{\alpha-1}E^x u|_{1,\infty,B_i^x} \lesssim h^{\alpha-1}|E^x u|_{\alpha,\infty,B_i^x} \lesssim h^{\alpha-1}|u|_{\alpha,\infty,B_i^x}.$$

Note that

$$\partial_y\Psi_{p,q} = \phi'_q = O(1),\quad \partial_x^r\Psi_{p,q} = \left(\frac{2}{h}\right)^r\phi_p^{(r)} = O(h^{-r}),\quad r\le p,$$

then

$$|I_1| = \left|\langle \partial_y^{-1}\partial_x(E^x u - Q^x_{\alpha-1}E^x u), \partial^2_{x,y}\Psi_{p,q}\rangle_{B_i^x}\right| \lesssim h^{\alpha-1}|u|_{\alpha,\infty,B_i^x}.$$

To estimate I_2, we first denote Gauss points $g^\tau_{r,l} = (g^x_{\tau,r}, g^y_{\tau,l}), r, l \in \mathbb{Z}_k$, for all $\tau \in B^x_i, i \in \mathbb{Z}_m$. Since for any fixed y, $\partial^2_{x,y}\Psi_{p,q}(\cdot, y) \in \mathbb{P}_{k-1}$, by the orthogonality (3.7), and the fact that $\Theta = \partial_x(Q^x_{\alpha-1}E^x(\partial^{-1}_y u))$, we have

$$\int_{x_{i-1}}^{x_i} \Theta\partial^2_{x,y}\Psi_{p,q}dx = \int_{x_{i-1}}^{x_i} (\partial_x E^x(\partial^{-1}_y u))\partial^2_{x,y}\Psi_{p,q}dx = 0.$$

Then

$$I_2 = \langle\Theta, \partial^2_{x,y}\Psi_{p,q}\rangle_{B^x_i} = -\sum_{\tau\in B^x_i}\sum_{l=1}^{k} A^y_{\tau,l} e^\tau_{p,q}(g^y_{\tau,l}),$$

where

$$e^\tau_{p,q}(y) = \int_{x_{i-1}}^{x_i} \Theta\partial^2_{x,y}\Psi_{p,q}dx - \sum_{r=1}^{k} A^x_{\tau,r}\left(\Theta\partial^2_{x,y}\Psi_{p,q}\right)(g^x_{\tau,r}, y)$$

is the error of the Gauss quadrature for calculating the integral of $\Theta\partial^2_{x,y}\Psi_{p,q}$ in $[x_{i-1}, x_i]$. By (3.15), there exists a point $\xi_i \in (x_{i-1}, x_i)$ such that

$$e^\tau_{p,q}(y) = c_k\frac{h^{2k+1}}{2^{2k+1}}\partial^{2k}_x\left(\Theta\partial^2_{x,y}\Psi_{p,q}\right)(\xi_i, y).$$

Note that

$$\|\partial^j_x\Theta\|_{\infty,B^x_i} \lesssim \|\partial^{j+1}_x E^x(\partial^{-1}_y u)\|_{\infty,B^x_i} \lesssim |u|_{\alpha-1,\infty,B^x_i}, \quad j < \alpha - 1.$$

Then by the Leibniz formula for derivatives,

$$|e^\tau_{p,q}| \lesssim h^{2k+1-p}|u|_{\alpha-1,\infty,B^x_i},\ 2 \le q \le k,$$

which implies

$$|I_2| \lesssim h^{2k+1-p}|u|_{\alpha-1,\infty,B^x_i}.$$

Combining I_1 with I_2, we obtain

$$|f_{p,q}| \lesssim h(|I_1| + |I_2|) \lesssim h^{\min(\alpha,2k+2-p)}\|u\|_{\alpha,\infty,B^x_i}.$$

Now recall from the standard regularity argument [1],

$$\|u\|_{\alpha,\infty,B^x_i} \lesssim |\ln h|^{\frac{1}{2}}\|u\|_{\alpha+1,B^x_i},$$

the desired estimate (3.20) follows. This finishes our proof. ∎

We next study properties of the matrix $A = (d-c)^2(D\otimes K) + h^2(K\otimes D)$. By Gauss quadrature and the orthogonality of Legendre polynomials, we have

$$d_{p,q} = (P_{p-1}, P_{q-1}) = 0, p \ne q, \quad d_{p,p} = \frac{2}{2p-1}, \quad p, q = 2, \dots, k.$$

In other words, D is a diagonal matrix. Similarly,

$$m_{p,q} = -(\partial^{-1}\phi_p, \phi'_q) = (\phi_p, \phi_q),\ p, q \le k, p+q \le 2k-1. \tag{3.21}$$

By the quasi-orthogonal property of Lobatto polynomials, $m_{p,q} \neq 0$ only when $p-q = 0, \pm 2$. Consequently, K is a five-diagonal matrix.

