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Superconvergence property of an over-penalized discontinuous Galerkin finite element gradient recovery method

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ABSTRACT

A polynomial preserving recovery method is introduced for over-penalized symmetric interior penalty discontinuous Galerkin solutions to a quasi-linear elliptic problem. As a post-processing method, the polynomial preserving recovery is superconvergent for the linear and quadratic elements under specified meshes in the regular and chevron patterns, as well as general meshes satisfying *Condition* (ϵ, σ). By means of the averaging technique, we prove the polynomial preserving recovery method for averaged solutions is superconvergent, satisfying similar estimates as those for conforming finite element methods. We deduce superconvergence of the recovered gradient directly from discontinuous solutions and naturally construct an *a posteriori* error estimator. Consequently, the *a posteriori* error estimator based on the recovered gradient is asymptotically exact. Extensive numerical results consistent with our analysis are presented.

1. Introduction

In recent years there have been superconvergence results of the gradient and gradient recovery schemes [9,12,20,23,31–33,36–38], while their contributions are based on finite element approximations and attract many researchers from the fields of modern engineering and scientific computation. The Zienkiewicz–Zhu (ZZ) error estimator [38] is referred to the superconvergence patch recovery (SPR), which is based on gradient recovery from the gradient of the finite element solution on patches in the discrete least-squares fitting sense. The robustness of the ZZ patch recovery is originated from its superconvergence under structured meshes. As a new strategy, polynomial preserving recovery (PPR) has first been introduced by Zhang in [24,34] with the use of the fitted finite element solution values to recover the gradient. The PPR keeps all known superconvergence properties of the ZZ patch recovery, out-performing the SPR in the cases of quadratic element at edge centers and linear element for the chevron mesh [36]. The PPR has superconvergence in mildly structured grids as well as anisotropic grids [34,35]. Therefore, for gradient recovery to finite element solutions, the PPR method is a good alternative.

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It is well known that if the recovered quantity better approximates the exact one, then it can be used in constructing asymptotically exact *a posteriori* error estimates (see [1]). The PPR becomes standard in finite element methods and has been adopted in some commercial softwares (COMSOL etc.) as a superconvergence tool.

We consider the following second-order quasi-linear elliptic problem

$$\begin{cases} -\nabla \cdot (a(x,u)\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$. We assume $0 < a_1 \le a(x, v) \le a_2$, $x \in \bar{\Omega}$, $v \in \mathbb{R}$ for some positive constants a_1, a_2 and $a(x, v) \in C_b^2(\bar{\Omega} \times \mathbb{R})$, where $C_b^2(\bar{\Omega} \times \mathbb{R})$ is the space of twice continuously differentiable functions on $\mathbb{R}$ whose first and second order derivatives are bounded in $\bar{\Omega} \times \mathbb{R}$. It holds from [15] that there exists a unique weak solution u to (1) and $u \in C^{2+\delta}$ with $\delta \in (0, 1)$ when $f \in C^{\delta}(\Omega)$ and the boundary $\partial\Omega$ is smooth. The equation (1), supplemented with the homogeneous Dirichlet boundary condition, describes an equilibrium state of a chemical species of the concentration u in a porous medium with a source term $f(x)$.

Interior penalty discontinuous Galerkin (IPDG) methods are a powerful simulation tool for solving linear or nonlinear equations (see e.g. [4,11,14,17,18,22,25,28]). There are some primal DG versions belonged to IPDG methods (see [3,19]), such as symmetric interior penalty Galerkin (SIPG), nonsymmetric interior penalty Galerkin (NIPG), incomplete interior penalty Galerkin (IIPG) as well as its corresponding over-penalized interior penalty methods. We are interested in an over-penalized symmetric interior penalty Galerkin (OPSIPG) method [27,30] to realize a gradient recovery. One of reasons is that its penalty parameters can be bounded above rather than sufficiently large in the usual SIPG method for refined grids. The OPSIPG method we use preserves the integral terms on hybrid multiplication of jump or average of test and trial functions on interior edges. The weakly over-penalized symmetric interior penalty method (WOPSIP) presented by Brenner in [6] ignores the hybrid multiplication terms. The OPSIPG and WOPSIP methods produce an ill-conditioned discrete system, which results from the over-penalization terms. Fortunately, it can be remedied by a simple block-diagonal preconditioner (see [6]) and a multilevel preconditioner in [8]. Now the main question lies in how to implement the PPR technique into discontinuous Galerkin solutions under the framework of discontinuous Galerkin finite element methods.

In the present work, we aim to the PPR technique based on discontinuous Galerkin solutions and its theoretical analysis. The PPR technique is good for arbitrary order Lagrange finite elements, then for simplicity, we would focus on the linear and quadratic elements, which are widely used in practice. Several steps for the PPR are needed: we choose a patch including necessary or enough points first, and then by the fitted solution values recover the gradient, and further construct an *a posteriori* error estimate in the energy norm. Due to the PPR partial to the symmetry of patches, we shall consider some specified meshes in the regular and chevron patterns, as well as general meshes satisfying *Condition* (ϵ, σ). In case that the resultant DG system becomes ill-conditioned from over-penalized parameters, it is important to use a simple block-diagonal preconditioner remedying the problem. To the best of our knowledge, this is the first theoretical superconvergence proof for the PPR implemented on discontinuous Galerkin solutions to nonlinear elliptic problems. Furthermore, the proposed method can be used to solve time-dependent diffusion problems, e.g., the problem discussed in [13]. Our method can be applied to the spatial discretization part at each time level while maintaining the time discretization part unchanged.

The remainder of this paper is organized as follows. In Section 2 we introduce the over-penalized interior penalty discontinuous Galerkin (OPIPDG) formulas in the broken Sobolev space to approximate elliptic equations. In Section 3, we state and prove some preliminary lemmas for OPSIPG scheme analogous to those appear in the usual SIPG method. In Section 4, the gradient recovery operator will be constructed for OPSIPG solutions, thereafter we prove the main results for the gradient recovery, which can be used to define an *a posteriori* error estimator. In the last section, several numerical examples are given to illustrate superconvergence of the gradient recovery for linear and quadratic elements in some structured meshes as well as unstructured meshes, and also show that an *a posteriori* error estimator is asymptotically exact for a corner singularity problem.

2. The over-penalized discontinuous Galerkin method

Let $\mathcal{E}_h$ be a subdivision of Ω into disjoint open elements such that $\bar{\Omega} = \bigcup_{i=1}^{N_h} \bar{E}_i$, where E_i is a triangle in 2D and N_h is the number of all elements. We let $h_k := diam(\bar{E}_k)$ and $h := \max_{E \in \mathcal{E}_h} h_k$. It is assumed that the family of subdivisions $\mathcal{E}_h$ is shape regular and each element $E \in \mathcal{E}_h$ shall be an affine image of a standard reference element. Assume that the mesh $\mathcal{E}_h$ is quasi-uniform: for all $E_i \in \mathcal{E}_h$, there exists a constant $\tau > 0$, independent of h, such that

$$h \le \tau h_i, \tag{2}$$

where ρ_i denotes the diameter of the largest circle inscribed in E_i. We introduce the set of all edges of the mesh $\mathcal{E}_h$ by

$$\mathcal{F}_h := \{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\},$$

where

$$\begin{cases} e_i \subset \Omega, & \text{if } 1 \le i \le P_h, \\ e_i \subset \partial\Omega, & \text{if } P_h + 1 \le i \le M_h. \end{cases}$$

On each edge e_i $(1 \le i \le M_h)$ of $\mathcal{E}_h$, we fix a unit outer normal vector $\mathbf{n}_i$:

$$\begin{cases} \text{the unit normal vector on } e_i, \text{ pointing from } E_k \text{ to } E_j, \text{ if } e_i = \partial E_k \cap \partial E_j, \\ \text{the unit normal vector on } e_i, \text{ pointing outward of } \Omega, \text{ if } P_h + 1 \le i \le M_h, \end{cases}$$

then we denote the average on e_i and the jump across e_i below: For $v \in H^s(\mathcal{E}_h)$, $s > \frac{1}{2}$,

$$\{v\}_{e_i} := \begin{cases} \frac{1}{2}(v|_{E_j})|_{e_i} + \frac{1}{2}(v|_{E_k})|_{e_i}, \text{ if } e_i = \partial E_j \cap \partial E_k, \ 1 \le i \le P_h, \\ (v|_{E_k})|_{e_i}, \text{ if } e_i = \partial E_k \cap \partial \Omega, \ P_h + 1 \le i \le M_h, \end{cases}$$

$$[v]_{e_i} := \begin{cases} (v|_{E_k})|_{e_i} - (v|_{E_j})|_{e_i}, \text{ if } e_i = \partial E_j \cap \partial E_k, \ 1 \le i \le P_h, \\ (v|_{E_k})|_{e_i}, \text{ if } e_i = \partial E_k \cap \partial \Omega, \ P_h + 1 \le i \le M_h. \end{cases}$$

For brevity, we drop the subscript e_i of these two operators throughout this paper. And a definition for general meshes is given as follows.

Definition 2.1. Triangulation $\mathcal{E}_h$ satisfies *Condition* (ϵ, σ) if $\mathcal{E}_h$ can be separated into two parts

$$\mathcal{E}_h = \mathcal{E}_{0,h} \cup \mathcal{E}_{1,h}, \quad \bigcup_{K \in \mathcal{E}_{i,h}} \bar{K} = \bar{\Omega}_{i,h}, \quad \bar{\Omega} = \bar{\Omega}_{0,h} \cup \bar{\Omega}_{1,h},$$

such that the following conditions exist:

1. Any two triangles that share a common edge in $\Omega_{0,h}$ form a convex quadrilateral which is an ϵ-perturbation from a parallelogram.

2. $\Omega_{1,h}$ has a small measure: $|\Omega_{1,h}| = O(h^\sigma)$, $\sigma > 0$.

