A New Adaptive Mixed Finite Element Method Based on Residual Type A Posterior Error Estimates for the Stokes Eigenvalue Problem

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In this article, we combine mixed finite element method, multiscale discretization, and Rayleigh quotient iteration to propose a new adaptive algorithm based on residual type a posterior error estimates for the Stokes eigenvalue problem. Both reliability and efficiency of the error indicator are proved. The efficiency of the algorithm is also investigated using Chen's Innovation Finite Element Method (iFEM) package. Numerical results are satisfying. Numer Methods Partial Differential Eq 31: 31–53, 2015

Keywords: Stokes eigenvalue problem; mixed finite element; Rayleigh quotient iteration; a posterior error estimates; adaptive algorithm

I. INTRODUCTION

Adaptive methods and residual type a posterior estimates, which were first proposed by Babuska and Rheinboldt [1, 2], have gained an enormous attention in scientific engineering computing. More and more researchers entered this field and obtained many good results, most of which have been systemically summarized in [3, 4].

Deriving a-posterior estimates for the Stokes equations has received much attention. The work of [5, 6] builds its basic foundation for mathematical analysis of practical methods. Based on their work, many researchers have made significant advances, see [4, 7–15] and the references therein. In recent years, adaptive algorithms for the Stokes eigenvalue problem are also hot topics; for example, [16] discusses the a posteriori error estimation. It also investigates adaptive mesh refinement of discontinuous Galerkin finite element approximations of the hydrodynamic stability problem associated with the incompressible Stokes eigenvalue problem. Reference [17] presents

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a posteriori error analysis of stable finite elements for the Stokes eigenvalue problem and gives an adaptive algorithm (see Algorithm 4.1). It is worthwhile to note that both adaptive algorithms in [16, 17] require to solve an eigenvalue problem in every iteration step.

Xu [18, 19] first introduces the two-grid method used for discretization for nonsymmertic and nonlinear elliptic problems. At present, this idea is successfully applied to the Stokes problem and eigenvalue problem, see [20–25], and so forth. Rayleigh quotient iteration is also a basic approach for solving the matrix eigenvalue problem (see Lecture 27.4 in [26]). It is meaningful to combine the above two methods for solving the Stokes eigenvalue problem.

Hinged on the work mentioned above, in this article, combining mixed finite element method, multiscale discretization and Rayleigh quotient iteration, we propose a new adaptive method based on residual type a posterior error estimates for the Stokes eigenvalue problem, and the special features of our method are:

1. Compared with the algorithm in [17] (see Algorithm 4.1), our algorithm (Algorithm 4.2) only need to solve an eigenvalue problem on a coarser grid at the first step and then to solve a linear algebraic system on fine grids at each step that follows. Therefore, Algorithm 4.2 is more efficient.
2. Literature [25] establishes a multiscale discretization scheme (see Scheme 3.1) based on the Rayleigh quotient iteration for mixed variational formulation of eigenvalue problems, but it does not discuss adaptive meshes and a posteriori error estimates. For Scheme 3.1, we improve the a priori error estimates given in [25] and obtain new estimates (see Theorem 3.2). Furthermore, based on the work of [17], we give the a posteriori error indicator for Scheme 3.1 and prove its global reliability and local efficiency (see Theorem 3.3), thus a new adaptive algorithm is constructed.
3. Our algorithm is convenient to extend to the existing finite element softwares. Algorithm 4.2 is performed under Chen's iFEM package (see [27] for details) and numerical results show its efficiency is much higher than Algorithm 4.1.

This article is organized as follows. Section II introduces basic knowledge of mixed finite element approximations for the Stokes eigenvalue problem. Section III is devoted to a priori and a posterior error estimates of mixed finite elements for mutiscale discretization. The adaptive Algorithm 4.2 for the Stokes eigenvalue problem is proposed in Section IV.A. In Section IV.B, some numerical experiments using mini element are performed to support our theory. In Section IV.C, a shifted-inverse iteration Scheme combing Mini element and P_2– P_1 element is proposed.

Throughout this article, we refer to [28–31] for the basic theory of finite elements.

II. PRELIMINARIES

Consider the Stokes eigenvalue problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla \sigma &= \lambda \mathbf{u}, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subset R^N (N = 2, 3)$ is a bounded polygonal domain, $\mathbf{u} = (u_1, \cdots, u_N)$ is the velocity of the flow, and σ is the pressure.

Denote $U = H_0^1(\Omega)^N$, $V = L_0^2(\Omega) = \{v \in L^2(\Omega) : \int_\Omega v = 0\}$, $H = L^2(\Omega)^N$. Let $D \subset \Omega$. Denote

$$||\sigma||_{m,D} = \left(\sum_{|\alpha|\le m} \int_D |\partial^\alpha \sigma|^2\right)^{\frac{1}{2}}, \quad \alpha = (\alpha_1, \cdots, \alpha_N),\ |\alpha| = \alpha_1 + \cdots + \alpha_N.$$

For simplicity, write $||\sigma||_{m,\Omega} = ||\sigma||_m$. For any vector function $\mathbf{w} = (w_1, \cdots, w_N)$, denote

$$||\mathbf{w}||_{m,D} = \left(\sum_{i=1}^N ||w_i||_{m,D}^2\right)^{\frac{1}{2}}, \quad |\mathbf{w}|_{m,D} = \left(\sum_{i=1}^N |w_i|_{m,D}^2\right)^{\frac{1}{2}}.$$

Likewise write $||\mathbf{w}||_{m,\Omega} = ||\mathbf{w}||_m$, $|\mathbf{w}|_{m,\Omega} = |\mathbf{w}|_m$. Denote

$$a(\mathbf{w}, \mathbf{v}) = \int_\Omega \sum_{i=1}^N \nabla w_i \cdot \nabla v_i, \quad b(\mathbf{v}, q) = -\int_\Omega \operatorname{div} \mathbf{v} q, \quad (\mathbf{w}, \mathbf{v}) = \int_\Omega \sum_{i=1}^N w_i v_i.$$

Let $||\mathbf{w}||_a^2 = a(\mathbf{w}, \mathbf{w}) = |\mathbf{w}|_1^2$.

The mixed variational form of (2.1) is given by: find $(\lambda, \mathbf{u}, \sigma) \in R \times U \times V$, with $||\mathbf{u}||_a = 1$, such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \sigma) = \lambda(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in U, \tag{2.2}$$

$$b(\mathbf{u}, q) = 0, \quad \forall q \in V. \tag{2.3}$$

Let $\{\pi_h\}_{h>0}$ be a sequence of decompositions of Ω into elements κ. We also assume $\{\pi_h\}_{h>0}$ is regular: there exists a constant $\gamma > 0$ such that

$$h_\kappa \le \gamma \rho_\kappa, \quad \forall \kappa \in \pi_h,$$

where h_κ is the diameter of κ, and ρ_κ is the diameter of the biggest ball contained in κ, $h = \max h_\kappa$. We denote the mixed finite element spaces by U_h and V_h. Then the discrete problem of (2.2)–(2.3) reads: find $(\lambda_h, \mathbf{u}_h, \sigma_h) \in R \times U_h \times V_h$, $||\mathbf{u}_h||_a = 1$, such that

$$a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, \sigma_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}), \quad \forall \mathbf{v} \in U_h, \tag{2.4}$$

$$b(\mathbf{u}_h, q) = 0, \quad \forall q \in V_h. \tag{2.5}$$

Consider the source problem associated with the Stokes eigenvalue problem (2.2)–(2.3) and its discrete mixed finite element form:

Find $(\mathbf{w}, \varphi) \in U \times V$, such that

$$a(\mathbf{w}, v) + b(\mathbf{v}, \varphi) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in U, \tag{2.6}$$

$$b(\mathbf{w}, q) = 0, \quad \forall q \in V. \tag{2.7}$$

Find $(\mathbf{w}_h, \varphi_h) \in U_h \times V_h$, such that

$$a(\mathbf{w}_h, v) + b(\mathbf{v}, \varphi_h) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in U_h, \tag{2.8}$$

$$b(\mathbf{w}_h, q) = 0, \quad \forall q \in V_h. \tag{2.9}$$

Problem (2.6)–(2.7) admits a unique solution $(\mathbf{w}, \varphi) \in U \times V$ (e.g., see [32]), and the following a prior error estimate is valid:

$$||\mathbf{w}||_1 + ||\varphi||_0 \leq C_P||\mathbf{f}||_0, \tag{2.10}$$

where C_P is a positive constant independent of $\mathbf{f}$.

Throughout this article, C denotes a generic positive constant independent of the mesh size h, which may not be the same at each occurrence.

Assume that the mixed finite element spaces $U_h \subset U$ and $V_h \subset V$ satisfy the inf-sup condition, that is, there exists $\beta > 0$ independent of h such that

$$\sup_{\mathbf{v}_h \in U_h} \frac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_1} \geq \beta||q_h||_0, \quad \forall q_h \in V_h.$$

Then (2.8)–(2.9) also admits a unique solution $(\mathbf{w}_h, \varphi_h) \in U_h \times V_h$ and the following error estimate is valid (see [29, 33, 34]):

$$||\mathbf{w} - \mathbf{w}_h||_1 + ||\varphi - \varphi_h||_0 \leq C \left(\inf_{\mathbf{v} \in U_h} ||\mathbf{w} - \mathbf{v}||_1 + \inf_{q \in V_h} ||\varphi - q||_0 \right). \tag{2.11}$$

We assume that the following regularity is valid: for any $\mathbf{f} \in L^2(\Omega)^N$, $(\mathbf{w}, \varphi) \in H^{1+r}(\Omega)^N \times H^r(\Omega)$ satisfies

$$||\mathbf{w}||_{1+r} + ||\varphi||_r \leq C||\mathbf{f}||_0, \tag{2.12}$$

for some $r \in (0, 1]$ depending on Ω.