LEMMA 3.3. *The matrix K is symmetric and positive definite.*

Proof. Let $K_1 = (m^1_{p,q})_{(k-1)\times(k-1)}$ with $m^1_{p,q} = (\phi_p, \phi_q), p, q = 2, \ldots, k$. By (3.21),

$$m^1_{p,q} = m_{p,q}, \quad p, q \le k, p + q \le 2k - 1.$$

We next study the relationship between $m^1_{k,k}$ and $m_{k,k}$. Denoting

$$e_k = m_{k,k} - m^1_{k,k},$$

we have from (3.15) and the Leibniz formula for derivatives

$$e_k = c_k((\partial^{-1}\phi_k)\phi'_k)^{(2k)}(\xi) = c_k \binom{2k}{k-1} \|\phi_k\|^2_{k,\infty} > 0.$$

Then

$$K = K_1 + K_2,$$

where $K_2 = (m^2_{p,q})_{(k-1)\times(k-1)}, p, q = 2, \ldots, k$, with

$$m^2_{k,k} = e_k > 0, \quad m^2_{p,q} = 0 \text{ otherwise.}$$

Since K_1 is symmetric and positive definite, K is also symmetric and positive definite. ∎

Note that since both D and K are symmetric and positive definite and independent of h, then both $D \otimes K$ and $D \otimes K$ are positive definite. By the definition of A, we have

$$\det(A) = \det((d-c)^2(D \otimes K)) + O(h^2).$$

Therefore, when h is sufficiently small, $\det A$ is positive and uniformly bounded from below. In other words, when h is sufficiently small,

$$0 < \det(A)^{-1} \lesssim C, \tag{3.22}$$

where C is independent of h.

With the estimate of F and properties of A, we are now ready to estimate $\mathcal{L}_{B^x_i}(E^x u)$.

LEMMA 3.4. *Assume $u \in H^{\alpha+1}(\Omega), \alpha = k+2$ (or $2k$). Then for sufficiently small h and all $i \in \mathbb{Z}_m$*

$$\|X_r\|_\infty \lesssim h^{k+2+\max(0,\alpha-k-r)} |\ln h|^{\frac{1}{2}} \|u\|_{\alpha+1,B^x_i}, \quad r = 2, \ldots, k. \tag{3.23}$$

Consequently,

$$\left\|\mathcal{L}_{B^x_i}(E^x u)\right\|_{0,\infty} \lesssim h^{k+2} |\ln h|^{\frac{1}{2}} \|u\|_{\alpha+1,B^x_i}. \tag{3.24}$$

Proof. Note that since

$$\left\|\mathcal{L}_{B^x_i}(E^x u)\right\|_{0,\infty} \lesssim \sum_{r=2}^{k} \|X_r\|_\infty,$$

then (3.24) follows from (3.23). We next show (3.23). When $u \in H^{k+3}(\Omega)$, by (3.19), (3.22), and Cramer's rule,

$$\|X_r\|_\infty \lesssim h^{k+2}|\ln h|^{\frac{1}{2}}\|u\|_{k+3,B_i^x}, \quad r = 2\ldots,k.$$

Then (3.23) is valid for $\alpha = k+2$. To prove (3.23) for $\alpha = 2k$, we rewrite A in its block matrix form $A = (A_{r,l})_{(k-1)\times(k-1)}$, where each

$$A_{r,l} = (d-c)^2 d_{r,l}K + h^2 m_{r,l}D, \quad r,l = 2,\ldots,k,$$

is a $(k-1)\times(k-1)$ matrix. Let $A'_{r,l} = A_{r,l}h^{-|r-l|}$ and

$$Y_r = X_r h^{r-2k-2}|\ln h|^{-\frac{1}{2}}\|u\|^{-1}_{2k+1,B_i^x}, \quad F'_r = F_r h^{r-2k-2}|\ln h|^{-\frac{1}{2}}\|u\|^{-1}_{2k+1,B_i^x}.$$