Throughout the paper, $W^{k,p}(\omega)$ will signify the usual Sobolev space on polygonal Lipschitz domain ω, of differentiability index k and integrability index p, equipped with the norm $\|\cdot\|_{W^{k,p}(\omega)}$ and seminorm $|\cdot|_{W^{k,p}(\omega)}$. When $p = 2$, we shall write $W^{k,2}(\omega) = H^k(\omega)$ and suppress the index p in the notation of the norm and seminorm, writing $\|\cdot\|_{k,\omega}$ and $|\cdot|_{k,\omega}$, respectively. Note that, by default, $H^0(\omega)$ denotes $L^2(\omega)$ with the L^2 inner product $(\cdot,\cdot)$ and the standard L^2-norm $\|\cdot\|_{0,\omega}$, or equivalently, $\|\cdot\|_{L^2(\omega)}$. We define as usual the standard L^∞-norm $\|\cdot\|_{\infty,\omega}$ on ω. Then we assign to the subdivision $\mathcal{E}_h$ the broken Sobolev space of real orders s:

$$H^s(\mathcal{E}_h) = \left\{ v \in L^2(\Omega) : \ v|_{E_i} \in H^s(E_i), \ 1 \le i \le N_h \right\},$$

equipped with the broken Sobolev space norm,

$$|||v|||_s := \left(\sum_{i=1}^{N_h} \|v\|_{s,E_i}^2 \right)^{1/2}.$$

We also introduce a space of test functions

$$\mathcal{D}_r(\mathcal{E}_h) = \left\{ v \in L^2(\Omega) : v|_{E_i} \in \mathbb{P}_r(E_i), \ 1 \le r \le 2, \ 1 \le i \le N_h \right\}, \tag{3}$$

where $\mathbb{P}_r(E_i)$ denotes the set on E_i of all polynomials of (total) degree at most r on E_i. It is clear that $\mathcal{D}_r(\mathcal{E}_h) \not\subset H^1(\Omega)$. We thus introduce the interior super penalty term $J_\beta^\sigma : \mathcal{D}_r(\mathcal{E}_h) \times \mathcal{D}_r(\mathcal{E}_h) \to \mathbb{R}$ in the form:

$$J_\beta^\sigma(v, w) = \sum_{k=1}^{M_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [v][w] ds, \qquad \beta > 1, \tag{4}$$

which penalizes the jump of the functions across the edges e_k, $1 \le k \le M_h$. Here the penalty parameter σ_k is a nonnegative real number to be chosen and $|e_k|$ is the Lebesgue measure of the edge e_k. We also define the energy norm on $\mathcal{D}_r(\mathcal{E}_h)$ throughout this paper:

$$\|v\|_{DG} = \left(\sum_{k=1}^{N_h} \|\nabla v\|_{0,E_k}^2 + J_\beta^\sigma(v, v) \right)^{1/2}, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{5}$$

Aiming to study the strong solution $u \in C^2(\bar{\Omega} \times [0, T])$, satisfying the regularity conditions, of the problem (1), we proceed element by element as appears in [27]. The weak formulation of the problem (1) reads as follows: Find $u \in H^s(\mathcal{E}_h)$, $s \ge 2$, such that

$$A_\lambda(u; u, v) = (f, v), \quad \forall v \in H^s(\mathcal{E}_h), \tag{6}$$

where $A_\lambda(\rho; v, w)$ is bilinear in the last two terms:

$$A_\lambda(\rho; v, w) = \sum_{j=1}^{N_h} \int_{E_j} a(\rho)\nabla v \cdot \nabla w dx - \sum_{k=1}^{M_h} \int_{e_k} \{a(\rho)\nabla v \cdot \mathbf{n}_k\}[w] ds$$
$$+ \lambda \sum_{k=1}^{M_h} \int_{e_k} \{a(\rho)\nabla w \cdot \mathbf{n}_k\}[v] ds + J_\beta^\sigma(v, w), \quad v, w \in H^s(\mathcal{E}_h). \tag{7}$$

The OPIPDG approximation of (1) is to find $u_h \in \mathcal{D}_r(\mathcal{E}_h)$ such that

$$A_\lambda(u_h; u_h, v_h) = (f, v_h), \quad \forall v_h \in \mathcal{D}_r(\mathcal{E}_h). \tag{8}$$

Moreover, depending on the coefficient λ, the discontinuous Galerkin method (8) is referred to OPSIPG if $\lambda = -1$; OPNIPG if $\lambda = 1$; or OPIIPG if $\lambda = 0$. In the rest of paper, we will use the parameter $\lambda = -1$ in A_λ. For the choice of penalty parameters σ_k of these discontinuous formulations, the reader is referred to Georgoulis and Süli [16], Rivière et al. [28], Song [29], and the references therein. By analogous arguments as in [18] and Brouwer's fixed theorem, the consistence of (6) and the existence and uniqueness of the OPSIPG scheme (8) can be shown.

3. Some estimates of the OPSIPG scheme

In this section, before embarking on gradient recovery and its superconvergence analysis, we state and prove some preliminary results of (8) with $\lambda = -1$. And we denote by C a generic positive constant. From (8), the OPSIPG approximation of u is to find $u_h \in \mathcal{D}_r(\mathcal{E}_h)$ such that

$$A_{-1}(u_h; u_h, v_h) = (f, v_h), \quad \forall v_h \in \mathcal{D}_r(\mathcal{E}_h). \tag{9}$$

There exists a positive constant $C_{\Omega,\tau}$ depending on Ω and τ in [29, Lemma 3.4] such that

$$\sum_{k=1}^{M_h} |e_k| \left\| \left\{ \frac{\partial v}{\partial \mathbf{n}_k} \right\} \right\|_{0,e_k}^2 \le C_{\Omega,\tau} |||\nabla v|||_0^2, \quad \forall v \in H^1(\mathcal{E}_h). \tag{10}$$

Analogously to the proof of Lemma 3.5 in [29], we note that A_{-1} is coercive. Indeed, if a positive constant δ is such that $\frac{a_1}{C_{\Omega,\tau}} > \delta > \min_k \left\{ \frac{a_2^2 |e_k|^{\beta-1}}{\sigma_k} \right\}$ and the penalty parameters σ_k satisfy $\sigma_k > C_{\Omega,\tau} \frac{a_2^2}{a_1} |e_k|^{\beta-1}$, then for any $v, \rho \in \mathcal{D}_r(\mathcal{E}_h)$,

$$A_{-1}(\rho; v, v) \ge \alpha_0 \Big(|||\nabla v|||_0^2 + J_0^\sigma(v, v) \Big), \tag{11}$$

where $\alpha_0 = \min_k \left\{ a_1 - C_{\Omega,\tau}\delta, 1 - \frac{a_2^2 |e_k|^{\beta-1}}{\delta \sigma_k} \right\}$. Thus, in OPSIPG scheme, the penalty parameters σ_k may be small quantities on edges, due to the ratio $\frac{a_2^2}{a_1}$ or $\frac{a_2^2}{\delta}$ multiplying with a small factor $|e_k|^{\beta-1}$. The choice of penalty parameters is significantly different from that in the usual SIPG scheme. For simplicity, we may take small constant penalty parameters for OPSIPG scheme. Based on the estimate (11), then we have the following lemma for (8).

Lemma 3.1. *Assume that* $\frac{a_1}{C_{\Omega,\tau}} > \delta > \min_k \left\{ \frac{a_2^2 |e_k|^{\beta-1}}{\sigma_k} \right\}$ *and* $\{\sigma_k\} > C_{\Omega,\tau} \frac{a_2^2}{a_1} |e_k|^{\beta-1}$, *then it holds that*

$$A_{-1}(\rho; v, v) \ge \alpha_0 \|v\|_{DG}^2, \quad \forall \rho, v \in \mathcal{D}_r(\mathcal{E}_h). \tag{12}$$

The following continuity lemma can be proven similarly as in [29].

Lemma 3.2. *There exists a constant* $\beta_0 > 0$ *such that*

$$A_{-1}(\rho; v, u) \le \beta_0 \|v\|_{DG} \|u\|_{DG}, \quad \forall \rho, v, u \in \mathcal{D}_r(\mathcal{E}_h). \tag{13}$$

By Brouwer's fixed point theorem, the existence of discrete solution of SIPG scheme has been shown in [18] for sufficiently small h, so does OPSIPG scheme. We need the following a priori error estimates for solutions of OPSIPG scheme (9) in the broken H^1 and L^2 norms, which can be derived analogously from the proofs in [18, Theorems 4.9 and 4.10] and [7, Theorem 2.6].

Lemma 3.3. *Let* $u \in W^{1,\infty}(\Omega) \cap H^{r+1}(\Omega)$ *and* $u_h \in \mathcal{D}_r(\mathcal{E}_h)$ *be the solutions of (6) and (9), respectively. For sufficiently small h and for any* $\beta \ge 1$, *there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{DG} \le Ch^r |||v|||_{r+1}, \qquad \|u - u_h\|_{L^2(\mathcal{E}_h)} \le Ch^{r+1} |||v|||_{r+1}. \tag{14}$$

The Aubin–Nitsche lifting technique is well suited to the analysis of the DG method for linear problems, since OPSIPG scheme is symmetric. But for the quasilinear elliptic equation, we will use the nonlinear elliptic projection $\pi_h : H^2(\Omega) \cap H_0^1(\Omega) \to \mathcal{D}_r(\mathcal{E}_h)$ of u defined by

$$A_{-1}(u; u, v_h) = A_{-1}(u; \pi_h u, v_h), \quad \forall v_h \in \mathcal{D}_r(\mathcal{E}_h). \tag{15}$$

For any $u \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$, the Lax–Milgram Theorem leads to the uniqueness and existence of $\pi_h u$ in the linear variational form (15). By a minor modification of Theorem 4.5 in [2], we recover an approximation property of $\pi_h u$ in the following lemma.

Lemma 3.4. *There exists a positive constant C independent of h such that*

$$\|u - \pi_h u\|_{L^2(\mathcal{E}_h)} + h\|u - \pi_h u\|_{DG} \le C(u)h^{r+1}, \tag{16}$$

for $u \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$.