We adopt the following assumptions for finite element space:

(A1) $\forall \mathbf{z} \in H^{1+r}(\Omega)^N \cap H_0^1(\Omega)^N$ and $v \in H^r(\Omega) \cap L_0^2(\Omega)$, there hold

$$\inf_{\mathbf{v_h} \in U_h} ||\mathbf{z} - \mathbf{v}_h||_1 \leq Ch^r|\mathbf{z}|_{r+1},$$

$$\inf_{q_h \in V_h} ||v - q_h||_0 \leq Ch^r|v|_r.$$

We deduce from (2.11)–(2.12) and (A1),

$$||\mathbf{w} - \mathbf{w}_h||_1 + ||\varphi - \varphi_h||_0 \leq Ch^r(|\mathbf{w}|_{r+1} + |\varphi|_r). \tag{2.13}$$

Using Aubin–Nitche technique and (A1), we have

$$||\mathbf{w} - \mathbf{w}_h||_0 \leq Ch^{2r}(|\mathbf{w}|_{r+1} + |\varphi|_r). \tag{2.14}$$

Define linear bounded operators (see [1]):

$$T : H \to U \subset H, \quad S : H \to V \subset L^2(\Omega), \quad \forall \mathbf{f} \in H,$$

$$a(T\mathbf{f}, \mathbf{v}) + b(\mathbf{v}, S\mathbf{f}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in U,$$

$$b(T\mathbf{f}, q) = 0, \quad \forall q \in V.$$

$$T_h : H \to U_h \subset H, \quad S_h : H \to V_h \subset L^2(\Omega), \quad \forall \mathbf{f} \in H,$$

$$a(T_h\mathbf{f},\mathbf{v}) + b(\mathbf{v}, S_h\mathbf{f}) = (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in U_h,$$
$$b(T_h\mathbf{f}, q) = 0, \quad \forall q \in V_h.$$

Then (2.2)–(2.3) and (2.4)–(2.5) can be written in operator forms

$$\lambda T\mathbf{u} = \mathbf{u}, \quad \sigma = S(\lambda\mathbf{u}), \tag{2.15}$$

and

$$\lambda_h T_h\mathbf{u}_h = \mathbf{u}_h, \quad \sigma_h = S_h(\lambda_h\mathbf{u}_h), \tag{2.16}$$

respectively. It is easy to know both T and T_h are self-adjoint (e.g., see [35]). From (2.10) and the compact injection $U \hookrightarrow H$, we can conclude that T is completely continuous.

Suppose that λ and λ_h are the kth eigenvalue of (2.15) and (2.16), respectively; throughout this article $\mu = \frac{1}{\lambda}$ also denote the kth eigenvalue of T. The algebraic multiplicity of λ is equal to q, $\lambda = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$. Let $M(\lambda)$ be the space spanned by all eigenfunctions corresponding to λ and $M_h(\lambda)$ be the direct sum of eigenfunctions corresponding to all eigenvalues of (2.16) that converge to λ. Let $\widehat{M}(\lambda) = \{\mathbf{v} : \mathbf{v} \in M(\lambda), ||\mathbf{v}||_a = 1\}$. Now we introduce the following crucial quantity:

$$\delta_h = \delta_h(\lambda) = \sup_{\mathbf{u}\in\widehat{M}(\lambda)} \left(\inf_{\mathbf{v}\in U_h} ||\mathbf{u} - \mathbf{v}||_1 + \inf_{q\in V_h} ||S(\lambda\mathbf{u}) - q||_0 \right).$$

A priori estimates for the Stokes eigenvalue problem are critical for our analysis. The convergence and error estimate for mixed element method of eigenvalue problem have been studied in [35–38], from which we easily know

$$||T - T_h||_0 \to 0 \quad (h \to 0),$$

and the following results are valid.

Lemma 2.1. *Let $(\lambda_h, \mathbf{u}_h, \sigma_h)$ be the kth eigenpair of (2.4)–(2.5) with $||\mathbf{u}_h||_a = 1$, λ be the kth eigenvalue of (2.2)–(2.3). Then $\lambda_h \to \lambda (h \to 0)$ and there exists an eigenpair $(\mathbf{u}, \sigma)$ corresponding to λ such that*

$$|\lambda_h - \lambda| \le C\delta_h(\lambda)^2, \tag{2.17}$$
$$||\mathbf{u}_h - \mathbf{u}||_0 \le Ch^r\delta_h(\lambda), \tag{2.18}$$
$$||\sigma_h - \sigma||_0 \le C\delta_h(\lambda), \tag{2.19}$$
$$||\mathbf{u}_h - \mathbf{u}||_1 \le C\delta_h(\lambda), \tag{2.20}$$

let $\mathbf{u} \in \widehat{M}(\lambda)$, then there exists $\mathbf{u}_h \in M_h(\lambda)$ such that

$$||\mathbf{u}_h - \mathbf{u}||_1 \le C\delta_h(\lambda). \tag{2.21}$$

III. MULTISCALE DISCRETIZATION SCHEME BASED ON RAYLEIGH QUOTIENT ITERATION

In this section, we prove a prior and a posterior error estimates for multiscale discretization Scheme 3.1 provided in [24, 25]. The main theoretical results are given in Theorems 3.1–3.3. Priori to our argument, we give the following crucial condition related to degree of approximation of eigenfunctions.

Condition 3.1. There exists a properly small positive number ϵ such that $\delta_{h_i} = O(\delta_{h_{i-1}}^{t_i})$ $(i = 1, 2, \cdots)$ with $t_i \in (1, 2-\epsilon]$ and $\delta_{h_i} \to 0 (i \to \infty)$.

Condition 3.1 could be easily satisfied. For example, for the problem having smooth solution, using the uniform mesh $h_0 = \sqrt{2}/8$, $h_1 = \sqrt{2}/32$, $h_2 = \sqrt{2}/64$, and $h_3 = \sqrt{2}/128$, we have $h_i = h_{i-1}^{t_i}$, that is, $\delta_{h_i} = O(\delta_{h_{i-1}}^{t_i})$, where $t_1 \approx 1.80$, $t_2 \approx 1.22$, $t_3 \approx 1.18$. As for the problem having nonsmooth solution, the condition could be satisfied when the local refinement is made near the singular points.

The following scheme is proposed in [24, 25].

Scheme 3.1. Multiscale Discretization.

Step 1. Solve the eigenvalue problem (2.2)–(2.3) on $U_H \times V_H$: find $(\lambda_H, \mathbf{u}_H, \sigma_H) \in R \times U_H \times V_H$, with $||\mathbf{u}_H||_a = 1$ such that

$$a(\mathbf{u}_H, \mathbf{v}) + b(\mathbf{v}, \sigma_H) = \lambda_H(\mathbf{u}_H, \mathbf{v}), \quad \forall \mathbf{v} \in U_H,$$
$$b(\mathbf{u}_H, q) = 0, \quad \forall q \in V_H.$$

Step 2. $\mathbf{u}^{h_0} \Leftarrow \mathbf{u}_H, \lambda^{h_0} \Leftarrow \lambda_H, i \Leftarrow 1$.

Step 3. Solve an equation on $U_{h_i} \times V_{h_i}$: find $(\mathbf{u}', \sigma') \in U_{h_i} \times V_{h_i}$ such that

$$a(\mathbf{u}', \mathbf{v}) + b(\mathbf{v}, \sigma') - \lambda^{h_{i-1}}(\mathbf{u}', \mathbf{v}) = (\mathbf{u}^{h_{i-1}}, \mathbf{v}), \quad \forall \mathbf{v} \in U_{h_i},$$
$$b(\mathbf{u}', q) = 0, \ \forall q \in V_{h_i}.$$

Set $\mathbf{u}^{h_i} = \mathbf{u}'/||\mathbf{u}'||_a$, $\sigma^{h_i} = \sigma'/||\mathbf{u}'||_a$.

Step 4. Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a(\mathbf{u}^{h_i}, \mathbf{u}^{h_i})}{(\mathbf{u}^{h_i}, \mathbf{u}^{h_i})}.$$

Step 5. If $i = l$ (l denotes the final level of the multiscale scheme), then output $(\lambda^{h_i}, \mathbf{u}^{h_i}, \sigma^{h_i})$, stop; else, $i \Leftarrow i + 1$, and return to Step 3.

Lemma 3.1. *For any nonzero* $\mathbf{u}, \mathbf{v} \in U$,

$$\left\| \frac{\mathbf{u}}{||\mathbf{u}||_a} - \frac{\mathbf{v}}{||\mathbf{v}||_a} \right\|_a \le 2\frac{||\mathbf{u}-\mathbf{v}||_a}{||\mathbf{u}||_a}, \quad \left\| \frac{\mathbf{u}}{||\mathbf{u}||_a} - \frac{\mathbf{v}}{||\mathbf{v}||_a} \right\|_a \le 2\frac{||\mathbf{u}-\mathbf{v}||_a}{||\mathbf{v}||_a}. \tag{3.1}$$

Proof. See [24]. ■

Denote $\text{dist}(\mathbf{w}, W) = \inf_{\mathbf{v}\in W}||\mathbf{w}-\mathbf{v}||_a$. The following lemma is a crucial property of the shifted-inverse iteration in finite element method (see Lemma 4.2 in [39]), which is a development of Theorem 3.2 in [24].

