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Short note

Convergence of a p-version/hp-version method for fractional differential equations

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ABSTRACT

Recently, M. Zayernouri and G.E. Karniadakis (2014) [10] proposed a new spectral method for fractional differential equations and observed an exponential rate of convergence. In this paper, we will prove a convergence rate of their spectral method and thus, provide an explanation for their observations.

1. Introduction

In this paper, we consider a new spectral/spectral element method proposed by M. Zayernouri and G.E. Karniadakis [10] for the Fractional Initial Value Problem (FIVP) of order $\nu \in (0, 1)$

$$\begin{cases} {}_0D_t^{\nu} u(t) = f(t), \quad t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{1}$$

where ${}_0D_t^{\nu}$ denotes the left-sided Riemann–Liouville fractional derivative defined as

$${}_0D_t^{\nu} u(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t \frac{u(s)ds}{(t-s)^{\nu}}.$$

The essential ingredient of their method is the selection of bases, which they named “Jacobi polyfractonomials”. These polyfractonomials are eigen-functions of a class of fractional Sturm–Liouville eigen-problems (FSLP) of two different kinds [9] and explicitly obtained as: for $x \in [-1, 1]$,

$${}^{(1)}P_n^{\alpha,\beta,\mu}(x) = (1+x)^{-\beta+\mu-1} P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(x), \quad \mu \in (0,1),\ \alpha \in (-1, 2-\mu),\ \beta \in (-1, \mu-1); \tag{2}$$

$${}^{(2)}P_n^{\alpha,\beta,\mu}(x) = (1-x)^{-\alpha+\mu-1} P_{n-1}^{-\alpha+\mu-1,\beta-\mu+1}(x), \quad \mu \in (0,1),\ \alpha \in (-1, 1-\mu),\ \beta \in (-1, 2\mu-1). \tag{3}$$

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Here $P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(x)$ is the standard Jacobi polynomials. In particular $\alpha=\beta=-1$ results in a simpler form

$$^{(1)}P_n^{\mu}(x)=(1+x)^{\mu}P_{n-1}^{-\mu,\mu}(x);$$
$$^{(2)}P_n^{\mu}(x)=(1-x)^{\mu}P_{n-1}^{\mu,-\mu}(x),\quad x\in[-1,1]. \tag{4}$$

Through the affine mapping $x=2t/T-1$, one easily obtains the shifted "Jacobi polyfractonomials" on $[0,T]$:

$$^{(1)}P_n^{\mu}(t)=\left(\frac{2}{T}\right)^{\mu}t^{\mu}P_{n-1}^{-\mu,\mu}(t),\qquad {}^{(2)}P_n^{\mu}(t)=\left(\frac{2}{T}\right)^{\mu}(T-t)^{\mu}P_{n-1}^{\mu,-\mu}(t).$$

These two types of "Jacobi polyfractonomials" are then carefully chosen as trial space bases or test space bases of spectral Petrov–Galerkin methods for (1). It is noteworthy that both types of bases have singularity at either $t=0$ or $t=T$. Therefore, the method is able to produce highly accurate approximation even if the true solution is non-smooth. Because of the unique feature, these bases have been also successfully employed in spectral collocation method [11] and in spectral method for high dimensional fractional PDEs [12]. Furthermore, these two types of "Jacobi fractonomials" have the following properties, which are directly used in the aforementioned paper [10].

$${}_0D_t^{\mu}\left({}^{(1)}P_n^{\mu}(t)\right)={}_tD_T^{\mu}\left({}^{(2)}P_n^{\mu}(t)\right)=\left(\frac{2}{T}\right)^{\mu}\frac{\Gamma(n+\mu)}{\Gamma(n)}P_{n-1}(t), \tag{5}$$

where $P_n(x)$ is the shifted Legendre polynomial on $[0,T]$.

In their algorithms, the following identity is equivalently crucial [5]. For $0<\xi<1$, $u\in H^1([0,T])$, and $w\in H^{\xi/2}([0,T])$,

$$\left({}_0D_t^{\xi}u,w\right)_{[0,T]}=\left({}_0D_t^{\xi/2}u,{}_tD_T^{\xi/2}w\right)_{[0,T]}. \tag{6}$$

Now, we are ready to state their algorithms.

Algorithm 1. Approximate u of the FIVP by u_N,

$$u_N:=\sum_{k=1}^{N}a_k\,{}^{(1)}P_k^{\mu}(t),\quad \mu=\nu/2. \tag{7}$$

Plugging (7) into (1), multiplying both sides by the test function ${}^{(2)}P_n^{\mu}(t)$ and applying (5), one obtains a diagonal system

$$\begin{aligned}&\sum_{k=1}^{N}a_k\left(\frac{2}{T}\right)^{2\mu}\left(\frac{\Gamma(k+\mu)}{\Gamma(k)}\right)\left(\frac{\Gamma(n+\mu)}{\Gamma(n)}\right)\int_0^T P_{k-1}\big(x(t)\big)P_{n-1}\big(x(t)\big)dt\\&=\sum_{k=1}^{N}a_k\left(\frac{2}{T}\right)^{2\mu-1}\left(\frac{\Gamma(k+\mu)}{\Gamma(k)}\right)^2\frac{2}{2k-1}\delta_{nk}\\&=\int_0^T f(t)\,{}^{(2)}P_n^{\mu}\big(x(t)\big)dt.\end{aligned} \tag{8}$$