Then both $A'_{r,l}$ and F'_r are independent of h. Multiplying the rth equation of (3.18) with the factor $h^{r-2k-2}|\ln h|^{-\frac{1}{2}}\|u\|^{-1}_{2k+1,B_i^x}$, we have for all $r = 2,\ldots,k$,

$$h^4 A'_{r,r-2}Y_{r-2} + A'_{r,r}Y_r + A'_{r,r+2}Y_{r+2} = F'_r, \tag{3.25}$$

where we use the notation $A_{2,0} = A_{3,1} = A_{k-1,k+1} = A_{k,k+2} = 0$. Let $B = (B_{r,l})_{(k-1)\times(k-1)}$ with

$$B_{r,l} = A'_{r,l}, \quad r \le l, \quad B_{r,l} = h^4 A'_{r,l} \ \text{ otherwise.}$$

Then (3.25) can be written as a linear system $BY = F'$. A direct calculation yields

$$det(B) = \prod_{r=2}^{k} det(A'_{r,r}) + O(h^4),$$

which means that B is uniformly bounded from below. By Cramer's rule, each entry of Y is bounded independent of h. In other words, $\|Y_r\|_\infty \lesssim 1$. Consequently,

$$\|X_r\|_\infty \lesssim h^{2k+2-r}|\ln h|^{\frac{1}{2}}\|u\|_{2k+1,B_i^x}, \quad r = 2,\ldots,k.$$

The proof is completed. ∎

To prove Proposition 3.1, we still need to estimate the residual

$$-\langle \partial_y^{-1}\partial_x w, \partial^2_{x,y}v\rangle_{B_i^x} - a_h(\mathcal{L}_{B_i^x}(w), \Pi v), \quad w \in H_0^1(\Omega)$$

for a general function $v \in U_h$. Note that when $v \in U_h(B_i^x)$, the above residual equals zero. The estimate of the above residual for a general $v \in U_h$ is a difficult task, however. In the following lemma, we will use the inequalities in (3.23), especially $\|X_2\|_\infty = O(h^{2k})$, to estimate the above residual.

Lemma 3.5. *Assume that $u \in H^{\alpha+1}(\Omega), \alpha = k+2$ (or $2k$). Then for a general function $v \in U_h$ and $i \in \mathbb{Z}_m$,*

$$\left|\langle \partial_y^{-1}\partial_x E^x u, \partial^2_{x,y}v\rangle_{B_i^x} + a_h(\mathcal{L}_{B_i^x}(E^x u), \Pi v)\right| \lesssim h^\alpha \|u\|_{\alpha+1,B_i^x}\|v\|_{1,B_i^x}. \tag{3.26}$$

Proof. Note that for all $B_i^x \subset \Omega, i \in \mathbb{Z}_m$,

$$\phi_0(x) = \frac{x_i - x}{h}, \quad \phi_1(x) = \frac{x - x_{i-1}}{h} \notin U_h(B_i^x).$$

Then a general function $v \in U_h$ has the decomposition

$$v(x,y) = v_h(x,y) + \tilde{v}(x,y), \quad (x,y) \in B_i^x,$$

where $v_h \in U_h(B_i^x)$ and $\tilde{v}(x,y) = v(x_{i-1},y)\phi_0(x) + v(x_i,y)\phi_1(x)$. Let

$$J_1 = \langle \partial_y^{-1}\partial_x E^x u, \partial_{x,y}^2 \tilde{v}\rangle_{B_i^x}, \quad J_2 = a_h(\mathcal{L}_{B_i^x}(E^x u), \Pi\tilde{v}).$$

Then we have, from (3.3),

$$\langle \partial_y^{-1}\partial_x E^x u, \partial_{x,y}^2 v\rangle_{B_i^x} + a_h(\mathcal{L}_{B_i^x}(E^x u), \Pi v) = J_1 + J_2.$$

We next estimate J_1 and J_2 separately. Let $\Phi = \partial_y^{-1}\partial_x E^x u$. By (3.9) and the fact that $\partial_y^{k+1}\Phi = \partial_x E^x(\partial_y^k u)$, we have for all $(x,y) \in \tau_{i,j}, (i,j) \in \mathbb{Z}_m \times \mathbb{Z}_n$

$$\begin{aligned}|(\Phi - Q_k^y\Phi)(x,y)| &\lesssim h^k \int_{y_{j-1}}^{y_j} \left|\partial_x E^x(\partial_y^k u)(x,y)\right| dy \\ &\lesssim h^{\alpha-1}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} \left|\partial_x^{\alpha+1-k}E^x(\partial_y^k u)(x,y)\right| dxdy \lesssim h^\alpha |u|_{\alpha+1,\tau_{i,j}}.\end{aligned}$$