Lemma 3.2. *Suppose that $(\mu, \mathbf{u})$ are the kth eigenpair of T and μ_h is the kth eigenvalue of T_h. $(\mu_0, \mathbf{u}_0)$ is an approximate pair of $(\mu, \mathbf{u})$, where μ_0 is not an eigenvalue of T_h, $\mathbf{u}_0 \in U_h$, $||\mathbf{u}_0||_a = 1$, $dist(\mathbf{u}_0, M_h(\mu)) \leq 1/2$, $|\mu_0 - \mu_{j,h}| \geq \rho/2 (j \neq k, k+1, \cdots, k+q-1)$. Let $\mathbf{u}^s \in U_h, \mathbf{u}^h \in U_h$ satisfy*

$$(\mu_0 - T_h)\mathbf{u}^s = \mathbf{u}_0, \quad \mathbf{u}^h = \mathbf{u}^s/||\mathbf{u}^s||_a.$$

Then

$$dist(\mathbf{u}^h, M_h(\mu)) \leq \frac{4}{\rho} \max_{k\leq j\leq k+q-1} |\mu_0 - \mu_{j,h}| dist(\mathbf{u}_0, M_h(\mu)),$$

where $\rho = \min_{\mu_j \neq \mu} |\mu_j - \mu|$ is the separation constant of the eigenvalue μ.

Note that in our later argument we actually take in Lemma 3.2 $\mu = \frac{1}{\lambda}, \mu_{j,h} = \frac{1}{\lambda_{j,h}}, M_h(\mu) = M_h(\lambda), \mu_h = \frac{1}{\lambda_h}$.

Let $\lambda_{j,h_l}(j = k, k+1, \cdots, k+q-1)$ be eigenvalues of T_{h_l} that converge to λ, and $(\mathbf{u}_{j,h_l}, \sigma_{j,h_l})$ be the eigenfunction corresponding to λ_{j,h_l} such that $\{\mathbf{u}_{j,h_l}\}_{j=k}^{k+q-1}$ is an orthonormal system of $M_{h_l}(\lambda)$ in the sense of norm $||\cdot||_a$.

From lemma 2.1, we know there exists $\{\mathbf{u}_j^0\}_k^{k+q-1} \subset M(\lambda)$ such that $\mathbf{u}_{j,h_l} - \mathbf{u}_j^0$ satisfies (2.18) and (2.20).

Let

$$\mathbf{u} = \sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_l})\mathbf{u}_j^0, \tag{3.2}$$

$$\sigma = S(\lambda\mathbf{u}). \tag{3.3}$$

Lemma 3.3. *Suppose that $||T - T_h||_0 \to 0 (h \to 0)$, H is properly small, and Condition 3.1 holds. Let $(\lambda^{h_l}, \mathbf{u}^{h_l})$ be the discrete solution for the kth eigenpair of (2.2)–(2.3) obtained by Scheme 3.1, let $\mathbf{u}^{h_{l-1}}$ approximate $\overline{\mathbf{u}} \in \widehat{M}(\lambda)$ and $\lambda^{h_{l-1}}$ approximate λ. Then there hold*

$$||\mathbf{u}^{h_l} - \mathbf{u}||_a \leq C(|\lambda^{h_{l-1}} - \lambda|^2 + |\lambda^{h_{l-1}} - \lambda| ||\mathbf{u}^{h_{l-1}} - \overline{\mathbf{u}}||_a + ||(T - T_{h_l})|_{M(\lambda)}||_a), \tag{3.4}$$

$$|\lambda^{h_l} - \lambda| \leq C(||\mathbf{u}^{h_l} - \mathbf{u}||_a^2 + ||\mathbf{u}^{h_l} - \mathbf{u}||_a \inf_{v\in V_{h_l}} ||\sigma - v||_0), \tag{3.5}$$

where $\mathbf{u}$ and σ are defined by (3.2) and (3.3), respectively, and C is independent of the mesh size and l.

Proof. See the proof of Theorem 3.3 in [25] for the case where the assumption

$$\max_{k\leq j\leq k+q-1} \left| \frac{\mu_{j,h} - \mu_h}{\mu_0 - \mu_{j,h}} \right| \leq \frac{1}{2}$$

is considered. It is essential to note that (3.4) and (3.5) are still valid using Lemma 3.2 to slightly modify this proof when deleting this assumption. ■

Theorem 3.1. *Suppose that H is properly small, and Condition 3.1 holds. Let* $(\lambda^{h_l}, \mathbf{u}^{h_l})$ *be the discrete solution for the kth eigenpair of (2.2)–(2.3) obtained by Scheme 3.1. Then there hold*

$$||\mathbf{u}^{h_l} - \mathbf{u}||_a \le C\delta_{h_l}, \tag{3.6}$$

$$|\lambda^{h_l} - \lambda| \le C\delta_{h_l}^2, \tag{3.7}$$

where $l \ge 1$ *and* $\mathbf{u}$ *is defined by (3.2).*

Proof. As discussed before, we know $||T - T_h||_0 \to 0(h \to 0)$, thus the conditions of Lemma 3.3 hold.

When $l = 1$, noting that $u^{h_0} = u_H$ and $\lambda^{h_0} = \lambda_H$, by (2.17), (2.20), and (3.1) there exists $\overline{\mathbf{u}} \in \widehat{M}(\lambda)$ such that

$$||\mathbf{u}^{h_0} - \overline{\mathbf{u}}||_a \le C\delta_{h_0},$$

$$|\lambda^{h_0} - \lambda| \le C\delta_{h_0}^2.$$

Substituting the above two formulae into (3.4) yields

$$||\mathbf{u}^{h_1} - \mathbf{u}||_a \le C(\delta_{h_0}^4 + \delta_{h_0}^3 + \delta_{h_1}) \le C\delta_{h_1}, \tag{3.8}$$

where $\mathbf{u} = \sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_1}, \mathbf{u}_{j,h_1})\mathbf{u}_j^0$ with $\mathbf{u}_{j,h_1} - \mathbf{u}_j^0$ satisfying (2.18) and (2.20), and substituting (3.8) into (3.5) yields

$$|\lambda^{h_1} - \lambda| \le C\delta_{h_1}^2, \tag{3.9}$$

thus (3.6)–(3.7) hold for $l = 1$.

Suppose (3.6)–(3.7) hold for $l-1$, that is, there hold

$$||\mathbf{u}^{h_{l-1}} - \mathbf{u}||_a \le C\delta_{h_{l-1}}, \tag{3.10}$$

$$|\lambda^{h_{l-1}} - \lambda| \le C\delta_{h_{l-1}}^2, \tag{3.11}$$

where $\mathbf{u} = \sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_{l-1}}, \mathbf{u}_{j,h_{l-1}})\mathbf{u}_j^0$ with $\mathbf{u}_{j,h_{l-1}} - \mathbf{u}_j^0$ satisfying (2.18) and (2.20).

Let $\overline{\mathbf{u}} = \mathbf{u}/||\mathbf{u}||_a \in \widehat{M}(\lambda)$. Using (3.1) and (3.10) we deduce that $\mathbf{u}^{h_{l-1}}$ approximates $\overline{\mathbf{u}}$ and there holds

$$||\mathbf{u}^{h_{l-1}} - \overline{\mathbf{u}}||_a \le C\delta_{h_{l-1}}. \tag{3.12}$$

Substituting (3.11)–(3.12) into (3.4)–(3.5), from Condition 3.1, we get

$$||\mathbf{u}^{h_l} - \mathbf{u}||_a \le C(\delta_{h_{l-1}}^3 + \delta_{h_l}) \le C\delta_{h_l},$$

and

$$|\lambda^{h_l} - \lambda| \le C\delta_{h_l}^2,$$

then (3.6)–(3.7) follow. The proof concludes. ■

We set

$$\mathbf{u}^* = \sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_j})\mathbf{u}_{j,h_l}, \tag{3.13}$$

$$\sigma^* = \sum_{j=k}^{k+q-1} \lambda_{j,h_l} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_l}) S_{h_l}\mathbf{u}_{j,h_l}. \tag{3.14}$$

Lemma 3.4. *Under the conditions of Theorem 3.1, there hold*

$$||\mathbf{u}^{h_l} - \mathbf{u}^*||_a \leq C\delta_{h_l}^{3/t_l}, \tag{3.15}$$

$$||\sigma^{h_l} - \sigma^*||_0 \leq C\delta_{h_l}^{2/t_l}. \tag{3.16}$$

Proof. Denote

$$\mu_0 = \frac{1}{\lambda^{h_{l-1}}}, \quad \mathbf{u}_0 = \frac{\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}}}{||\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}}||_a},$$

$$\mathbf{u}^s = \frac{\lambda^{h_{l-1}}\mathbf{u}'}{||\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}}||_a}, \quad \mu_{h_l} = \frac{1}{\lambda_{h_l}}.$$

Then Step 3 of Scheme 3.1 ($i = l$) is equivalent to (see [25]):

$$(\mu_0 - T_{h_l})\mathbf{u}^s = \mathbf{u}_0, \quad \mathbf{u}^{h_l} = \mathbf{u}^s/||\mathbf{u}^s||_a.$$

It is easy to verify the other conditions of Lemma 3.2 (e.g., see [25]), thus we have

$$\text{dist}(\mathbf{u}^{h_l}, M_{h_l}(\lambda)) \leq C \max_{k\leq j\leq k+q-1} |\lambda_{j,h_l} - \lambda^{h_{l-1}}| dist(\mathbf{u}_0, M_{h_l}(\lambda)). \tag{3.17}$$

By (3.1), (3.6), and (3.7), there exists $\overline{\mathbf{u}} \in \widehat{M}(\lambda)$ such that

$$\begin{aligned} ||\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}} - \overline{\mathbf{u}}||_a &\leq ||(\lambda^{h_{l-1}} - \lambda)T_{h_l}\mathbf{u}^{h_{l-1}} + \lambda T_{h_l}(\mathbf{u}^{h_{l-1}} - \overline{\mathbf{u}}) + \lambda(T_{h_l} - T)\overline{\mathbf{u}}||_a \\ &\leq C(|\lambda^{h_{l-1}} - \lambda| + ||\mathbf{u}^{h_{l-1}} - \overline{\mathbf{u}}||_0 + ||(T_{h_l} - T)|_{M(\lambda)}||_a) \\ &\leq C(\delta_{h_{l-1}}^2 + \delta_{h_{l-1}} + \delta_{h_l}) \\ &\leq C\delta_{h_{l-1}}, \end{aligned} \tag{3.18}$$

which together with (3.1) yields

$$\begin{aligned} \text{dist}(\mathbf{u}_0, \widehat{M}(\lambda)) &\leq ||\mathbf{u}_0 - \overline{\mathbf{u}}||_a \\ &\leq 2||\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}} - \overline{\mathbf{u}}||_a \\ &\leq C\delta_{h_{l-1}}, \end{aligned} \tag{3.19}$$

from the triangle inequality, (3.19) and (2.21) we deduce

$$\begin{aligned} \text{dist}(\mathbf{u}_0, M_{h_l}(\lambda)) &\leq \text{dist}(\mathbf{u}_0, \widehat{M}(\lambda)) + C||(T - T_{h_l})|_{M(\lambda)}||_a \\ &\leq C\delta_{h_{l-1}}. \end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.17), from (3.7) and (2.17) we have

$$\mathrm{dist}(\mathbf{u}^{h_l}, M_{h_l}(\lambda)) \le C\delta^3_{h_{l-1}},$$

noting $||\mathbf{u}^{h_l} - \mathbf{u}^*||_a = dist(\mathbf{u}^{h_l}, M_{h_l}(\lambda))$, thus (3.15) follows.