We apply analogously the proof of superconvergence of an elliptic projection in [4, Theorem A.1] to the projection π_h in the broken H^1-norm, then the following lemma still holds.

Lemma 3.5. *Let* $u \in W^{r+1,\infty}(\Omega)$ *and* $u_h \in \mathcal{D}_r(\mathcal{E}_h)$ *be the solution of (6) and (8), respectively. For sufficiently small h, there exists a positive constant C independent of h such that*

$$\|u_h - \pi_h u\|_{DG} \le C(u)h^{r+1}. \tag{17}$$

Let V_h be a space of continuous piecewise polynomials of order r ($r = 1, 2$) associated with $\mathcal{E}_h$. Let z be any interior node of the Lagrange finite element space $\mathbb{P}_r$ associated with triangulation $\mathcal{E}_h$ and let $\mathcal{T}_z$ be the set of the triangles in $\mathcal{E}_h$ that share the node z in their closures. A linear map $\mathcal{R} : \mathcal{D}_r(\mathcal{E}_h) \to V_h$ can be constructed by averaging techniques [5,17], i.e., for $v \in \mathcal{D}_r(\mathcal{E}_h)$,

$$\mathcal{R}v = \frac{1}{|\mathcal{T}_z|} \sum_{E \in \mathcal{E}_h} v|_E(z), \tag{18}$$

where $|\mathcal{T}_z|$ is the cardinality of triangles in $\mathcal{T}_z$. For all boundary nodes z, we set $\mathcal{R}v(z) = 0$. For uniformly simplicially reducible meshes or quasiuniform quadrilateral meshes obtained by hierarchical refinement, the following result provides a link between discontinuous piecewise polynomial functions and functions in $H_0^1(\Omega)$ (see [25]).

Lemma 3.6. *There exists a constant C, independent of h, and a linear operator* $\mathcal{R} : \mathcal{D}_r(\mathcal{E}_h) \to H_0^1(\Omega)$ *such that, for all* $u_h \in \mathcal{D}_r(\mathcal{E}_h)$ *and* $s \in \{0, 1\}$,

$$\|\nabla^s(u_h - \mathcal{R}u_h)\|_{L^2(\Omega)} \le Ch^{\frac{1}{2}-s}\Big(\sum_{e \in \Gamma_{int} \cup \partial\Omega} \|[u_h]\|_{L^2(e)}^2\Big)^{\frac{1}{2}}, \tag{19}$$

where $\nabla^0 = id$ *and* $\nabla^1 = \nabla$.

4. The gradient recovery operator

4.1. Construction of DG gradient recovery and its primal properties

We introduce a gradient recovery operator $G_h : \mathcal{D}_r(\mathcal{T}_h) \to V_h \times V_h$. For any r-th order finite elements, all we need is to define $G_h u_h$ at each node z_i of the triangulation $\mathcal{E}_h$:

$$G_h u_h(z_i) = \sum_j \mathbf{C}_{ij} u_h(z_{ij}), \quad \sum_j \mathbf{C}_{ij} = 0,$$

where $\mathbf{C}_{ij}$ are coefficients of some finite difference schemes and $\{z_{ij}\}$ are all local points on a patch of elements around z_i. In some special situations, we are referred to [36] for the choices of $\mathbf{C}_{ij}$. It means that the recovered gradient at z_i can be regarded as a linear combination of neighbor values of discontinuous finite element solutions [30].

First, we present definition of the PPR for discontinuous approximations. By $\mathcal{N}_h$ we denote the set of all interior nodes in $\mathcal{E}_h$. Let $z \in \mathcal{N}_h$ be a mesh vertex and let $\mathcal{E}_{h,z}$ denote a patch of mesh elements around z. Set $p_z \in \mathbb{P}_{r+1}$ be the polynomial that best fits discontinuous solutions u_h at the mesh nodes in $\mathcal{E}_{h,z}$ in the local discrete least-squares sense:

$$\sum_{z_i \in \mathcal{N}_h \cap \mathcal{E}_{h,z}} |(u_h - p_z)(z_i)|^2 = \min_{p \in \mathbb{P}_{r+1}(\mathcal{E}_{h,z})} \sum_{z_i \in \mathcal{N}_h \cap \mathcal{E}_{h,z}} |(u_h - p)(z_i)|^2. \tag{20}$$

Here p_z is called the least-squares polynomial approximation (LSPA) of u_h at z. Then it is well defined by

$$(G_h u_h)(z) \triangleq \nabla p_z(z).$$

We denote by N_z the number of mesh nodes in the patch $\mathcal{E}_{h,z}$. For an internal mesh vertex z and an order r, there are N_z ($\geq m := \frac{(r+2)(r+3)}{2}$) points required in an element patch $\mathcal{E}_{h,z}$ including the mesh vertex z itself. To fit a polynomial of degree $r+1$, in the least-squares sense, we select points distributed around z on the ball $B_h(z) = \{x \in \mathcal{E}_h : |x - z| \leq h\}$. If the number of points (including z) is less than m, we search further points and proceed this process on a larger circle until more than or identical to m points being chosen. Then the patch $\mathcal{E}_{h,z}$ is well defined and must have at least m points distributed around z in a way that leads to a unique p_z. Next, to define $\mathcal{E}_{h,z}$ at a boundary mesh vertex z, we set

$$\mathcal{E}_{h,z} \triangleq \mathcal{E}_{h,z_0} \cup \mathcal{L}_{z,n_0}, \tag{21}$$

where z_0 is the closest internal vertex to z and $\mathcal{L}_{z,n_0}$ is the union of mesh elements in the first $n_0 \in \mathbb{Z}^+$ layers around z including the internal mesh vertex z_0. This definition ensures the uniqueness of p_z as shown in Lemma 3.6 of [36].

Let h_z be the length of the longest edge attached to z. Taking the local coordinates (x, y) with z as the origin, the fitting polynomial is

$$p_{r+1}(x, y; z) = \mathbf{P}^T\mathbf{a} = (1, x, y, \cdots, x^{r+1}, x^r y, x^{r-1}y^2, \cdots, y^{r+1})\mathbf{a}$$

with $\mathbf{a}^T = (a_1, a_2, \cdots, a_m)$. With a scaling augment by $h = h_i$, set

$$\hat{\mathbf{P}}^T = (1, \xi, \eta, \cdots, \xi^{r+1}, \xi^r\eta, \xi^{r-1}\eta^2, \cdots, \eta^{r+1}),$$

the fitting polynomial becomes

$$\hat{p}_{r+1}(\xi, \eta) = \hat{\mathbf{P}}^T\hat{\mathbf{a}},$$

where $\hat{\mathbf{a}}^T = (a_1, ha_2, ha_3, \cdots, h^{k+1}a_m)$. The coefficient vector $\hat{\mathbf{a}}$ is determined by the linear system

$$A^T A\hat{\mathbf{a}} = A^T\mathbf{b}, \tag{22}$$

where $\mathbf{b}^T = (u_h(z_{i1}), u_h(z_{i2}), \cdots, u_h(z_{in}))$ and

$$A = \begin{pmatrix} 1 & \xi_1 & \eta_1 & \cdots & \eta_1^{r+1} \\ 1 & \xi_2 & \eta_2 & \cdots & \eta_2^{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_n & \eta_n & \cdots & \eta_n^{r+1} \end{pmatrix}.$$

The uniqueness condition for the linear system (22) is $\text{Rank}(A) = m$, so it holds when $n \leq m$ and all the n sampling points are not on the same conic curve for the linear element [24]. For linear or quadratic discontinuous/continuous finite elements in the regular and chevron patterns, the reader is referred to [30,36] about the weights of recovery gradient on vertices, edge nodes and internal nodes for more details.

Especially, if one takes the average of the discontinuous Galerkin solution at each node, i.e., for any a node $z \in \mathcal{N}_h$, the average $\bar{u}_h(z) := \mathcal{R}u_h(z)$, then $G_h\bar{u}_h(z)$ is the same as in [36]. It is well known that G_h has some important properties:

(1) $\|G_h\bar{u}_h\|_{L^2(\tau)} \leq C\|\nabla\bar{u}_h\|_{L^2(\omega_\tau)}$, $\forall \bar{u}_h \in V_h$;
(2) if $p \in \mathbb{P}_{r+1}(\omega_{z_i})$, then $(G_h p_I)(z) = \nabla p(z)$, $\forall z \in \mathcal{N}_h$,

where $p_I \in V_h$ is the Lagrangian interpolation of p, ω_z is an element patch associated with node z and ω_τ denotes the union of element patches associated with three vertices of element τ. We have

$$\|\nabla u - G_h u_I\|_{L^2(\Omega)} \leq Ch^{r+1}|u|_{r+2,\Omega}, \qquad r = 1, 2. \tag{23}$$

We need the following boundedness assumption on G_h: When there are no two adjacent angles on an element patch adding up to exceed π, it is assumed that

$$\|G_h v\|_{L^2(\mathcal{E}_h)} \lesssim \|v\|_{DG}, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{24}$$

Remark 1. As a post-processing technique, PPR has been implemented by COMSOL Multiphysics [39,40] for any order finite elements. Therefore, PPR for higher-order elements can also be applied to problems with corner singularities similar to what we have done for linear element. However, theoretical analysis for higher-order elements would be much involved.

The difference between $G_h u_h$ and $G_h\bar{u}_h$ has been presented in the following lemma.