Then by the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}|\langle \Phi - Q_k^y\Phi, \partial_{x,y}^2\tilde{v}\rangle_{B_i^x}| &\lesssim \langle \Phi - Q_k^y\Phi, \Phi - Q_k^y\Phi\rangle_{B_i^x}^{\frac{1}{2}} \langle \partial_{x,y}^2\tilde{v}, \partial_{x,y}^2\tilde{v}\rangle_{B_i^x}^{\frac{1}{2}} \\ &\lesssim h^\alpha |u|_{\alpha+1,B_i^x}\|\partial_x\tilde{v}\|_{0,B_i^x}.\end{aligned}$$

Here in the last step, we have used the inverse inequality

$$\|\partial_{x,y}^2\tilde{v}\|_{0,B_i^x} \lesssim h^{-1}\|\partial_x\tilde{v}\|_{0,B_i^x}.$$

Note that $(Q_k^y\Phi)\partial_{x,y}^2\tilde{v}(x,\cdot) \in \mathbb{P}_{2k-1}$, by Gauss quadrature and integration by parts, we obtain

$$\begin{aligned}\langle Q_k^y\Phi, \partial_{x,y}^2\tilde{v}\rangle_{B_i^x} &= -\sum_{\tau\in B_i^x}\sum_{l=1}^k A_{\tau,l}^x \int_c^d \big((\partial_y Q_k^y\Phi)\partial_x\tilde{v}\big)(g_{\tau,l}^x, y)dy \\ &= -\sum_{\tau\in B_i^x}\sum_{l=1}^k A_{\tau,l}^x \int_c^d \left(\partial_x E^x(\partial_y Q_k^y \partial_y^{-1}u)\partial_x\tilde{v}\right)(g_{\tau,l}^x, y)dy.\end{aligned}$$

Let $\Upsilon = \partial_y Q_k^y \partial_y^{-1}u$. Since $\tilde{v}$ is linear with respect to x, we have

$$\sum_{\tau\in B_i^x}\sum_{l=1}^k A_{\tau,l}^x \int_c^d (\partial_x Q_\alpha^x E^x\Upsilon)\partial_x\tilde{v}(g_{\tau,l}^x, y)dy = \int_{B_i^x}(\partial_x Q_\alpha^x E^x\Upsilon)\partial_x\tilde{v}dxdy = 0.$$

Consequently,

$$\begin{aligned}|\langle Q_k^y\Phi, \partial_{x,y}^2\tilde{v}\rangle_{B_i^x}| &= \sum_{\tau\in B_i^x}\sum_{l=1}^k A_{\tau,l}^x \int_c^d \left|\big((\partial_x E^x\Upsilon - \partial_x Q_\alpha^x E^x\Upsilon)\partial_x\tilde{v}\big)(g_{\tau,l}^x, y)\right| dy \\ &\lesssim h^\alpha\|\partial_x^{\alpha+1}E^x\Upsilon\|_{0,B_i^x}\|\partial_x\tilde{v}\|_{0,B_i^x} \lesssim h^\alpha|u|_{\alpha+1,B_i^x}\|\partial_x\tilde{v}\|_{0,B_i^x}.\end{aligned}$$

Note that

$$\partial_x \tilde{v} = \frac{v(x_i, y) - v(x_{i-1}, y)}{h} = h^{-1} \int_{x_{i-1}}^{x_i} \partial_x v(x, y) dx,$$

and thus we have

$$\|\partial_x \tilde{v}\|_{0,B_i^x} \lesssim \|v\|_{1,B_i^x}. \tag{3.27}$$

Then

$$\begin{aligned} |J_1| &= \left| \langle \Phi - Q_k^y \Phi, \partial_{x,y}^2 \tilde{v} \rangle_{B_i^x} + \langle Q_k^y \Phi, \partial_{x,y}^2 \tilde{v} \rangle_{B_i^x} \right| \\ &\lesssim h^\alpha |u|_{\alpha+1,B_i^x} \|v\|_{1,B_i^x}. \end{aligned}$$