We split $\mathbf{u}'$ as follows:

$$\begin{aligned}\mathbf{u}' &= (\lambda^{h_{l-1}})^{-2}((\lambda^{h_{l-1}})^{-1} - T_{h_l})^{-1}(\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}}) \\ &= (\lambda^{h_{l-1}})^{-2}\Big\{((\lambda^{h_{l-1}})^{-1} - T_{h_l})^{-1}(\lambda^{h_{l-1}} T_{h_l}\mathbf{u}^{h_{l-1}} - \bar{\mathbf{u}}) \\ &\quad +((\lambda^{h_{l-1}})^{-1} - T_{h_l})^{-1}(\bar{\mathbf{u}} - \Sigma) + ((\lambda^{h_{l-1}})^{-1} - T_{h_l})^{-1}\Sigma\Big\},\end{aligned} \tag{3.21}$$

where $\Sigma = \sum_{j=k}^{k+q-1} a(\bar{\mathbf{u}}, \mathbf{u}_{j,h_l})\mathbf{u}_{j,h_l}$. By calculations (see Proposition 2.32 in [40]),

$$\begin{aligned}||((\lambda^{h_{l-1}})^{-1} - T_{h_l})^{-1}||_a &= \left[\min_{z\in\sigma(T_{h_l})}|(\lambda^{h_{l-1}})^{-1} - z|\right]^{-1} \\ &= \left\{\min_{k\le j\le k+q-1}|(\lambda^{h_{l-1}})^{-1} - (\lambda_{j,h_l})^{-1}|\right\}^{-1} \\ &= \left\{\min_{k\le j\le k+q-1}\left|\frac{\lambda^{h_{l-1}} - \lambda + \lambda - \lambda_{j,h_l}}{\lambda^{h_{l-1}}\lambda_{j,h_l}}\right|\right\}^{-1} \\ &\le C|\lambda^{h_{l-1}} - \lambda|^{-1}.\end{aligned} \tag{3.22}$$

Since $||\bar{\mathbf{u}} - \Sigma||_a = \mathrm{dist}(\bar{\mathbf{u}}, M_{h_l}(\lambda))$, by (2.21), there exists $\bar{\mathbf{u}}_{h_l} \in M_{h_l}(\lambda)$ such that

$$||\bar{\mathbf{u}} - \Sigma||_a \le ||\bar{\mathbf{u}} - \bar{\mathbf{u}}_{h_l}||_a \le C\delta_{h_l}. \tag{3.23}$$

It remains to estimate the third term of (3.21). By calculations, we have

$$\begin{aligned}\left|\left|(\mu_0 - T_{h_l})^{-1}\sum_{j=k}^{k+q-1} a(\bar{\mathbf{u}}, \mathbf{u}_{j,h_l})\mathbf{u}_{j,h_l}\right|\right|_a &= \left|\left|\sum_{j=k}^{k+q-1} a(\bar{\mathbf{u}}, \mathbf{u}_{j,h_l})((\lambda^{h_{l-1}})^{-1} - (\lambda_{j,h_l})^{-1})^{-1}\mathbf{u}_{j,h_l}\right|\right|_a \\ &= \left(\sum_{j=k}^{k+q-1} a(\bar{\mathbf{u}}, \mathbf{u}_{j,h_l})^2\left|\frac{\lambda^{h_{l-1}} - \lambda_{j,h_l}}{\lambda^{h_{l-1}}\lambda_{j,h_l}}\right|^{-2}\right)^{1/2} \\ &\ge C\min_{k\le j\le k+q-1}\left\{|\lambda^{h_{l-1}} - \lambda_{j,h_l}|^{-1}\right\} \\ &\ge C|\lambda^{h_{l-1}} - \lambda|^{-1}.\end{aligned} \tag{3.24}$$

From (3.18), (3.22), (3.23), and (3.24), we see that at the right-hand side of (3.21) the third term is the dominant term. Then from (2.17), (3.21), and (3.24), we deduce that

$$||\mathbf{u}'||_a \ge C|\lambda^{h_{l-1}} - \lambda|^{-1} \ge C\delta^{-2}_{h_{l-1}}. \tag{3.25}$$

Recalling (3.13) and (3.14), by a simple calculation, we get

$$\lambda S_{h_l}\mathbf{u}^* - \sigma^* = \sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_l})(\lambda - \lambda_{j,h_l})S_{h_l}\mathbf{u}_{j,h_l},$$

$$\lambda^{h_{l-1}}\mathbf{u}^{h_l} - \lambda\mathbf{u}^* = (\lambda^{h_{l-1}} - \lambda)\mathbf{u}^{h_l} + \lambda(\mathbf{u}^{h_l} - \mathbf{u}^*),$$

$$\sigma^{h_l} - \sigma^* = \frac{S_{h_l}\mathbf{u}^{h_{l-1}}}{||\mathbf{u}'||_a} + S_{h_l}(\lambda^{h_{l-1}}\mathbf{u}^{h_l} - \lambda\mathbf{u}^*) + (\lambda S_{h_l}\mathbf{u}^* - \sigma^*),$$

from the boundedness of S_{h_l}, (2.17), (3.7), (3.15), and (3.25), we yield

$$\begin{aligned} ||\sigma^{h_l} - \sigma^*||_0 &\le \frac{||S_{h_l}\mathbf{u}^{h_{l-1}}||_0}{||\mathbf{u}'||_a} + C\left(|\lambda^{h_{l-1}} - \lambda| + ||\mathbf{u}^{h_l} - \mathbf{u}^*||_0 + \sum_{j=k}^{k+q-1}|\lambda - \lambda_{j,h_l}|\right) \\ &\le C(\delta_{h_{l-1}}^2 + \delta_{h_{l-1}}^3 + \delta_{h_l}^2) \\ &\le C\delta_{h_{l-1}}^2, \end{aligned}$$

thus (3.16) holds. ∎

Theorem 3.2. *Under the conditions of Theorem 3.1, there hold*

$$||\mathbf{u} - \mathbf{u}^{h_l}||_0 \le C(h_l^r\delta_{h_l} + \delta_{h_l}^{3/t_l}), \tag{3.26}$$

$$||\sigma^{h_l} - \sigma||_0 \le C\delta_{h_l}, \tag{3.27}$$

where $\mathbf{u}$ *and* σ *are defined by (3.2) and (3.3), respectively.*

Proof. Recalling (3.13) and (3.2), by (2.18) we see it holds

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}^*||_0 &= \left\|\sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_l})(\mathbf{u}_j^0 - \mathbf{u}_{j,h_l})\right\|_0 \\ &\le C\sum_{j=k}^{k+q-1}||\mathbf{u}_j^0 - \mathbf{u}_{j,h_l}||_0 \\ &\le Ch_l^r\delta_{h_l}. \end{aligned}$$

Therefore, using the triangle inequality and (3.15) we have

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}^{h_l}||_0 &\le ||\mathbf{u} - \mathbf{u}^*||_0 + ||\mathbf{u}^* - \mathbf{u}^{h_l}||_0 \\ &\le C(h_l^r\delta_{h_l} + \delta_{h_l}^{3/t_l}), \end{aligned}$$

which is (3.26). By a simple calculation, from the boundness of S_{h_l}, (2.17) and (2.18) we deduce

$$\begin{aligned} ||\sigma - \sigma^*||_0 &= \left\|\sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_l})(S(\lambda\mathbf{u}_j^0) - S_{h_l}(\lambda_{j,h_l}\mathbf{u}_{j,h_l}))\right\|_0 \\ &\le C\sum_{j=k}^{k+q-1}||(S(\lambda\mathbf{u}_j^0) - S_{h_l}(\lambda_{j,h_l}\mathbf{u}_{j,h_l}))||_0 \\ &\le C\max_{k\le j\le k+q-1}\left\{||(S - S_{h_l})(\lambda\mathbf{u}_j^0)||_0 + ||S_{h_l}(\lambda\mathbf{u}_j^0 - \lambda_{j,h_l}\mathbf{u}_{j,h_l})||_0\right\} \end{aligned}$$

$$\leq C \max_{k\leq j\leq k+q-1} \left\{ ||(S - S_{h_l})(\lambda \mathbf{u}_j^0)||_0 + |\lambda_{j,h_l} - \lambda| + ||\mathbf{u}_{j,h_l} - \mathbf{u}_j^0||_0 \right\}$$

$$\leq C\delta_{h_l},$$

which, together with (3.16), yields (3.27). ∎

Now we are ready to discuss the a posterior error estimates for the multiscale discretization. Literature [17] has discussed a posterior error estimate of stable finite elements for the Stokes eigenvalue problem. First of all, we introduce some notations used in [17] as follows.