Lemma 4.1. *Consider an interior patch $\omega_z \subset\subset \Omega_{0,h} \subset \Omega$. It holds that*

$$\|G_h(u_h - \bar{u}_h)\|^2_{W^{-r,q}(\Omega_{0,h})} \lesssim |\ln h| h^{r+\beta-1}\|u\|^2_{W^{r+1,\infty}(\Omega)}. \tag{25}$$

Proof. Setting ω_z define a subdomain including a patch with a node z as the center and a vector function $\mathbf{w} \in C_0^\infty(\omega_z)^2$ with $\|\mathbf{w}\|_{H^1(\omega_z)} = 1$, we have

$$\begin{aligned}(G_h(u_h - \bar{u}_h), \mathbf{w}) &= (u_h - \bar{u}_h, G_h^* \mathbf{w}) \\ &\lesssim \|u_h - \bar{u}_h\|_{\infty,(\omega_z+h)} \|G_h^* \mathbf{w}\|_{L^1(\omega_z+h)} \\ &\lesssim \|u_h - \bar{u}_h\|_{\infty,(\omega_z+h)},\end{aligned} \tag{26}$$

where $\omega_z + h$ defines a subdomain that stretches out h from ω_z and $\|G_h^*\mathbf{w}\|_{L^1(\omega_z+h)}$ and $\|\mathbf{w}\|_{H^1(\omega_z+h)}$ are bounded uniformly with respect to h. From the discrete Sobolev inequality [5] for piecewise polynomial functions w_h with respect to $\mathcal{E}_h$, we have

$$\|w_h\|^2_{\infty,\Omega} \le (1 + |\ln h|)\Big(\sum_{E\in\mathcal{E}_h} \|w_h\|^2_{H^1(E)} + \sum_{e\in\Gamma_{int}} |e|^{-1} \|[w_h]\|^2_{L^2(e)}\Big).$$

Taking $w_h = u_h - \bar{u}_h$ leads to

$$\|u_h - \bar{u}_h\|^2_{\infty,\Omega} \le (1 + |\ln h|)\Big(\sum_{E\in\mathcal{E}_h} \|u_h - \bar{u}_h\|^2_{H^1(E)} + \sum_{e\in\Gamma_{int}} |e|^{-1} \|[u_h]\|^2_{L^2(e)}\Big),$$

which gives by Lemma 3.6 and the continuity of u

$$\begin{aligned}\|u_h - \bar{u}_h\|^2_{\infty,\Omega} &\lesssim |\ln h| \int_{\Gamma_{int}\cup\partial\Omega} h^{-1} |[u_h]|^2 ds \\ &= |\ln h| \int_{\Gamma_{int}\cup\partial\Omega} h^{-1} |[u - u_h]|^2 ds \\ &\lesssim |\ln h| h^{\beta-1} \|u - u_h\|^2_{DG} \\ &\lesssim |\ln h| h^{r+\beta-1} \|u\|^2_{W^{r+1,\infty}(\Omega)}. \qquad \square\end{aligned} \tag{27}$$

4.2. The main results for the DG gradient recovery

The analysis of averaging techniques plays an important role in a posteriori error control in [10]. Due to its importance, we shall first prove the superconvergence between the gradient and the recovery gradient from the averaged values of OPSIPG solutions under reasonable smoothness conditions. We decompose

$$\nabla u - G_h \bar{u}_h = (\nabla u - G_h u_I) + G_h(u_I - \bar{u}_h), \tag{28}$$

where $u_I \in V_h$ is the interpolation of u and $V_h \subset H^1(\Omega)$ represents the C^0 finite element space consisting of piecewise polynomials of order r on $\mathcal{E}_h$.

First, applying (23) to the first term on the RHS of (28) gives

$$\|\nabla u - G_h u_I\|_{L^2(\Omega)} \lesssim h^{r+1} |u|_{r+2,\Omega}. \tag{29}$$

To bound the second term on the right hand side of (28), it follows from the triangle inequality

$$\|G_h(u_I - \bar{u}_h)\|_{L^2(\mathcal{E}_h)} \lesssim \|\nabla(u_I - \pi_h u)\|_{L^2(\mathcal{E}_h)} + \|\nabla(\pi_h u - \bar{u}_h)\|_{L^2(\mathcal{E}_h)}, \tag{30}$$

where the boundedness of G_h has been used. To estimate the RHS terms of (30), we will apply the following critical lemma.

Lemma 4.2. *Let $u \in W^{r+1,\infty}(\Omega) \cap H^{r+2}(\Omega) \cap H_0^1(\Omega)$ and u_I be the solution of (6) and the Lagrange interpolation of u in V_h, respectively. Assume that $\mathcal{E}_h$ satisfies* Condition (ϵ, σ), *the following estimates hold.*

(1) If $r = 1$, it holds that

$$\sum_{j=1}^{N_h} \int_{E_j} \nabla e_I \cdot \nabla v_h dx \lesssim E_\beta^1 \|v_h\|_{DG}, \qquad \forall\, v_h \in \mathcal{D}_1(\mathcal{E}_h)$$

where

$$\begin{aligned}E_\beta^1 :=& h^{3/2}(\|u\|_{H^3(\Omega_{0,h})} + \epsilon) + h\epsilon(\|u\|_{H^2(\Omega_{0,h})} + \epsilon) + h^{1+\sigma/2}|u|_{W^{2,\infty}(\Omega)} \\ &+ h^{\frac{\beta+1}{2}} \|u\|_{W^{2,\infty}(\Omega)}.\end{aligned}$$

(2) If $r = 2$, it holds that

$$\sum_{j=1}^{N_h}\int_{E_j}\nabla e_I\cdot\nabla v_h dx \lesssim E_\beta^2\|v_h\|_{DG}, \qquad \forall v_h\in\mathcal{D}_2(\mathcal{E}_h)$$

where

$$E_\beta^2 := h^{5/2}(\|u\|_{H^4(\Omega_{0,h})}+\epsilon)+h^2\epsilon(\|u\|_{H^3(\Omega_{0,h})}+\epsilon)+h^{2+\sigma/2}|u|_{W^{3,\infty}(\Omega)} + h^{\frac{\beta+3}{2}}\|u\|_{W^{3,\infty}(\Omega)}.$$

Proof. Let Q_h be the L^2 projection onto V_h such that

$$(w_h, Q_h v_h) = (w_h, v_h), \quad v_h\in\mathcal{D}_r(\mathcal{E}_h),\ w_h\in V_h.$$

Due to $\|v_h - Q_h v_h\|_{L^2(\mathcal{E}_h)} \le \|v_h - \mathcal{R}v_h\|_{L^2(\mathcal{E}_h)}$, it holds from Lemma 3.6

$$\|v_h - Q_h v_h\|_{L^2(\mathcal{E}_h)} \lesssim h^{\frac{1}{2}}\Big(\sum_{j=1}^{M_h}\|[v_h]\|^2_{L^2(e_j)}\Big)^{\frac{1}{2}}, \tag{31}$$

and analogously, by the inverse inequality,

$$\|\nabla(v_h - Q_h v_h)\|_{L^2(\mathcal{E}_h)} \lesssim h^{-\frac{1}{2}}\Big(\sum_{j=1}^{M_h}\|[v_h]\|^2_{L^2(e_j)}\Big)^{\frac{1}{2}}. \tag{32}$$

Notice that $v_h\in\mathcal{D}_r(\mathcal{E}_h)$ does not apply since it is discontinuous and is not included in the continuous finite element space V_h. It is useful and applicable to analyze the error of the L^2 projection by a decomposition of v_h and the inverse inequality. We have

$$\begin{aligned}
&\sum_{j=1}^{N_h}\int_{E_j}\nabla(u_I-u)\cdot\nabla v_h dx\\
&\le \sum_{j=1}^{N_h}\int_{E_j}\nabla(u_I-u)\cdot Q_h(\nabla v_h)dx\\
&\quad + \sum_{j=1}^{N_h}\int_{E_j}\nabla(u_I-u)\cdot(\nabla v_h - Q_h(\nabla v_h))dx\\
&\lesssim \sum_{j=1}^{N_h}\int_{E_j}\nabla(u_I-u)\cdot Q_h(\nabla v_h)dx\\
&\quad + h^{r+\frac{\beta-1}{2}}\|u\|_{W^{r+1,\infty}(\Omega)}\Big(\sum_{j=1}^{M_h}\frac{\sigma_0}{|e_j|^\beta}\|[v_h]\|^2_{L^2(e_j)}\Big)^{\frac{1}{2}},
\end{aligned} \tag{33}$$

where for the last inequality we have used (32). According to the analysis in [21,35], we can directly use the well-known superconvergence results to estimate the first term in the right-hand side of (33). The case of linear elements ($r = 1$) employs these conclusions [35, (2.10) and (2.12)], while the results [21, (2.5) and (2.7)] have been applied to the case of quadratic elements ($r = 2$). Therefore, inserting these known results to (33) completes the proof of this lemma. □

Theorem 4.3. *For $\lambda = -1$, let $u\in W^{r+1,\infty}(\Omega)\cap H^{r+2}(\Omega)\cap H_0^1(\Omega)$ and u_h be the solution of (6) and (8), respectively. Assume that $\bar{u}_h$ is a continuous approximation of u_h, i.e., $\bar{u}_h = \mathcal{R}u_h$ and $\mathcal{E}_h$ satisfies Condition (ϵ,σ), then there is a constant C independent of h and r such that*

$$\|\nabla u - G_h\bar{u}_h\|_{L^2(\Omega)} \le C\big(h^{r+1}\|u\|_{H^{r+2}(\Omega)} + h^{(r+\frac{\beta-1}{2})}\|u\|_{H^{r+1}(\Omega)} + E_\beta^r\big), \tag{34}$$

where the terms E_β^r are specified as follows.