As for J_2, recalling the bilinear form $a_h(\cdot, \Pi \cdot)$, we have

$$J_2 = -\langle \partial_y^{-1} \partial_x \mathcal{L}_{B_i^x}(E^x u), \partial_{x,y}^2 \tilde{v} \rangle_{B_i^x} - \langle \partial_x^{-1} \partial_y \mathcal{L}_{B_i^x}(E^x u), \partial_{x,y}^2 \tilde{v} \rangle_{B_i^x}.$$

Note that since

$$\int_{x_{i-1}}^{x_i} (\partial_{x,y}^2 \tilde{v}) \partial_y^{-1} \partial_x \mathcal{L}_{B_i^x}(E^x u) dx = 0,$$

then

$$\langle \partial_y^{-1} \partial_x \mathcal{L}_{B_i^x}(E^x u), \partial_{x,y}^2 \tilde{v} \rangle_{B_i^x} = 0.$$

Therefore,

$$\begin{aligned} J_2 = -\langle \partial_x^{-1} \partial_y \mathcal{L}_{B_i^x}(E^x u), \partial_{x,y}^2 \tilde{v} \rangle_{B_i^x} &= -\int_{B_i^x} \frac{\partial^2 \tilde{v}}{\partial x \partial y} \partial_x^{-1} \partial_y \mathcal{L}_{B_i^x}(E^x u) dx dy \\ &= \int_{B_i^x} \frac{\partial \tilde{v}}{\partial x} \partial_x^{-1} \partial_y^2 \mathcal{L}_{B_i^x}(E^x u) dx dy. \end{aligned}$$

Here in the last step, we have used integration by parts and the fact that $\partial_x \tilde{v}(x, c) = \partial_x \tilde{v}(x, d) = 0$. Since for any given y, $\partial_x \tilde{v}(\cdot, y) \in \mathbb{P}_0$, by the properties of Lobatto polynomials, we have $\partial_x^{-1} \Psi_{p,q}(\cdot, y) \perp \mathbb{P}_{p-4}, p > 4$, and

$$\int_{x_{i-1}}^{x_i} \partial_x^{-1} \Psi_{p,q}(x, y) dx = 0, \ p = 2, 4.$$

Then

$$\begin{aligned} |J_2| &= \left| \sum_{q=2}^{k} w_{3,q} \int_{B_i^x} \partial_x \tilde{v} \partial_x^{-1} \partial_y^2 \Psi_{3,q} dx dy \right| \\ &\lesssim h \|X_3\|_\infty \int_{B_i^x} |\partial_x \tilde{v}| \, dx dy \lesssim h^{\alpha+\frac{1}{2}} |\ln h|^{\frac{1}{2}} \|u\|_{\alpha+1,B_i^x} \|v\|_{1,B_i^x}, \end{aligned}$$

where in the second and last step, we have used the fact that $\partial_x^{-1} \partial_y^2 \Psi_{3,q} = O(h)$ and (3.27), respectively.

The desired statement (3.26) follows by combining the estimates of J_1 with J_2. ∎

Recall the definitions of $\mathcal{L}_{B_i^x}$ and $\tilde{\mathcal{L}}_{B_i^x}$ in (3.3) and (3.4); it is easy to see that the only difference between the two operators lies in the right-hand side. Consequently, following the same arguments as we did for $\mathcal{L}_{B_i^x}(E^x u)$, we can show that the estimate for $\mathcal{L}_{B_i^x}(E^x u)$ in (3.24) is still valid for $\tilde{\mathcal{L}}_{B_i^x}(E^x u)$, and

$$\left|\langle \partial_x^{-1}\partial_y E^x u, \partial_{x,y}^2 v\rangle_{B_i^x} + a_h(\tilde{\mathcal{L}}_{B_i^x}(E^x u), \Pi v)\right| \lesssim h^\alpha \|u\|_{\alpha+1,B_i^x}\|v\|_{1,B_i^x}. \tag{3.28}$$

With all the preparations, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. As a direct consequence of (3.24), we have

$$\|\mathcal{L}^x(E^x u)\|_{0,\infty} \lesssim h^{k+2}|\ln h|^{\frac{1}{2}}\|u\|_{\alpha+1}.$$

By the same arguments, we can prove that the above inequality still holds true for $\tilde{\mathcal{L}}^x(E^x u)$, $\mathcal{L}^y(E^y u)$, and $\tilde{\mathcal{L}}^y(E^y u)$. Then (3.1) follows.