Define the residuals:

$$\tilde{R}_{\kappa,1}(\mathbf{u}_h, \sigma_h) = \Delta \mathbf{u}_h - \nabla \sigma_h + \lambda_h \mathbf{u}_h, \tag{3.28}$$

$$\tilde{R}_{\kappa,2}(\mathbf{u}_h) = \operatorname{div} \mathbf{u}_h, \tag{3.29}$$

$$\tilde{R}_{\partial\kappa}(\mathbf{u}_h, \sigma_h) = [[(\nabla \mathbf{u}_h - \sigma_h I) \cdot \mathbf{n}_\kappa]]|_{\partial\kappa}, \tag{3.30}$$

where $[[\cdot]]|_{\partial\kappa}$ denotes the jump across $\partial\kappa$ and I is 2×2 identity matrix.

For $\kappa \in \pi_h$, define the local error indicator

$$\tilde{\eta}_h(\mathbf{u}_h, \sigma_h, \kappa) = \left\{ h_\kappa^2 ||\tilde{R}_{\kappa,1}(\mathbf{u}_h, \sigma_h)||_{0,\kappa}^2 + ||\tilde{R}_{\kappa,2}(\mathbf{u}_h)||_{0,\kappa}^2 + \frac{1}{2} h_\kappa ||\tilde{R}_{\partial\kappa}(\mathbf{u}_h, \sigma_h)||_{0,\partial\kappa}^2 \right\}^{1/2}.$$

Finally, the global error indicator is given by

$$\tilde{\eta}_h(\mathbf{u}_h, \sigma_h, \Omega) = \left\{ \sum_{\kappa\in\pi_h} \tilde{\eta}_h^2(\mathbf{u}_h, \sigma_h, \kappa) \right\}^{1/2}. \tag{3.31}$$

We denote by $\omega(\kappa)$ the union of all elements having at least one edge (for $N = 2$)—or one face (for $N = 3$)—in common with κ. Similarly, for a given edge (for $N = 2$) E—or face (for $N = 3$), the set $\omega(E)$ stands for the union of the elements containing E. The following conclusion was proved in [17].

Lemma 3.5. *Let $(\lambda_h, u_h, \sigma_h)$ be an eigenpair of (2.4)–(2.5), then there exists an eigenpair (λ, u, σ) of (2.2)–(2.3) such that*

$$||\mathbf{u} - \mathbf{u}_h||_1 + ||\sigma - \sigma_h||_0 \leq C(\tilde{\eta}_h(\mathbf{u}_h, \sigma_h, \Omega) + |\lambda - \lambda_h| + \lambda||\mathbf{u} - \mathbf{u}_h||_0), \tag{3.32}$$

$$C\tilde{\eta}_h(\mathbf{u}_h, \sigma_h, \kappa) \leq |\mathbf{u} - \mathbf{u}_h|_{1,\omega(\kappa)} + ||\sigma - \sigma_h||_{0,\omega(\kappa)} + \sum_{\kappa'\subset\omega(\kappa)} h_{\kappa'}^{1/2}(|\lambda - \lambda_h| + \lambda||\mathbf{u} - \mathbf{u}_h||_{0,\kappa'}). \tag{3.33}$$

We proceed to introduce some notations of the residuals for $(\mathbf{u}^*, \sigma^*)$ as transition. Define the residuals:

$$\widehat{R}_{\kappa,1}(\mathbf{u}^*, \sigma^*) = \Delta \mathbf{u}^* - \nabla \sigma^* + \sum_{j=k}^{k+q-1} a(\mathbf{u}^{h_l}, \mathbf{u}_{j,h_l})\lambda_{j,h_l} \mathbf{u}_{j,h_l}, \tag{3.34}$$

$$\widehat{R}_{\kappa,2}(\mathbf{u}^*) = div\, \mathbf{u}^*, \tag{3.35}$$

$$\widehat{R}_{\partial\kappa}(\mathbf{u}^*, \sigma^*) = [[(\nabla \mathbf{u}^* - \sigma^* I) \cdot \mathbf{n}_\kappa]]|_{\partial\kappa}. \tag{3.36}$$

For $\kappa \in \pi_{h_l}$, define the local error indicator

$$\widehat{\eta}_{h_l}(\mathbf{u}^*, \sigma^*, \kappa) = \left\{ h_\kappa^2 ||\widehat{R}_{\kappa,1}(\mathbf{u}^*, \sigma^*)||_{0,\kappa}^2 + ||\widehat{R}_{\kappa,2}(\mathbf{u}^*)||_{0,\kappa}^2 + \frac{1}{2} h_\kappa ||\widehat{R}_{\partial\kappa}(\mathbf{u}^*, \sigma^*)||_{0,\partial\kappa}^2 \right\}^{1/2}.$$

Finally, the global error indicator is given by

$$\widehat{\eta}_{h_l}(\mathbf{u}^*, \sigma^*, \Omega) = \left\{ \sum_{\kappa \in \pi_{h_l}} \widehat{\eta}_{h_l}^2(\mathbf{u}^*, \sigma^*, \kappa) \right\}^{1/2}. \tag{3.37}$$

We can use the same argument to Theorems 3.1–3.2 in [17] to prove the following lemma.

Lemma 3.6. *Let* $\mathbf{u}$ *and* σ *be defined in (3.2) and (3.3), respectively, then there hold*

$$||\mathbf{u} - \mathbf{u}^*||_1 + ||\sigma - \sigma^*||_0 \le C(\widehat{\eta}_{h_l}(\mathbf{u}^*, \sigma^*, \Omega) + \sum_{j=k}^{k+q-1} ||\lambda \mathbf{u}_j^0 - \lambda_{j,h_l} \mathbf{u}_{j,h_l}||_0), \tag{3.38}$$

$$\begin{aligned} C\widehat{\eta}_{h_l}(\mathbf{u}^*, \sigma^*, \kappa) \le{}& |\mathbf{u} - \mathbf{u}^*|_{1,\omega(\kappa)} + ||\sigma - \sigma^*||_{0,\omega(\kappa)} \\ &+ \sum_{j=k}^{k+q-1} \sum_{\kappa' \subset \omega(\kappa)} h_{\kappa'}^{1/2} ||\lambda \mathbf{u}_j^0 - \lambda_{j,h_l} \mathbf{u}_{j,h_l}||_{0,\kappa'}. \end{aligned} \tag{3.39}$$

Now replacing $\mathbf{u}_h$ and σ_h by $\mathbf{u}^{h_l}$ and σ^{h_l} in (3.28)–(3.30), respectively, we define the residuals for $\mathbf{u}^{h_l}$ and σ^{h_l} as follows:

$$R_{\kappa,1}(\mathbf{u}^{h_l}, \sigma^{h_l}) = \Delta \mathbf{u}^{h_l} - \nabla \sigma^{h_l} + \lambda^{h_l} \mathbf{u}^{h_l}, \tag{3.40}$$

$$R_{\kappa,2}(\mathbf{u}^{h_l}) = div\, \mathbf{u}^{h_l}, \tag{3.41}$$

$$R_{\partial\kappa}(\mathbf{u}^{h_l}, \sigma^{h_l}) = [[(\nabla \mathbf{u}^{h_l} - \sigma^{h_l} I) \cdot \mathbf{n}_\kappa]]|_{\partial\kappa}. \tag{3.42}$$

For $\kappa \in \pi_{h_l}$, define the local error indicator

$$\eta_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \kappa) = \left\{ h_\kappa^2 ||R_{\kappa,1}(\mathbf{u}^{h_l}, \sigma^{h_l})||_{0,\kappa}^2 + ||R_{\kappa,2}(\mathbf{u}^{h_l})||_{0,\kappa}^2 + \frac{1}{2} h_\kappa ||R_{\partial\kappa}(\mathbf{u}^{h_l}, \sigma^{h_l})||_{0,\partial\kappa}^2 \right\}^{1/2}.$$

The global error indicator is given by

$$\eta_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \Omega) = \left\{ \sum_{\kappa \in \pi_{h_l}} \eta_{h_l}^2(\mathbf{u}^{h_l}, \sigma^{h_l}, \kappa) \right\}^{1/2}. \tag{3.43}$$

Theorem 3.3. *Let* $\mathbf{u}$ *and* σ *be defined by (3.2) and (3.3), respectively. Under the conditions of Theorem 3.1, there hold*

$$||\mathbf{u} - \mathbf{u}^{h_l}||_1 + ||\sigma - \sigma^{h_l}||_0 \le C(\eta_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \Omega) + \delta_{h_l}^{2/t_l} + h_l^r \delta_{h_l}), \tag{3.44}$$

$$C\eta_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \kappa) \le |\mathbf{u} - \mathbf{u}^{h_l}|_{1,\omega(\kappa)} + ||\sigma - \sigma^{h_l}||_{0,\omega(\kappa)}$$

$$+\sum_{j=k}^{k+q-1}\sum_{\kappa'\subset\omega(\kappa)} h_\kappa^{1/2}(||\mathbf{u}_{j,h_l}-\mathbf{u}_j^0||_{0,\kappa}+\delta_{h_l}^2||\mathbf{u}_{j,h_l}||_{0,\kappa})$$
$$+||\mathbf{u}^*-\mathbf{u}^{h_l}||_{1,\omega(\kappa)}+||\sigma^*-\sigma^{h_l}||_{0,\omega(\kappa)}, \tag{3.45}$$
$$|\lambda^{h_l}-\lambda|\le C(\eta_{h_l}^2(\mathbf{u}^{h_l},\sigma^{h_l},\Omega)+\delta_{h_l}^{4/t_l}+h_l^{2r}\delta_{h_l}^2). \tag{3.46}$$