(1) If $r = 1$, we have

$$E_\beta^1 = h^{3/2}(\|u\|_{H^3(\Omega)} + \epsilon) + h\epsilon(\|u\|_{H^2(\Omega_{0,h})} + \epsilon) + h^{1+\sigma/2}|u|_{W^{2,\infty}(\Omega)} + h^{\frac{\beta+1}{2}}\|u\|_{W^{2,\infty}(\Omega)}.$$

(2) If $r = 2$, we have

$$E_\beta^2 = h^{5/2}(\|u\|_{H^4(\Omega_{0,h})} + \epsilon) + h^2\epsilon(\|u\|_{H^3(\Omega)} + \epsilon) + h^{2+\sigma/2}|u|_{W^{3,\infty}(\Omega)} + h^{\frac{\beta+3}{2}}\|u\|_{W^{3,\infty}(\Omega)}.$$

Proof. By the above boundedness property (1) of G_h and (17), under the assumption that $u \in W^{r+2,\infty} \cap H_0^1(\Omega)$, we have

$$\|\nabla(u_I - \pi_h u)\|_{L^2(\mathcal{E}_h)} = \sum_{E\in\mathcal{E}_h} \|\nabla(u_I - \pi_h u)\|_{L^2(E)} \le \|u_I - \pi_h u\|_{DG}. \tag{35}$$

Note that

$$\begin{aligned} &A_{-1}(u; u_I, v_h) - A_{-1}(u; \pi_h u, v_h) \\ &= A_{-1}(u; u_I - \pi_h u, v_h) \\ &= A_{-1}(u; u_I - u, v_h), \quad \forall v_h \in \mathcal{D}_r(\mathcal{E}_h), \end{aligned} \tag{36}$$

where we have used the definition of π_h for the second identity and Lemma 3.2 for the last inequality. Moreover, applying Lemma 3.1 in (36) with $v_h = u_I - \pi_h u$ taken also gives

$$A_{-1}(u; u_I - \pi_h u, v_h) \ge \alpha_0 \|u_I - \pi_h u\|_{DG}\|v_h\|_{DG}, \tag{37}$$

therefore, it follows from (36) and (37)

$$\|u_I - \pi_h u\|_{DG} \le \frac{A_{-1}(u; u_I - u, v_h)}{\alpha_0\|v_h\|_{DG}}. \tag{38}$$

Let $\overline{a(u)}|_E$ be a piecewise constant function $\overline{a(u)} = \frac{1}{|E|}\int_E a(x,u)dx$ on each element $E \in \mathcal{E}_h$. Then we have $0 < a_1 \le \overline{a(u)} \le a_2$ on E. It follows from the assumption of $a(\cdot)$ that

$$|a(u) - \overline{a(u)}| \le Ch. \tag{39}$$

We define $\bar{A}_{-1}(u; u_I - u, v_h)$ replacing the term $a(u)$ by $\overline{a(u)}$ in $A_{-1}(u; u_I - u, v_h)$, then it is easy to show that

$$|A_{-1}(u; u_I - u, v_h) - \bar{A}_{-1}(u; u_I - u, v_h)| \lesssim h\|u_I - u\|_{1,\Omega}\|v_h\|_{DG}. \tag{40}$$

Therefore, we shift our analysis to $\bar{A}_{-1}(u; u_I - u, v_h)$. According to the formula of $A_{-1}(u; v, w)$, we can write

$$\begin{aligned} &\bar{A}_{-1}(u; u_I - u, v_h) \\ &= \sum_{j=1}^{N_h}\int_{E_j} \overline{a(u)}\nabla(u_I - u)\cdot\nabla v_h dx - \sum_{k=1}^{M_h}\int_{e_k}\{\overline{a(u)}\nabla(u_I - u)\cdot\mathbf{n}_k\}[v_h]ds \\ &:= I_1 + I_2. \end{aligned} \tag{41}$$

By the trace lemma and Cauchy–Schwarz inequality, I_2 is bounded

$$\begin{aligned} I_2 &\le \left(\frac{C}{\sigma_0}\sum_{k=1}^{N_h}|e_k|^\beta h_k^{2r-1}\|u\|^2_{H^{r+1}(E_k)}\right)^{\frac{1}{2}} J_\beta^\sigma(v_h, v_h)^{\frac{1}{2}} \\ &\le \left(\frac{C}{\sigma_0}\sum_{k=1}^{N_h}h^{2r-1+\beta}\|u\|^2_{H^{r+1}(E_k)}\right)^{\frac{1}{2}} J_\beta^\sigma(v_h, v_h)^{\frac{1}{2}} \\ &\lesssim \frac{h^{r+\frac{\beta-1}{2}}}{\sigma_0^{\frac{1}{2}}}\|u\|_{H^{r+1}(\Omega)} J_\beta^\sigma(v_h, v_h)^{\frac{1}{2}}, \end{aligned} \tag{42}$$

where σ_0 means the maximum penalty parameter of $\{\sigma_k\}_{k=1}^{M_h}$. Taking $\beta > 1$ in (38) for $r = 1$, or 2, applying Lemma 4.2 to the term I_1 and using (41)–(42), we get the following result for $\|u_I - \pi_h u\|_{DG}$:

$$\|u_I - \pi_h u\|_{DG} \lesssim E_\beta^r + \left(h^{r+1} + h^{r+\frac{\beta-1}{2}}\right)\|u\|_{H^{r+1}(\Omega)}. \tag{43}$$

Here, the superconvergence of the error between u_I and $\pi_h u$ depends on β. Note that (43) shows superconvergence between u_I and $\pi_h u$.

Next, we bound the second term on the right hand side of (30). This term can be decomposed into the two parts $\|\nabla(\pi_h u - u_h)\|_{L^2(\mathcal{E}_h)}$ and $\|\nabla(u_h - \bar{u}_h)\|_{L^2(\mathcal{E}_h)}$. For the first part, we have

$$\|\nabla(\pi_h u - u_h)\|_{L^2(\mathcal{E}_h)} \le \|\pi_h u - u_h\|_{DG} \lesssim h^{r+1}\|u\|_{H^{r+1}(\Omega)}, \tag{44}$$

where Lemma 3.5 has been used for the last inequality. For the second part $\|\nabla(u_h - \bar{u}_h)\|_{L^2(\mathcal{E}_h)}$, it is observed from Lemma 3.6 that

$$\begin{aligned}\|\nabla(u_h - \bar{u}_h)\|_{L^2(\mathcal{E}_h)} &\lesssim \Big(\sum_{e\in\Gamma_{int}} |e|^{-1}\|[u_h]\|^2_{L^2(e)}\Big)^{1/2} \lesssim \frac{h^{\frac{\beta-1}{2}}}{\sigma_0^{\frac{1}{2}}} J^{\sigma}_{\beta}(u_h, u_h)^{\frac{1}{2}} \\ &= \frac{h^{\frac{\beta-1}{2}}}{\sigma_0^{\frac{1}{2}}} J^{\sigma}_{\beta}(u - u_h, u - u_h)^{\frac{1}{2}} \lesssim \frac{h^{\frac{\beta-1}{2}}}{\sigma_0^{\frac{1}{2}}}\|u - u_h\|_{DG}.\end{aligned} \tag{45}$$

From (44)–(45) we bound

$$\|\nabla(\pi_h u - \bar{u}_h)\|_{L^2(\mathcal{E}_h)} \lesssim h^{r+1}\|u\|_{H^{r+1}(\Omega)} + \frac{h^{\frac{\beta-1}{2}}}{\sigma_0^{\frac{1}{2}}}\|u - u_h\|_{DG}. \tag{46}$$

Finally, applying Lemma 4.2 and inserting (29)–(30), (35), (43) and (46) into (28), we obtain (34) and complete the proof. □

Remark 2. From Theorem 4.3, the superconvergence result holds under the conditions that $\beta > 1$, $\sigma > 0$ and $\epsilon =: O(h^{\alpha})$, for any $\alpha > 0$. By the definition of *Condition* (ϵ, σ), a convex quadrilateral formed by any two triangles at most has a perturbation of $O(h^{\alpha})$ from a parallelogram in $\Omega_{0,h}$. The requirement $\beta > 1$ means over-penalized DG methods to be a good alternative. However, the theoretical proof above cannot be applied directly to the corresponding SIPG scheme with $\beta = 1$, which is left behind in the near future as an open problem.

Further, we derive the superconvergence between the gradient and the recovery gradient from OPSIPG solutions under suitable conditions. As shown in the proof of Theorem 4.3, some above estimates are very useful to prove superconvergence of $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$ for the DG gradient recovery $G_h u_h$. Decompose the term $\nabla u - G_h u_h$ by

$$\nabla u - G_h u_h = (\nabla u - G_h u_I) + G_h(u_I - \pi_h u) + G_h(\pi_h u - u_h).$$

Then with the use of (24), we get the following main superconvergence theorem from (29), (43) and Lemma 3.5.

Theorem 4.4. *For $\lambda = -1$, let $u \in W^{r+1,\infty}(\Omega) \cap H^{r+2}(\Omega) \cap H^1_0(\Omega)$ and u_h be the solution of (6) and (8), respectively. Assume that $\mathcal{E}_h$ satisfies Condition (ϵ, σ) with $\sigma > 0$ and $\epsilon = O(h^{\alpha})$ $(\forall \alpha > 0)$, then there is a constant C independent of h and r such that*

$$\|\nabla u - G_h u_h\|_{L^2(\Omega)} \le C\big(h^{r+1}\|u\|_{H^{r+2}(\Omega)} + h^{(r+\frac{\beta-1}{2})}\|u\|_{H^{r+1}(\Omega)} + E^r_{\beta}\big), \tag{47}$$

where E^r_{β} $(r = 1, 2)$ are specified as in Theorem 4.3 and $\beta > 1$.

4.3. An a posteriori error estimator

We naturally define the global error estimator for discontinuous numerical solutions by

$$\Pi_h := \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}. \tag{48}$$

Without a loss of generality, the small quantity ϵ in *Condition* (ϵ, σ) is assumed to be of $O(h^{\alpha})$ with $\alpha > 0$. Then we get the following asymptotically exact result motivated by Theorem 5.1 in [33].

Theorem 4.5. *Under the assumptions of Theorem 4.4, and if there exists a constant $C(u) > 0$ such that*

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \ge C(u)h^r, \quad r = 1, 2. \tag{49}$$

Then it holds

$$\left|\frac{\Pi_h}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} - 1\right| \lesssim h^{\rho}, \quad \rho = \min\left\{\frac{1}{2}, \frac{\sigma}{2}, \alpha, \frac{\beta - 1}{2}\right\}. \tag{50}$$

Proof. Applying Theorem 4.4 and (49) completes the proof. □

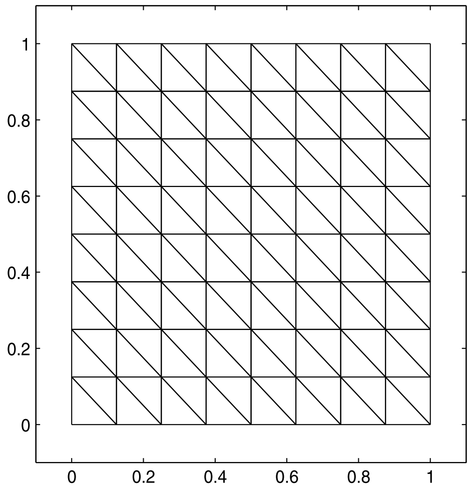

Fig. 1. An initial mesh in the regular pattern.