Now we prove (3.2). By the decomposition (3.12) and (3.13), we have

$$a_h(u-u_I-w_h,\Pi v) = a_h(E^x u - w_1, \Pi v) + a_h(E^y u - w_2, \Pi v) - a_h(E^y E^x u, \Pi v).$$

As a direct consequences of (3.26) and (3.28),

$$\left|\langle \partial_y^{-1}\partial_x E^x u, \partial_{x,y}^2 v\rangle + a_h(\mathcal{L}^x(E^x u), \Pi v)\right| \lesssim h^\alpha\|u\|_{\alpha+1}\|v\|_1,$$
$$\left|\langle \partial_x^{-1}\partial_y E^x u, \partial_{x,y}^2 v\rangle + a_h(\tilde{\mathcal{L}}^x(E^x u), \Pi v)\right| \lesssim h^\alpha\|u\|_{\alpha+1}\|v\|_1.$$

Note that since

$$a_h(E^x u - w_1, \Pi v) = -\langle \partial_y^{-1}\partial_x E^x u, \partial_{x,y}^2 v\rangle - \langle \partial_x^{-1}\partial_y E^x u, \partial_{x,y}^2 v\rangle - a_h(\mathcal{L}^x(E^x u) + \tilde{\mathcal{L}}^x(E^x u), \Pi v),$$

then

$$|a_h(E^x u - w_1, \Pi v)| \lesssim h^\alpha\|u\|_{\alpha+1}\|v\|_1.$$

Similarly, we have

$$|a_h(E^y u - w_2, \Pi v)| \lesssim h^\alpha\|u\|_{\alpha+1}\|v\|_1.$$

On the other hand, for all $(x,y)\in\tau_{i,j}, i\in\mathbb{Z}_m, j\in\mathbb{Z}_n$, noticing that

$$(\partial_x^{-1}\partial_y E^y E^x u)(x,y) = \int_a^x (\partial_y E^y E^x u)(x',y)dx' = \int_{x_{i-1}}^x (\partial_y E^y E^x u)(x',y)dx',$$

then we have, from (3.9),

$$\left|(\partial_x^{-1}\partial_y E^y E^x u)(x,y)\right| \lesssim h^k \int_{y_{j-1}}^{y_j}\left|\partial_y^{k+1}E^x u\right|dy = h^k\int_{y_{j-1}}^{y_j}\left|E^x(\partial_y^{k+1}u)\right|dy$$
$$\lesssim h^{\alpha-1}\int_{\tau_{i,j}}\left|\partial_x^{\alpha-k}\partial_y^{k+1}u\right|dxdy \lesssim h^\alpha\|u\|_{\alpha+1,\tau_{i,j}}.$$

Similarly, we obtain

$$\left|(\partial_y^{-1}\partial_x E^y E^x u)(x,y)\right| \lesssim h^{\alpha-1}\int_{\tau_{i,j}}\left|\partial_y^{\alpha-k}\partial_x^{k+1}u\right|dxdy \lesssim h^\alpha\|u\|_{\alpha+1,\tau_{i,j}}.$$

Then by (2.7),

$$|a_h(E^yE^xu,\Pi v)| \lesssim h^\alpha\|u\|_{\alpha+1}\left(\sum_{\tau\in\mathcal{T}_h} h^2\|\partial^2_{x,y}v\|^2_{0,\tau}\right)^{\frac{1}{2}} \lesssim h^\alpha\|u\|_{\alpha+1}\|v\|_1.$$

Here in the last step, we have used the inverse inequality. The statement (3.2) follows by combining the estimates for $a_h(E^xu-w_1,\Pi v)$, $a_h(E^yu-w_2,\Pi v)$, and $a_h(E^yE^xu,\Pi v)$. ∎

Remark 3.6. Functions w_1 and w_2 in (3.13) are designed to correct the errors of $a_h(E^xu,\Pi v)$ and $a_h(E^yu,\Pi v)$, respectively. Since $a_h(E^yE^xu,\Pi v)$ is of high order, a correction function for $a_h(E^yE^xu,\Pi v)$ is not necessary.

4. Superconvergence. In this section, we shall study superconvergence properties of u_h at three kinds of special points: nodes, and Gauss and Lobatto points.

Our first goal is to prove the $2k$*-conjecture*.