Proof. By the triangle inequality,

$$||\mathbf{u}-\mathbf{u}^{h_l}||_1+||\sigma-\sigma^{h_l}||_0\le ||\mathbf{u}-\mathbf{u}^*||_1+||\sigma-\sigma^*||_0$$
$$+||\mathbf{u}^*-\mathbf{u}^{h_l}||_1+||\sigma^*-\sigma^{h_l}||_0. \tag{3.47}$$

Using (3.38), from (3.47), we get

$$||\mathbf{u}-\mathbf{u}^{h_l}||_1+||\sigma-\sigma^{h_l}||_0 \le C(\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\Omega)+\sum_{j=k}^{k+q-1}||\lambda\mathbf{u}_j^0-\lambda_{j,h_l}\mathbf{u}_{j,h_l}||_0)$$
$$+||\mathbf{u}^*-\mathbf{u}^{h_l}||_1+||\sigma^*-\sigma^{h_l}||_0,$$
$$=C(\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\Omega)+\sum_{j=k}^{k+q-1}||\lambda\mathbf{u}_j^0-\lambda_{j,h_l}\mathbf{u}_{j,h_l}||_0)$$
$$+R_1+||\mathbf{u}^*-\mathbf{u}^{h_l}||_1+||\sigma^*-\sigma^{h_l}||_0, \tag{3.48}$$

where $R_1=C(\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\Omega)-\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\Omega))$.

By the triangle inequality,

$$|R_1|\le C\left(\sum_{\kappa\in\pi_{h_l}}(\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\kappa)-\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\kappa))^2\right)^{1/2}. \tag{3.49}$$

Using the triangle inequality,

$$|\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\kappa)-\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\kappa)| \le \big\{h_\kappa^2||\widehat{R}_{\kappa,1}(\mathbf{u}^*,\sigma^*)-R_{\kappa,1}(\mathbf{u}^{h_l},\sigma^{h_l})||_{0,\kappa}^2$$
$$+||\widehat{R}_{\kappa,2}(u^*,\sigma^*)-R_{\kappa,2}(\mathbf{u}^{h_l},\sigma^{h_l})||_{0,\kappa}^2$$
$$+\frac{1}{2}h_\kappa||\widehat{R}_{\partial\kappa}(u^*)-R_{\partial\kappa}(\mathbf{u}^{h_l})||_{0,\partial\kappa}^2\Big\}^{1/2}$$
$$\le\big\{h_\kappa^2||\Delta(\mathbf{u}^*-\mathbf{u}^{h_l})-\nabla(\sigma^*-\sigma^{h_l})||_{0,\kappa}^2$$
$$+h_\kappa^2||\lambda^{h_l}\mathbf{u}^{h_l}-\sum_{j=k}^{k+q-1}a(\mathbf{u}^{h_l},\mathbf{u}_{j,h_l})\lambda_{j,h_l}\mathbf{u}_{j,h_l}||_{0,\kappa}^2$$
$$+||\operatorname{div}(\mathbf{u}^*-\mathbf{u}^{h_l})||_{0,\kappa}^2$$
$$+\frac{1}{2}h_\kappa||[(\nabla(\mathbf{u}^*-\mathbf{u}^{h_l})-(\sigma^*-\sigma^{h_l})I)\cdot n_\kappa]||_{0,\partial\kappa}^2\Big\}^{1/2},$$

which together with the inverse estimates and by trace theorem yields

$$|\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\kappa)-\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\kappa)|\le C\bigg(|\mathbf{u}^*-\mathbf{u}^{h_l}|_{1,\kappa}+||\sigma^*-\sigma^{h_l}||_{0,\kappa}$$
$$+h_\kappa||\lambda^{h_l}\mathbf{u}^{h_l}-\sum_{j=k}^{k+q-1}a(u^{h_l},\mathbf{u}_{j,h_l})\lambda_{j,h_l}\mathbf{u}_{j,h_l}||_{0,\kappa}\bigg).$$

Since we have

$$\left|\left|\lambda^{h_l}\mathbf{u}^{h_l}-\sum_{j=k}^{k+q-1}a(\mathbf{u}^{h_l},\mathbf{u}_{j,h_l})\lambda_{j,h_l}\mathbf{u}_{j,h_l}\right|\right|_{0,\kappa}$$
$$\le||\lambda^{h_l}(\mathbf{u}^{h_l}-\mathbf{u}^*)||_{0,\kappa}+\left|\left|\sum_{j=k}^{k+q-1}a(\mathbf{u}^{h_l},\mathbf{u}_{j,h_l})(\lambda_{j,h_l}-\lambda^{h_l})\mathbf{u}_{j,h_l}\right|\right|_{0,\kappa}$$
$$\le C\left(||\mathbf{u}^{h_l}-\mathbf{u}^*||_{0,\kappa}+\sum_{j=k}^{k+q-1}|\lambda_{j,h_l}-\lambda^{h_l}|||\mathbf{u}_{j,h_l}||_{0,\kappa}\right),$$

thus by (2.17) and (3.7)

$$|\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\kappa)-\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\kappa)|\le C\bigg(||\mathbf{u}^*-\mathbf{u}^{h_l}||_{1,\kappa}+||\sigma^*-\sigma^{h_l}||_{0,\kappa}$$
$$+\sum_{j=k}^{k+q-1}h_\kappa\delta_{h_l}^2||\mathbf{u}_{j,h_l}||_{0,\kappa}\bigg). \tag{3.50}$$

Substituting the above formula into (3.49), from (2.17), (2.18), (3.15), and (3.16) we obtain

$$|R_1|\le C(||\mathbf{u}^*-\mathbf{u}^{h_l}||_1+||\sigma^*-\sigma^{h_l}||_0+\sum_{j=k}^{k+q-1}h_l\delta_{h_l}^2||\mathbf{u}_{j,h_l}||_0)\le C\delta_{h_l}^{2/t_l},$$

and

$$||\mathbf{u}^*-\mathbf{u}^{h_l}||_1+||\sigma^*-\sigma^{h_l}||_0+\sum_{j=k}^{k+q-1}||\lambda\mathbf{u}_j^0-\lambda_{j,h_l}\mathbf{u}_{j,h_l}||_0\ \le C(\delta_{h_l}^{2/t_l}+h_l^r\delta_{h_l}),$$

substituting the above two formulae into (3.48) yields (3.44).

From (3.39) we have

$$C\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\kappa)\le|\mathbf{u}-\mathbf{u}^{h_l}|_{1,\omega(\kappa)}+||\sigma-\sigma^{h_l}||_{0,\omega(\kappa)}$$
$$-C(\widehat{\eta}_{h_l}(\mathbf{u}^*,\sigma^*,\kappa)-\eta_{h_l}(\mathbf{u}^{h_l},\sigma^{h_l},\kappa))+|\mathbf{u}^*-\mathbf{u}^{h_l}|_{1,\omega(\kappa)}$$
$$+||\sigma^*-\sigma^{h_l}||_{0,\omega(\kappa)}+\sum_{j=k}^{k+q-1}\sum_{\kappa'\subset\omega(\kappa)}h_{\kappa'}^{1/2}||\lambda\mathbf{u}_j^0-\lambda_{j,h_l}\mathbf{u}_{j,h_l}||_{0,\kappa'}. \tag{3.51}$$

Using (2.17) and (3.7), we get

$$||\lambda_{j,h_l}\mathbf{u}_{j,h_l} - \lambda\mathbf{u}_j^0||_{0,\kappa'} \leq C(\delta_{h_l}^2||\mathbf{u}_{j,h_l}||_{0,\kappa'} + ||\mathbf{u}_{j,h_l} - \mathbf{u}_j^0||_{0,\kappa'}),$$

and substituting the above formula and (3.50) into (3.51) yields (3.45).

We deduce from (3.5)

$$|\lambda^{h_l} - \lambda| \leq C(||\mathbf{u}^{h_l} - \mathbf{u}||_a^2 + ||\mathbf{u}^{h_l} - \mathbf{u}||_a||\sigma - \sigma^{h_l}||_0),$$

which together with (3.44) yields (3.46). ■

Remark 3.1. From (3.44) and (3.46), we see $\eta_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \Omega)$ and $\eta_{h_l}^2(\mathbf{u}^{h_l}, \sigma^{h_l}, \Omega)$ are reliable indicators of (u^{h_l}, σ^{h_l}) and λ^{h_l}, respectively. From (3.15), (3.16), and (2.18) we obtain

$$||\mathbf{u}^* - \mathbf{u}^{h_l}||_1 + ||\sigma^* - \sigma^{h_l}||_0 + \sum_{j=k}^{k+q-1} ||\mathbf{u}_{j,h_l} - \mathbf{u}_j^0||_0 \leq C(\delta_{h_l}^{2/t_l} + h_l^r\delta_{h_l}),$$

therefore, from (3.45) we know $\eta_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \kappa)$ is a locally efficient indicator of (u^{h_l}, σ^{h_l}).

IV. ADAPTIVE ALGORITHMS AND NUMERICAL EXPERIMENTS

In this section, we mainly establish Adaptive algorithm 4.2 for multiscale discretization Scheme 3.1 and provide some related numerical results to support our theoretical results. In addition, we improve Adaptive algorithm 4.2 into Adaptive algorithm 4.3 in view of the higher accuracy of higher order finite elements.

A. Adaptive Algorithms

The following Algorithm 4.1 is fundamental (see e.g., the algorithm in [17]).

Algorithm 4.1.