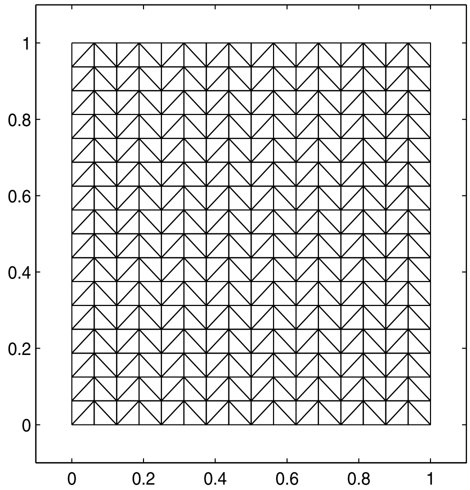

Fig. 2. An initial mesh in the chevron pattern.

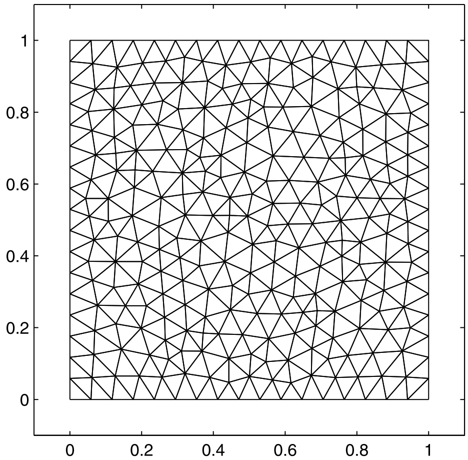

Fig. 3. An unstructured mesh.

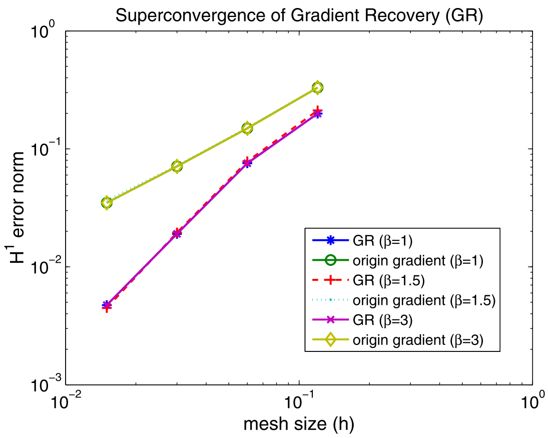

Fig. 4. A comparison of relative errors and GR in the H^1-seminorm, by linear elements in unstructured meshes, for *Case 2*.

5. Numerical tests

In this section, we carry out a series of numerical experiments to validate our code and to illustrate superconvergence of the DG gradient recovery method. We shall focus on the superiority of the recovery method over the OPSIPG method by comparing the two facets: on the one hand, a linear DG finite element on uniform triangular meshes of the regular and chevron patterns (see Figs. 1–2), as well as on unstructured mesh (see Fig. 3); on the other hand, a quadratic element on the uniform triangular mesh of the regular and chevron patterns. Moreover, we will consider a milestone problem with a corner singularity to verify if the *a posteriori* error estimator Π_h is asymptotically exact. As a post-processing technique, the PPR can be applied to any unstructured meshes. However, theoretical results are only available for very limited structured cases.

Case 1. We consider a nonlinear elliptic equation with a homogeneous boundary condition on the unit square $[0,1]^2$ satisfying $a(u) = u^2 + 1$ and the exact solution

$$u(x, y) = x(1-x)y(1-y). \tag{51}$$

Case 2. The second example to be used is the nonlinear elliptic equation with $a(u) = u^2 + 1$ and a homogeneous boundary condition on the unit square $\Omega = [0,1]^2$. The exact solution is chosen to be

$$u(x, y) = \sin(\pi x)\sin(\pi y). \tag{52}$$

We define $\|\nabla \cdot\|_{L^2(\Omega_{in})}$ as a discrete H^1 semi-norm in an interior region Ω_{in}. An initial mesh for the linear elements in *Cases 1 and 2* is produced by dividing the unit square into 2×2 uniform squares and only decomposing each subsquare into two triangles in the regular and chevron patterns. Then we intend to perform the numerical computation on multi-level meshes based on a similar process of uniform refinement, which preserves many advantages of initial meshes.

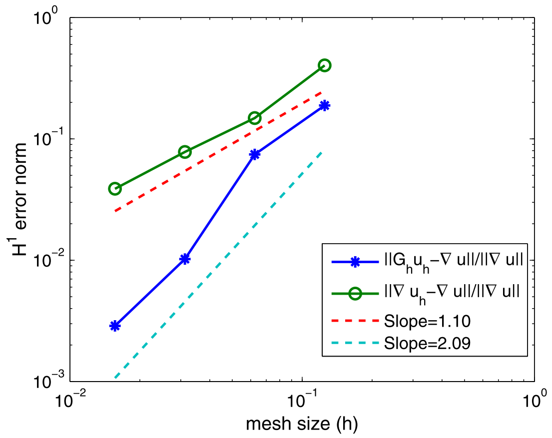

Fig. 5. Linear elements in the regular pattern, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}/\|\nabla u\|_{L^2(\Omega_{in})}$, $\Omega_{in} = [0.1, 0.9]^2$ for *Case 1*.

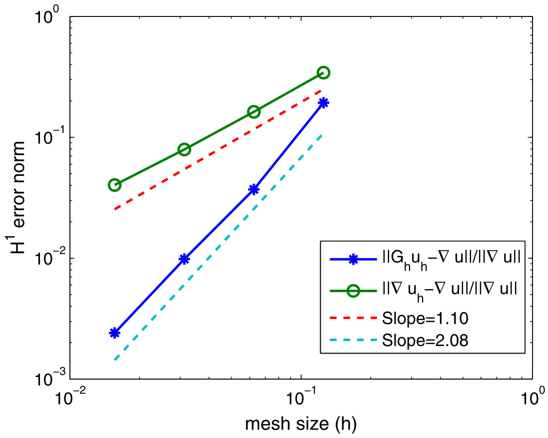

Fig. 6. Linear elements in the regular pattern, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}/\|\nabla u\|_{L^2(\Omega_{in})}$, $\Omega_{in} = [0.15, 0.85]^2$ for *Case 2*.

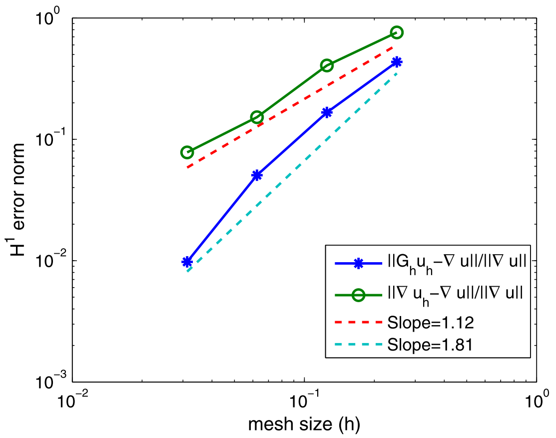

Fig. 7. Linear elements in the chevron pattern, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}/\|\nabla u\|_{L^2(\Omega_{in})}$ for *Case 1*.

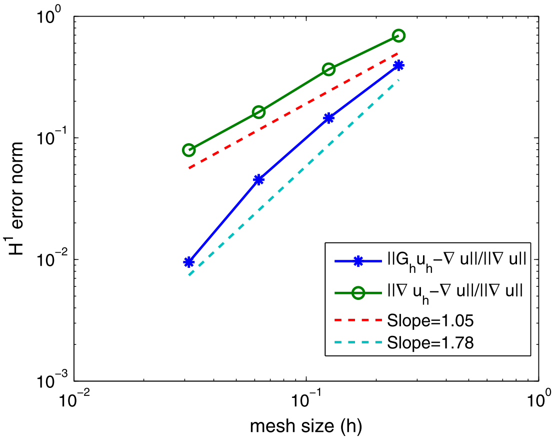

Fig. 8. Linear elements in the chevron pattern, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}/\|\nabla u\|_{L^2(\Omega_{in})}$ for *Case 2*.

First, we present a numerical comparison on H^1-seminorm relative errors in Fig. 4 for OPSIPG scheme ($\beta > 1$), including SIPG scheme ($\beta = 1$), with different values of β and the same penalty parameters σ_k taken. The illustrations imply that the SIPG and OPSIPG schemes have almost the same convergence order of H^1-seminorm errors and their gradient recovery superconverges with different values of β. Numerically, the PPR technique can be applied to SIPG and OPSIPG schemes to realize superconvergence. Consequently, we may choose $\beta = 3$, $\sigma_e = 1/19$ in the over-penalized term and the relative error norms for the following numerical computation.

In the regular pattern, Figs. 5–6 illustrate the performance of the OPSIPG gradient recovery method on the H^1-seminorm relative errors for *Cases 1 and 2*. They compare a larger than second order convergence rate of the PPR to a first-order convergent rate of the DG method without using recovery technique. Some H^1-seminorm errors and convergence rates are listed in Table 1. In the chevron pattern, the PPR for *Case 1* performs well with a superconvergence rate of order 1.81 in the inner region as in Fig. 7, while it also provides a superconvergent recovery of order 1.78 in Ω_{in} in Fig. 8 for *Case 2*. And Table 2 shows H^1-seminorm errors and convergence rates. For the unstructured meshes (see Fig. 3), the PPR for *Case 1* performs well with a superconvergence rate of order 1.96 in Fig. 9, and it has a superconvergent recovery of order 1.85 in Fig. 10 for *Case 2* and the convergence rates are listed in Table 3. In Figs. 11–12, we also illustrate the errors of the PPR in 3D based on SIPG and OPSIPG schemes, respectively.