THEOREM 4.1. *Let* $u\in H^{2k+1}(\Omega)$ *be the solution of* (1.1)*, and* u_h *be the solution of* (2.1)*. Then*

$$|(u-u_h)(P)| \lesssim h^{2k}|\ln h|^{\frac{1}{2}}\|u\|_{2k+1} \quad \forall P\in\mathcal{N}_h. \tag{4.1}$$

Proof. By [29], there hold

$$a_h(w,\Pi v)\lesssim \|w\|_1\|v\|_1,\ a_h(v,\Pi v)\gtrsim \|v\|_1^2, \quad w,v\in U_h. \tag{4.2}$$

For any $Q\in\Omega$, by the Lax–Milgram lemma, there exists $g_h\in U_h$ such that

$$a_h(v,\Pi g_h)=v(Q)\ \forall v\in U_h. \tag{4.3}$$

Choosing $v=g_h$, we have, from (4.2) and (4.3),

$$\|g_h\|_1^2\le |a_h(g_h,\Pi g_h)|=|g_h(Q)|\le\|g_h\|_\infty.$$

Since (cf. [31, p. 84, Theorem 2.8])

$$\|v\|_{0,\infty}\lesssim |\ln h|^{\frac{1}{2}}\|v\|_1, \quad v\in U_h,$$

we have

$$\|g_h\|_1\lesssim|\ln h|^{\frac{1}{2}}. \tag{4.4}$$

Letting $v=u_h-u_I-w_h\in U_h$ in (4.3) and using (3.2) and (4.4), we obtain

$$|(u_h-u_I-w_h)(Q)|=|a_h(u-u_I-w_h,\Pi g_h)|\lesssim h^{2k}|\ln h|^{\frac{1}{2}}\|u\|_{2k+1}. \tag{4.5}$$

Noticing that $w_h=0$ and $u_I=u$ at all nodes $P\in N_h$, then (4.1) follows. ∎

Remark 4.2. The inequality $||v||_{0,\infty}\le C|\ln h|^{\frac{1}{2}}\,||v||_1$ was first proved by Bramble in 1966, using the fundamental solution $w=\ln\sqrt{x^2+y^2}$ and Green's formula (see [11, p. 100]). Actually, by the imbedding inequality $||v||_{0,p}\le Cp^{1/2}||v||_1, 2\le p<\infty$, and the inverse estimate $||v||_{0,\infty}\le Ch^{-2/p}||v||_{0,p}$, taking $p=|\mathrm{lnh}|$, $h^{-2/p}=e^2$, we immediately have

$$||v||_{0,\infty}\le Ch^{-2/p}||v||_{0,p}\le C|\ln h|^{1/2}||v||_1.$$

We next discuss superconvergence of u_h at Gauss and Lobatto points.

THEOREM 4.3. *Let $u \in H^{k+3}(\Omega)$ be the solution of* (1.1)*, and u_h be the solution of* (2.1). *Then*

$$|(u-u_h)(P)| \lesssim h^{k+2}|\ln h|^{\frac{1}{2}}\|u\|_{k+3} \quad \forall P \in \mathcal{L}_h, \tag{4.6}$$

$$|\nabla(u-u_h)(Q)| \lesssim h^{k+1}|\ln h|^{\frac{1}{2}}\|u\|_{k+3} \quad \forall Q \in \mathcal{G}_h. \tag{4.7}$$

Proof. By (3.1)–(3.2) and (4.3), we have

$$\|u_I - u_h\|_{0,\infty} \lesssim h^{k+2}|\ln h|^{\frac{1}{2}}\|u\|_{k+3}.$$

By the inverse inequality,

$$|u_I - u_h|_{1,\infty} \lesssim h^{-1}\|u_I - u_h\|_{0,\infty} \lesssim h^{k+1}|\ln h|^{\frac{1}{2}}\|u\|_{k+3}.$$

On the other hand, by the definition of u_I, we have (see, e.g., [10, 31])

$$|(u-u_I)(P)| \lesssim h^{k+2}|u|_{k+2,\infty}, \quad |\nabla(u-u_I)(Q)| \lesssim h^{k+1}|u|_{k+2,\infty}, \quad P \in \mathcal{L}_h, Q \in \mathcal{G}_h.$$

The desired statements (4.6)–(4.7) then follow. ∎

Remark 4.4. As direct consequences of the above theorem, we have

$$|u_h - u_I|_1 \lesssim |u_h - u_I|_{1,\infty} \lesssim h^{k+1}|\ln h|^{\frac{1}{2}}\|u\|_{k+3},$$
$$\|u_I - u_h\|_0 \lesssim \|u_I - u_h\|_{0,\infty} \lesssim h^{k+2}|\ln h|^{\frac{1}{2}}\|u\|_{k+3}.$$

It was pointed out in [29] that the FV solution u_h is superclose to the Lobatto interpolation function $\tilde{u}_I$. The above inequalities clearly indicate the same for the interpolation function u_I, i.e., u_h is also superclose to u_I.