Step 1. Pick any initial mesh π_{h_0} with mesh size h_0.
Step 2. Solve (2.4)–(2.5) on π_{h_0} for discrete solution $(\lambda_{h_0}, u_{h_0}, \sigma_{h_0})$.
Step 3. Let $l = 0$.
Step 4. Compute the local indicators $\tilde{\eta}_{h_l}(u_{h_l}, \sigma_{h_l}, \kappa)$.
Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by **Marking Strategy E1** and parameter θ.
Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by Procedure **REFINE**.
Step 7. Solve (2.4)–(2.5) on $\pi_{h_{l+1}}$ for discrete solution $(\lambda_{h_{l+1}}, u_{h_{l+1}}, \sigma_{h_{l+1}})$.
Step 8. Let $l = l + 1$ and go to **Step** 4.

Marking Strategy E1. Given parameter $0 < \theta < 1$:

Step 1. Construct a minimal subset $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by selecting some elements in π_{h_l} such that

$$\sum_{\kappa\in\widehat{\pi}_{h_l}} \tilde{\eta}_{h_l}^2(\mathbf{u}_{h_l}, \sigma_{h_l}, \kappa) \geq \theta\tilde{\eta}_{h_l}^2(\mathbf{u}_{h_l}, \sigma_{h_l}, \Omega).$$

Step 2. Mark all the elements in $\widehat{\pi}_{h_l}$.

Based on Theorem 3.3, we give the following adaptive algorithm by modifying Step 4 and 7 of Algorithm 4.1:

Algorithm 4.2.

Step 1. Pick any initial mesh π_{h_0} with mesh size h_0.
Step 2. Solve (2.4)–(2.5) on π_{h_0} for discrete solution $(\lambda^{h_0}, u^{h_0}, \sigma^{h_0})$.
Step 3. Let $l = 0$.
Step 4. Compute the local indicators $\eta_{h_l}(u^{h_l}, \sigma^{h_l}, \kappa)$.
Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by **Marking Strategy E2** and parameter θ.
Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by Procedure **REFINE**.
Step 7. Find $(\mathbf{u}', \sigma') \in U_{h_{l+1}} \times V_{h_{l+1}}$ such that

$$a(\mathbf{u}', \mathbf{v}) + b(v, \sigma') - \lambda^{h_l}(\mathbf{u}', \mathbf{v}) = (\mathbf{u}^{h_l}, \mathbf{v}), \quad \forall \mathbf{v} \in U_{h_{l+1}},$$
$$b(\mathbf{u}', q) = 0, \quad \forall q \in V_{h_{l+1}}.$$

Set $\mathbf{u}^{h_{l+1}} = \mathbf{u}'/||\mathbf{u}'||_a$, $\sigma^{h_{l+1}} = \sigma'/||\mathbf{u}'||_a$.
Compute the Rayleigh quotient

$$\lambda^{h_{l+1}} = \frac{a(\mathbf{u}^{h_{l+1}}, \mathbf{u}^{h_{l+1}})}{(\mathbf{u}^{h_{l+1}}, \mathbf{u}^{h_{l+1}})}.$$

Step 8. Let $l = l + 1$ and go to **Step** 4.

Marking Strategy E2. Given parameter $0 < \theta < 1$:

Step 1. Construct a minimal subset $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by selecting some elements in π_{h_l} such that

$$\sum_{\kappa \in \widehat{\pi}_{h_l}} \eta^2_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \kappa) \geq \theta \eta^2_{h_l}(\mathbf{u}^{h_l}, \sigma^{h_l}, \Omega).$$

Step 2. Mark all the elements in $\widehat{\pi}_{h_l}$.

Note that when $|\lambda^{h_l} - \lambda|$ is too small, the algebraic system in Step 7 of Algorithm 4.2 is almost singular. Although one has no difficulty in solving it numerically (see Lecture 27.4 in [26]), one should think of selecting a proper integer $l_0 \geq 0$ and set $\lambda^{h_l} = \lambda^{h_{l_0}}$ when $l \geq l_0$ in Step 7; the theoretical analysis and numerical experiment of the algorithm for this case will be our next work.

B. Numerical Experiments

In this section, for the model Stokes eigenvalue problem (2.1), we give some numerical examples to illustrate the efficiency of Adaptive Algorithm 4.2 with the triangular Mini element by comparing it with Algorithm 4.1.

Mini element was established by Arnold et al. in 1984 (see [41]).

Let π_h be a regular triangulation of Ω under the meaning of [31], and

$$S^h = \left\{v \in C(\overline{\Omega}) : v|_\kappa \in P_1, \kappa \in \pi_h\right\}, \quad S^h_0 = S^h \cap H^1_0(\Omega).$$

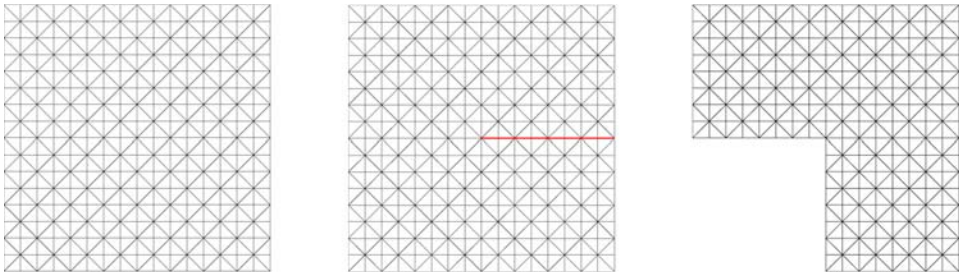

FIG. 1. Initial mesh on three domains.

For any $\kappa \in \pi_h$, let N_1, N_2 and N_3 be barycentric coordinates. Denote $B_h = \{v : v|_\kappa \in \text{span}\{N_1N_2N_3\}, \kappa \in \pi_h\}$, and let

$$U_h = (S_0^h \oplus B_h)^2, \qquad V_h = S^h \cap L_0^2(\Omega).$$

From [41], we know that Mini element satisfies inf-sup condition. From the interpolation theory in Sobolev space, we conclude that Condition (A1) is valid. Hence Scheme 3.1 and Theorem 3.2 are effective for Mini element.

We choose three two-dimensional domains to illustrate our algorithm: $\Omega = (-1,1)^2$, $\Omega = (-1,1)^2 \backslash \{0 \le x \le 1, y = 0\}$, and $\Omega = (-1,1)^2 \backslash [-1,0]^2$. Our program is compiled under the package of Chen. We set $\theta = 0.5$. In step 7 of Algorithm 4.1, we use internal command eigs in MATLAB to solve matrix eigenvalue problem. In step 7 of Algorithm 4.2, we use command $'\backslash'$ in MATLAB to solve the linear algebraic system. In numerical examples, for the sake of the unknown true eigenvalues, we compute the highly accurate values to replace them.

We use the following notations in our tables:

l: the lth iteration of Algorithms 4.1–4.2.
$\lambda_k^{h_l}$: the kth discrete eigenvalue from the lth iteration of Algorithms 4.2
λ_{k,h_l}: the kth discrete eigenvalue from the lth iteration of Algorithms 4.1

Figures 1–3 show the initial meshes on three domains and an adaptive mesh obtained by Algorithm 4.2. Figures 4–6 give the error indicator curves and the true error curves, corresponding to

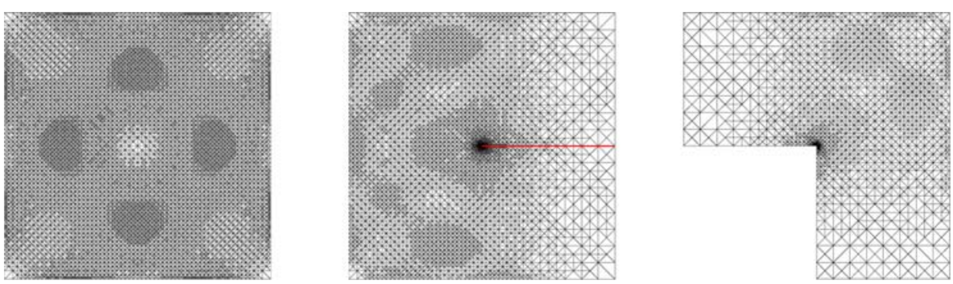

FIG. 2. Adaptive meshes for first eigenvalue by Algorithm 4.2. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

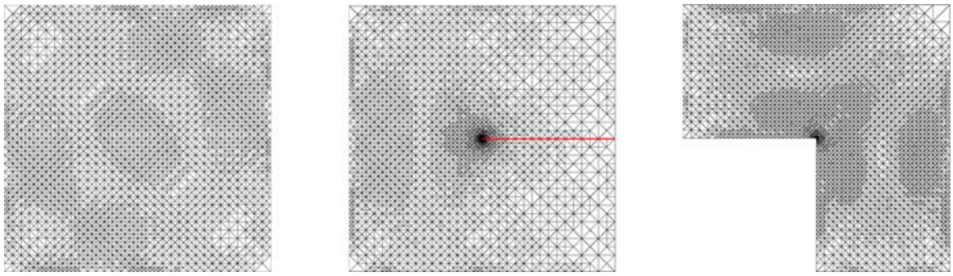

FIG. 3. Adaptive meshes for second eigenvalue by Algorithm 4.2. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

the smallest two eigenvalues obtained by Algorithms 4.1 and 4.2. In Figs. 4–6, for both Algorithms, the error indicator curves are almost parallel to the true error curves, and the error indicator curves almost coincide with each other, as well as the true error curves; all of these facts imply both error indicators are reliable and efficient and coincide with our theoretical results.

Next we compare Algorithm 4.2 with Algorithm 4.1 on accuracy and efficiency. Tables I–III lists the numerical results obtained by two algorithms on the three domains. All of the numerical results listed in Tables I–III indicate that, with less computational time and approximate number of degrees of freedom (DOF), Algorithm 4.2 can obtain almost the same numerical accuracy to Algorithm 4.1. In addition, Algorithm 4.2 can reach a higher numerical accuracy with more number of DOF, whereas Algorithm 4.1 could not proceed due to the limitation of computer memory ("-" in Tables I–III means that computations cannot be finished). Therefore, Algorithm 4.2 is more efficient.