We also consider the quadratic elements in the regular and chevron patterns with an inner region $[0.1, 0.9]^2$ for *Case 2*. Figs. 13 and 14 indicate more than third order convergence rates of the recovered gradient for *Case 2*. We also list the errors and convergence rates in Tables 4 and 5 for the regular and chevron patterns, respectively. Applying the quadratic elements in unstructured meshes, with a smaller inner region $[0.15, 0.85]^2$ for *Cases 1–2*, one observes from Figs. 15–16 that compared to the original gradients, the recovered gradients from the OPSIPG method are superconvergent. Table 6 shows the errors and convergence rates in the unstructured meshes for *Cases 1–2*.

Table 1
Convergence rates in the regular pattern by linear elements and the initial mesh size $h = 1.25e{-}1$.

Regular	Mesh	$\frac{\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order	$\frac{\|\nabla u-G_h u_h\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order
Case 1	h	4.030222e−1		1.884911e−1	
	$h/2$	1.485308e−1	1.4401	7.438193e−2	1.3415
	$h/4$	7.799522e−2	0.9293	1.022604e−2	2.8627
	$h/8$	3.880853e−2	1.0070	2.874884e−3	1.8307
Case 2	h	3.425271e−1		1.931751e−1	
	$h/2$	1.624057e−1	1.0766	3.705632e−2	2.3821
	$h/4$	7.924589e−2	1.0352	9.829886e−3	1.9145
	$h/8$	4.027035e−2	0.9766	2.406333e−3	2.0303

Table 2
Convergence rates in the chevron pattern by linear elements and the initial mesh size $h = 2.5e{-}1$.

Chevron	Mesh	$\frac{\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order	$\frac{\|\nabla u-G_h u_h\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order
Case 1	h	7.610685e−1		4.346563e−1	
	$h/2$	4.062792e−1	0.9056	1.663987e−1	1.3852
	$h/4$	1.521582e−1	1.4169	5.056267e−2	1.7185
	$h/8$	7.803728e−2	0.9633	9.779187e−3	2.3703
Case 2	h	6.923836e−1		3.950487e−1	
	$h/2$	3.657054e−1	0.9209	1.453494e−1	1.4425
	$h/4$	1.626811e−1	1.1686	4.534501e−2	1.6805
	$h/8$	7.926170e−2	1.0374	9.507066e−3	2.2539

Table 3
Convergence rates in the unstructured meshes by linear elements and the initial mesh size $\max\{h\} = 1.2e{-}1$.

Unstructured	Mesh	$\frac{\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order	$\frac{\|\nabla u-G_h u_h\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order
Case 1	h	3.439983e−1		1.167068e−1	
	$h/2$	1.554941e−1	1.1455	5.063971e−2	1.2045
	$h/4$	7.457448e−2	1.0601	1.243027e−2	2.0264
	$h/8$	3.677619e−2	1.0199	1.976579e−3	2.6528
Case 2	h	3.312666e−1		1.954807e−1	
	$h/2$	1.508032e−1	1.1353	7.786595e−2	1.3280
	$h/4$	7.221908e−2	1.0622	1.954964e−2	1.9939
	$h/8$	3.550672e−2	1.0243	4.280404e−3	2.1913

Table 4
Convergence rates in the regular pattern by quadratic elements and the initial mesh size $h = 2.5e{-}1$.

Regular	Mesh	$\frac{\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order	$\frac{\|\nabla u-G_h u_h\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order
Case 2	h	3.083754e−1		1.097187e−1	
	$h/2$	1.008987e−1	1.6118	1.957821e−2	2.4865
	$h/4$	8.501780e−3	3.5690	1.238307e−3	3.9828
	$h/8$	1.767713e−3	2.2659	5.773422e−5	4.4228

Table 5
Convergence rates in the chevron pattern by quadratic elements and the initial mesh size $h = 2.5e{-}1$.

Chevron	Mesh	$\frac{\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order	$\frac{\|\nabla u-G_h u_h\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order
Case 2	h	3.160556e−1		1.599464e−1	
	$h/2$	1.298687e−1	1.2831	2.492384e−2	2.6820
	$h/4$	1.078681e−2	3.5897	1.627671e−3	3.9366
	$h/8$	2.022879e−3	2.4148	1.681631e−4	3.2749

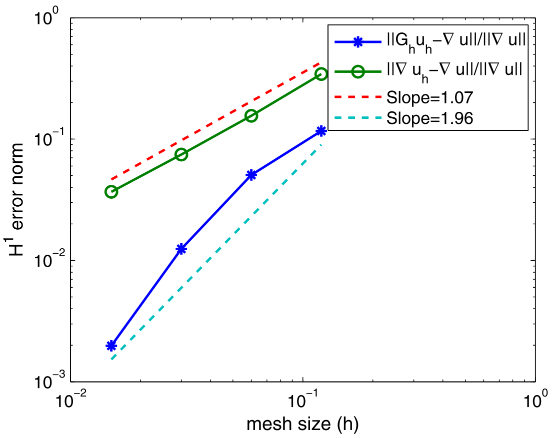

Fig. 9. Linear elements in the unstructured meshes, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}/\|\nabla u\|_{L^2(\Omega_{in})}$ for *Case 1*.

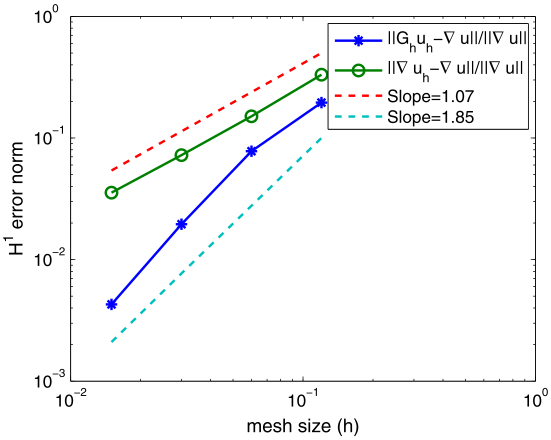

Fig. 10. Linear elements in the unstructured meshes, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}/\|\nabla u\|_{L^2(\Omega_{in})}$ for *Case 2*.

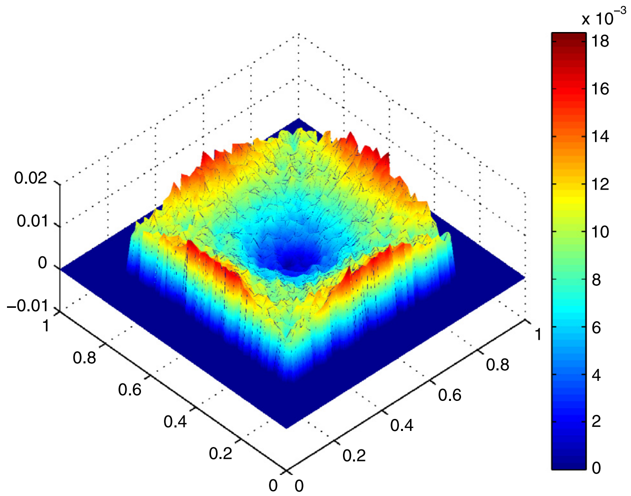

Fig. 11. Error $\nabla u - G_h u_h$ in 3D by SIPG gradient recovery method, linear elements in the unstructured meshes in $\Omega_{\text{in}} := [0.15, 0.85]^2$ for *Case 2*.

Table 6
Convergence rates in the unstructured meshes by quadratic elements and the initial mesh size $\max\{h\} = 1.2\text{e}{-1}$.

Unstructured	Mesh	$\frac{\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order	$\frac{\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}}{\|\nabla u\|_{L^2(\Omega_{in})}}$	Order
Case 1	h	1.409106e−1		7.933208e−2	
	$h/2$	2.446072e−3	5.8482	5.703833e−4	7.1198
	$h/4$	4.179735e−4	2.5490	6.599119e−5	3.1116
Case 2	h	3.831871e−2		2.536329e−2	
	$h/2$	2.685413e−3	3.8348	6.226057e−4	5.3483
	$h/4$	5.693536e−4	2.2377	7.518793e−5	3.0497

Case 3. The last example is a singularity problem setting in the L-shaped domain $\Omega = (0,1)^2 \backslash [1/2,1)^2$ with $a(u) = 3$. Under a polar coordinate system (r,θ) with the origin $(\frac{1}{2}, \frac{1}{2})$, the solution is

$$u(r) = \frac{1}{3} r^{\frac{2}{3}} \sin\left(\frac{2\theta + 5\pi}{3}\right), \quad \frac{\pi}{2} \le \theta \le 2\pi \,.$$

Note that the solution in *Case 3* has a corner singularity at the node $(1/2, 1/2)$ as well as the other five vertices of the L-shaped domain. Delaunay meshes generated by a software DISTMESH [26] have been considered as an adaptive mesh to simulate the convergence behavior. The domain is decomposed into two parts: one is the interior subdomain $\Omega_{in} := [0.08, 0.92] \times [0.08, 0.42] \cup [0.08, 0.42] \times [0.42, 0.92]$, and the other one is the boundary layer $\Omega_{ext} = \Omega \backslash \Omega_{in}$ with a width of

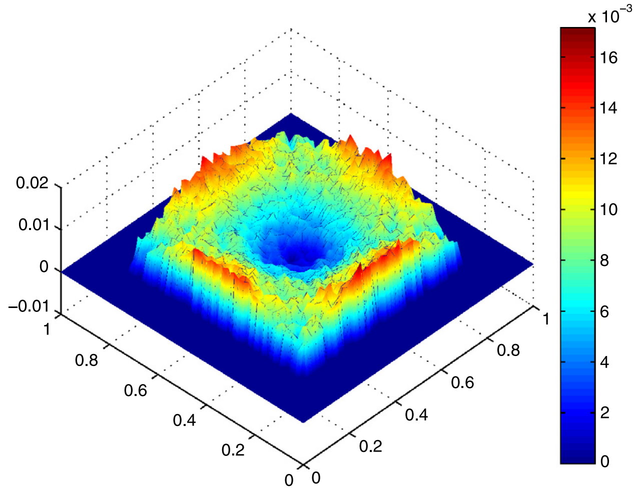

Fig. 12. Error $\nabla u - G_h u_h$ in 3D by OPSIPG gradient recovery method, linear elements in the unstructured meshes in $\Omega_{\text{in}} := [0.15, 0.85]^2$ for *Case 2*.