5. Numerical results. In this section, we present numerical examples to support our theoretical findings in the previous section.

We consider (1.1) with $\Omega = [0,1] \times [0,1]$ and the right-hand side

$$f(x,y) = [(5\pi^2 - 4y^2 - 3)\sin(\pi x)\sin(2\pi y) - 8\pi y \sin(\pi x)\cos(2\pi y) \\ - 2\pi\cos(\pi x)\sin(2\pi y)]e^{x-0.5+y^2}.$$

The exact solution is then

$$u(x,y) = \sin(\pi x)\sin(2\pi y)e^{x-0.5+y^2}, \ (x,y) \in \Omega.$$

We construct $\mathcal{T}_h$ with $h = 2^{-s}$, $s = 1, 2, \ldots, 8$, by dividing Ω into $h^{-1} \times h^{-1}$ squares, and solve (1.1) by the FV scheme (2.1) with $k = 3, 4$. For each h and k, we measure maximum errors at nodes, Lobatto points, and Gauss points (for gradient only), respectively. They are defined by

$$e_N = \max_{P \in \mathcal{N}_h} |(u-u_h)(P)|, \ \ e_L = \max_{P \in \mathcal{L}_h} |(u-u_h)(P)|, \ \ e_G = \max_{Q \in \mathcal{G}_h} |\nabla(u-u_h)(Q)|.$$

Numerical data are demonstrated in Table 1, and corresponding error curves are depicted in Figure 1 with log-log scale. We observe a convergence slope $k+1$ for e_G, $k+2$ for e_L, and $2k$ for e_N, respectively. These results confirm our theoretical findings in Theorems 4.1–4.3: The derivative error is superconvergent at all Gauss points and the function value error is superconvergent at all Lobatto points. Moreover, the approximation error at nodes converges with a rate h^{2k}, the $2k$-*conjecture* for our FV approximation is verified.

TABLE 1

	$k=3$			$k=4$		
N	e_G	e_L	e_N	e_G	e_L	e_N
2	2.699e-1	8.851e-3	1.327e-3	4.326e-2	2.044e-4	1.190e-5
4	2.897e-2	2.902e-4	2.761e-5	1.536e-3	5.354e-6	8.178e-8
8	2.224e-3	8.863e-6	5.743e-7	4.979e-5	1.092e-7	3.750e-10
16	1.660e-4	2.701e-7	1.056e-8	1.586e-6	2.397e-9	1.510e-12
32	1.117e-5	8.288e-9	1.919e-10	4.986e-8	4.340e-11	6.106e-15
64	7.222e-7	2.567e-10	3.309e-12	1.564e-9	7.257e-13	—

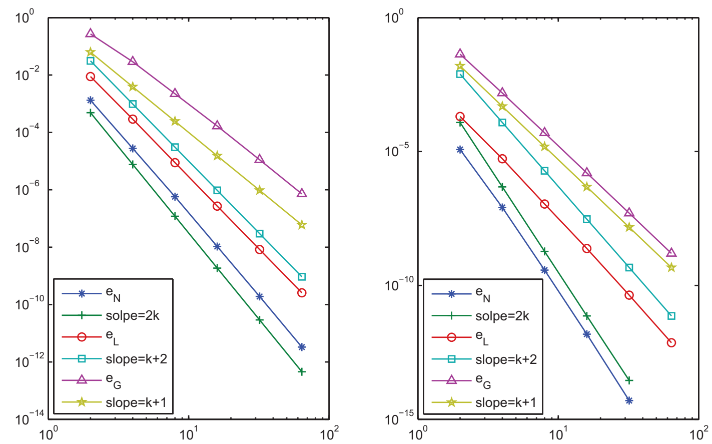

FIG. 1. *left:* $k = 3$, *right:* $k = 4$.

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