C. A Shifted-Inverse Iteration Scheme Combining Mini Element and $P_2 - P_1$ Element

As we know, generally higher-order finite elements have higher accuracy than lower-order finite elements. Therefore, we can combine Mini element and $P_2 - P_1$ element to obtain a new Shifted-Inverse Iteration Scheme as follows.

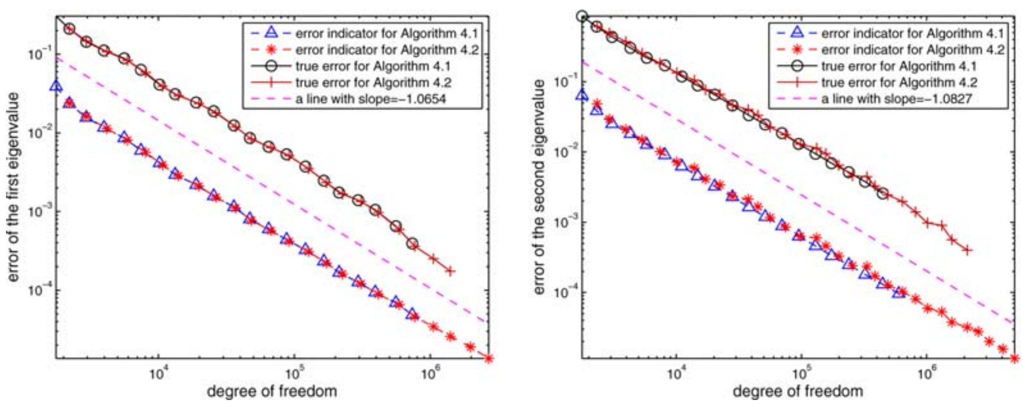

FIG. 4. Errors of smallest two eigenvalues by two Algorithms on $\Omega = (-1, 1)^2$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

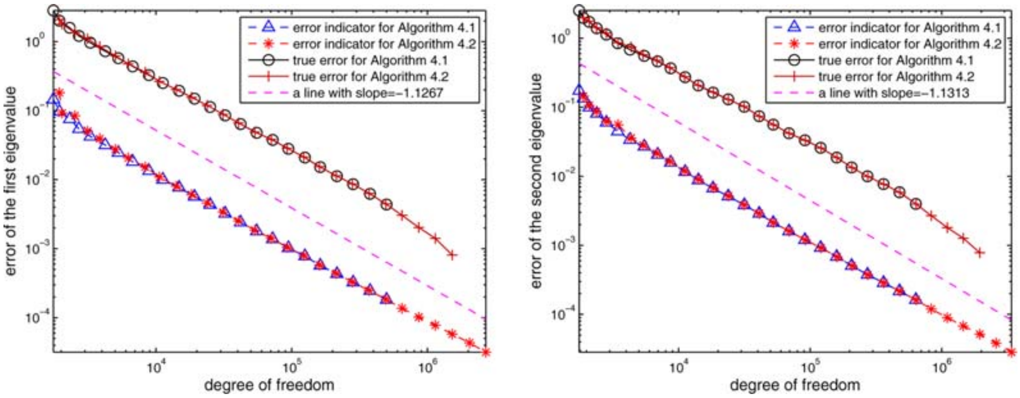

FIG. 5. Errors of smallest two eigenvalues by two Algorithms on $\Omega = (-1,1)^2 \setminus \{0 \le x \le 1, y = 0\}$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

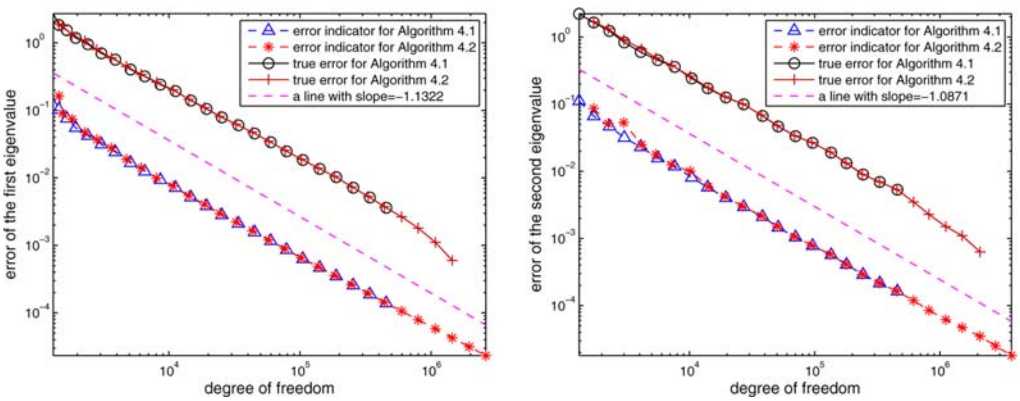

FIG. 6. Errors of smallest two eigenvalues by two Algorithms on $\Omega = (-1,1)^2 \setminus [-1,0]^2$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Algorithm 4.3. A Shifted-Inverse Iteration Scheme Combining Mini element and $P_2 - P_1$ element.

Step 1. Solve the eigenvalue problem (2.1) by Algorithm 4.2 with Mini elements for λ^{h_l}, u^{h_l} and an adaptive mesh π_{h_l}.

TABLE I. The numerical results on $\Omega = (-1,1)^2$

	Algorithm 4.2				Algorithm 4.1			
k	l	DOF	$\lambda_k^{h_l}$	cputime (s)	l	DOF	λ_{k,h_l}	cputime (s)
1	20	736,678	13.08673	98.1	20	736,652	13.08673	1559.7
1	23	1,852,929	13.08643	329.7	–	–	–	–
1	24	2,559,595	13.08634	820.6	–	–	–	–
2	21	452,736	23.03409	61.7	18	457,486	23.03385	350.8
2	28	2,675,404	23.03171	891.6	–	–	–	–
2	29	3,196,774	23.03152	1361.1	–	–	–	–

TABLE II. The numerical results on $\Omega = (-1,1)^2 \backslash \{0 \le x \le 1, y = 0\}$

	Algorithm 4.2				Algorithm 4.1			
k	l	DOF	$\lambda_k^{h_l}$	cputime (s)	l	DOF	λ_{k,h_l}	cputime (s)
1	23	500,202	29.92215	53.3	23	499,649	29.92217	264.6
1	28	2,063,023	29.91812	408.2	–	–	–	–
1	29	2,717,158	29.91779	811.7	–	–	–	–
2	25	640,003	32.59481	51.1	25	636793	32.59483	1105.9
2	30	2,593,509	32.59117	650.0	–	–	–	–
2	31	3,393,211	32.59085	1214.9	–	–	–	–

TABLE III. The numerical results on $\Omega = (-1,1)^2 \backslash [-1,0]^2$

	Algorithm 4.2				Algorithm 4.1			
k	L	DOF	$\lambda_k^{h_l}$	cputime (s)	l	DOF	λ_{k,h_l}	cputime (s)
1	23	450,345	32.13710	47.6	23	454781	32.13706	211.1
1	28	1,990,154	32.13369	311.3	–	–	–	–
1	29	2,658,631	32.13346	690.9	–	–	–	–
2	19	452,227	37.02433	32.3	19	457287	37.02427	146.6
2	25	2,770,585	37.01922	713.5	–	–	–	–
2	26	3,714,737	37.01896	1606.8	–	–	–	–

Step 2. Construct $P_2 - P_1$ element space $U'_{h_l} \times V_{h_l}$ on π_{h_l}. Solve an equation on $U'_{h_l} \times V_{h_l}$: find $(u', \sigma') \in U'_{h_l} \times V_{h_l}$ such that

$$a(\mathbf{u}', \mathbf{v}) + b(\mathbf{v}, \sigma') - \lambda^{h_l}(\mathbf{u}', \mathbf{v}) = (\mathbf{u}^{h_l}, \mathbf{v}), \quad \forall \mathbf{v} \in U'_{h_l},$$

$$b(\mathbf{u}', q) = 0, \quad \forall q \in V_{h_l}.$$

Set $\overline{\mathbf{u}}^{h_l} = \mathbf{u}'/||\mathbf{u}'||_a, \overline{\sigma}^{h_l} = \sigma'/||\mathbf{u}'||_a$.

Step 3. Compute the Rayleigh quotient $\overline{\lambda}^{h_l} = \frac{a(\overline{\mathbf{u}}^{h_l}, \overline{\mathbf{u}}^{h_l})}{(\overline{\mathbf{u}}^{h_l}, \overline{\mathbf{u}}^{h_l})}$.

We adopt Algorithm 4.3 to solve the Stokes eigenvalue problems (2.1). Numerical results show the high efficiency of Algorithm 4.3 (see Table IV). Note that in Table IV time23 denotes the total cpu time by Steps 2–3 of Algorithm 4.3.

TABLE IV. The numerical results by Algorithm 4.3 with $l = 19$

Domain	k	DOF	$\overline{\lambda}_k^{h_l}$	time23 (s)
$\Omega = (-1,1)^2$	1	716,037	13.08617	33.5
$\Omega = (-1,1)^2$	2	320,298	23.03110	15.9
$\Omega = (-1,1)^2 \backslash \{0 \le x \le 1, y = 0\}$	1	208,287	29.91692	9.9
$\Omega = (-1,1)^2 \backslash \{0 \le x \le 1, y = 0\}$	2	155,397	32.58815	7.4
$\Omega = (-1,1)^2 \backslash [-1,0]^2$	1	180,066	32.13243	9.4
$\Omega = (-1,1)^2 \backslash [-1,0]^2$	2	580,659	37.01834	29.5

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