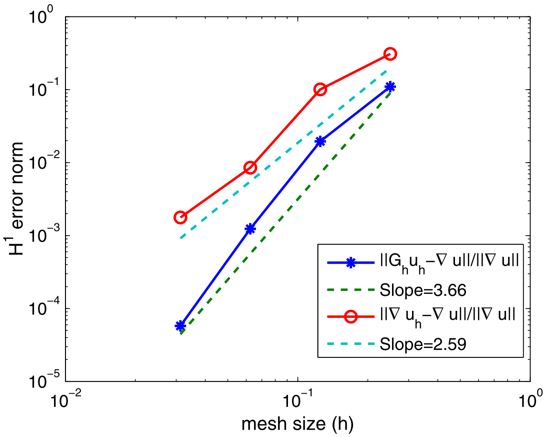

Fig. 13. Quadratic elements in the regular pattern, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{\text{in}})}/\|\nabla u\|_{L^2(\Omega_{in})}$, $\Omega_{\text{in}} := [0.1, 0.9]^2$ for *Case 2*.

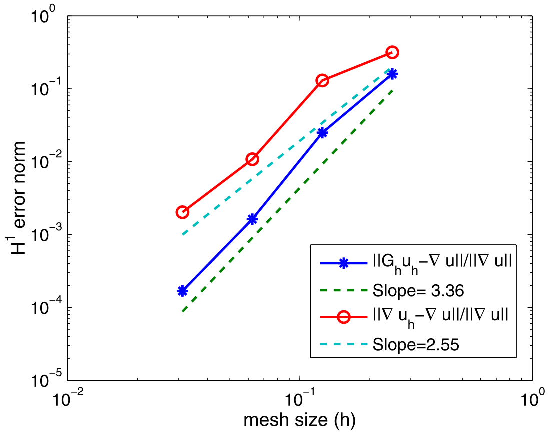

Fig. 14. Quadratic elements in the Chevron pattern, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{\text{in}})}/\|\nabla u\|_{L^2(\Omega_{in})}$, $\Omega_{\text{in}} := [0.1, 0.9]^2$ for *Case 2*.

0.08. For the penalty term in (4), we choose $\beta = 3$. As in [32], for the *a posteriori* error estimator $\Pi_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega_{in})}$ defined by (48), the ratio can be expressed as follows

$$\left| \frac{\Pi_h}{\|\nabla u - \nabla u_h\|_{L^2(\Omega_{in})}} - 1 \right| = \left| \frac{\|G_h u_h - \nabla u_h\|_{L^2(\Omega_{in})}}{\|\nabla u - \nabla u_h\|_{L^2(\Omega_{in})}} - 1 \right| \approx O(N^{order}),$$

where N is the number of degrees of freedom (DoFs) in OPSIPG scheme.

With the use of the gradient recovery method on a locally refined mesh (see Fig. 17), we observe in Fig. 18 that $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}$ is superconvergent, and the term $\|G_h u_h - \nabla u_h\|_{L^2(\Omega_{in})}/\|\nabla u - \nabla u_h\|_{L^2(\Omega_{in})}$ approaches 1 at the slope -2.47 in the interior subdomain. In Table 7, we present the absolute errors $\|\nabla(u - u_h)\|_{L^2(\Omega_{in})}$, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}$ and convergence orders with respect to DoFs on Delaunay meshes with linear elements. Consequently, the gradient recovery operator G_h is effective and the *a posteriori* error estimator Π_h is asymptotically exact under mildly structured grids. It is observed that the polynomial preserving recovery technique provides superconvergent recovered gradients for discontinuous solutions to the OPSIPG method.

6. Conclusions

In this work we employed the PPR to realize the gradient recovery technique based on the over-penalized interior penalty discontinuous Galerkin method for solving the second-order nonlinear elliptic problem. Superconvergence of the PPR implemented on the OPSIPG method has been rigorously proved. In this work, the gradient recovery method has superconvergence for linear and quadratic elements on the structured meshes in the regular and chevron patterns, even on the unstructured meshes. In addition, the gradient recovery method for the linear elements is well-performed for the corner

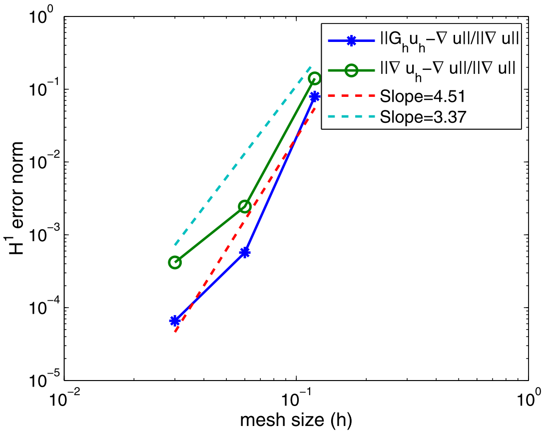

Fig. 15. Quadratic elements in unstructured meshes, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{\text{in}})}/\|\nabla u\|_{L^2(\Omega_{in})}$, $\Omega_{\text{in}} := [0.15, 0.85]^2$ for *Case 1*.

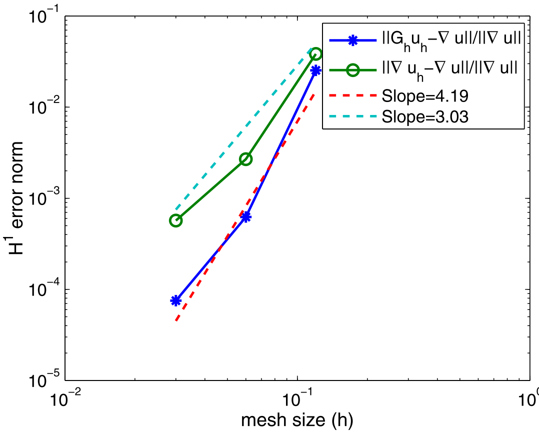

Fig. 16. Quadratic elements in unstructured meshes, $\|\nabla u - G_h u_h\|_{L^2(\Omega_{\text{in}})}/\|\nabla u\|_{L^2(\Omega_{in})}$, $\Omega_{\text{in}} := [0.15, 0.85]^2$ for *Case 2*.

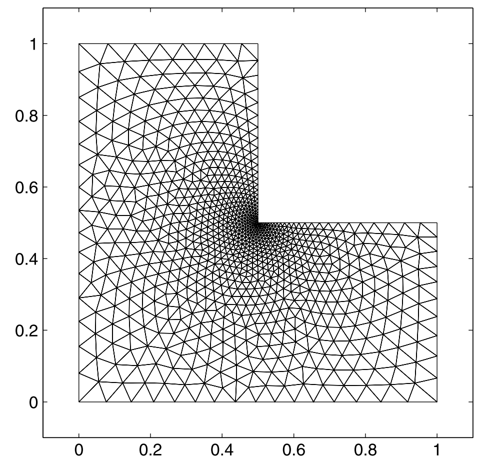

Fig. 17. A locally refined mesh in *Case 3*.

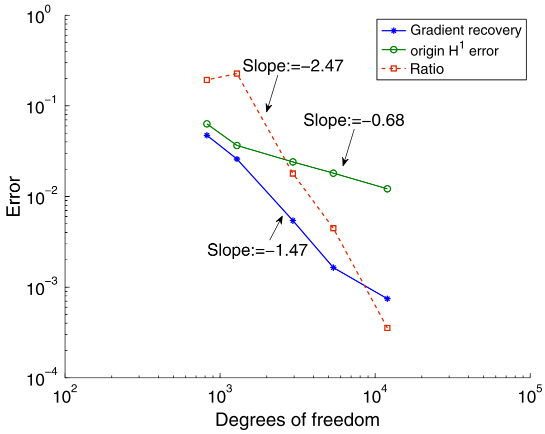

Fig. 18. Errors $\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}$, $\|\nabla u - \nabla u_h\|_{L^2(\Omega_{in})}$ and the ratio versus the number of DoFs for *Case 3*, linear elements in locally refined meshes.

Table 7
Convergence orders with respect to DoFs in Delaunay meshes by linear elements.

	DoFs	$\|\nabla(u-u_h)\|_{L^2(\Omega_{in})}$	Order	$\|\nabla u - G_h u_h\|_{L^2(\Omega_{in})}$	Order	The ratio
Case 3	822	6.331477e−2		4.732690e−2		1.932319e−1
	1284	3.657636e−2	−1.2303	2.589395e−2	−1.3522	2.272679e−1
	2949	2.404580e−2	−0.5044	5.423565e−3	−1.8801	1.791533e−2
	5376	1.810957e−2	−0.4722	1.642915e−3	−1.9889	4.456809e−3
	11 997	1.216340e−2	−0.4958	7.442289e−4	−0.9865	3.541773e−4

singularity problem with the inhomogeneous Dirichlet boundary condition, illustrating that the *a posteriori* error estimator based on the recovered gradient is asymptotically exact and efficient. Following the preconditioning technique in [5], it is straightforward to remedy the ill-conditioned system. However, related questions that remain open comprise construction of an optimal preconditioner of the resulting systems, and superconvergence analysis of the PPR for IPDG solutions in general meshes.

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