Denote $\gamma_k=(\frac{2}{T})^{2\mu-1}(\frac{\Gamma(k+\mu)}{\Gamma(k)})^2\frac{2}{2k-1}$, then

$$a_k=\frac{1}{\gamma_k}\int_0^T f(t)\,{}^{(2)}P_k^{\mu}(t)dt. \tag{9}$$

For the non-homogeneous case when $u_0\neq0$, $u(t)$ can be split as

$$u(t)=u_H(t)+u_0,$$

where $u_H(t)$ is the solution of the following homogeneous equation

$${}_0D_t^{\nu}u_H(t)=f(t)-\frac{u_0}{\Gamma(1-\nu)t^{\nu}}.$$

Note that in the statement of this algorithm in [10], $(\frac{n+\mu}{n})$ and $(\frac{k+\mu}{k})$ should be replaced by $(\frac{\Gamma(n+\mu)}{\Gamma(n)})$ and $(\frac{\Gamma(k+\mu)}{\Gamma(k)})$ and other part of the algorithm associated should be also modified. However, the result of numerical experiments is correct. We point it out here for the benefit of future users of the algorithm.

Obviously, Algorithm 1 is a p-version method, which gains its accuracy by increasing the degree of basis. Based upon it, M. Zayernouri and G.E. Karniadakis [10] proposed discontinuous spectral element methods (DSEM).

Partition the domain $[0, T]$ into N_{el} non-overlapping elements $I_e = [t_{e-1/2}, t_{e+1/2}]$. Choose trial and test space

$$V_h^N = \left\{ v : v|_{I_e} \in P_N(I_e), e = 1, \cdots, N_{el} \right\}. \tag{10}$$

In particular, on each I_e, they approximate u by

$$u_N^e = \sum_{n=1}^{N} C_n P_{n-1}^{\eta,0}\big(x^e(t)\big), \tag{11}$$

project it onto $v^e(t) = P_{n-1}^{0,\chi}$ and yield

$$\begin{aligned} &\left({}_{t_{e-1/2}^+}D_t^{\nu/2} u_N^e(t), {}_tD_{t_{e+1/2}^-}^{\nu/2} v^e(t) \right)_{I_e} - \frac{v^e(t_{e+1/2}^-) h_e^{1-\nu}}{(1-\nu)\Gamma(1-\nu)} \left[\!\left[u_N^e(t_{e-1/2}) \right]\!\right] \\ &\quad = \big(f(t), v(t)\big) - H_e. \end{aligned} \tag{12}$$

Here, $\eta = \chi = \nu/2$ and h_e is the length of I_e and H_e is defined as

$$H_e = v^e(t) F_e(t) \Big|_{t_{e-1/2}^+}^{t_{e+1/2}^-} - \left(F_e(t), \frac{d}{dt} v^e(t) \right)_{I_e}, \tag{13}$$

where $F_e(t)$ is the *history* function associated with element I_e

$$F_e(t) = \sum_{i=1}^{e-1} \sum_{\delta=0}^{N} \tau_\delta (t-s)^{\delta+1-\nu} u_N^{(\delta)i}(s) \Big|_{t_{i-1/2}^+}^{t_{i+1/2}^-} \tag{14}$$

in which $\tau_\delta = -1/\Gamma(\delta+2-\nu)$, $\delta = 0, \cdots, N$ and $u_N^{(\delta)i}$ represents the δ-th derivative of the solution in I_i to be only evaluated at the boundaries of I_i. Plugging (11) into (12) and after some calculations, one obtains

Algorithm 2 *(DSEM: Multi-Element).* (See [10].) On each I_e, a linear system is obtained as

$$\mathbf{S}_e C_e = \mathbf{F}_e$$

where $\mathbf{S}_e$ denotes the $N \times N$ local stiffness matrix

$$\begin{aligned} \mathbf{S}_e[k,n] &= \frac{\Gamma(k+1)\Gamma(n+1)}{\Gamma(k-\nu/2+1)\Gamma(n-\nu/2+1)} \int_{I_e} (t-t_{e-1/2})^{-\nu/2}(t_{e+1/2}-t)^{-\nu/2} \\ &\quad \times P_n^{\nu,-\nu/2}\big(x_e(t)\big) P_k^{-\nu/2,\nu}\big(x_e(t)\big) dt + \frac{(-1)^{n+1} h_e^{1-\nu}}{(1-\nu)\Gamma(1-\nu)}, \\ \mathbf{F}_e[k] &= \int_{I_e} f(t) P_k^{0,\nu/2}\big(x_e(t)\big) dt - \frac{h_e^{1-\nu}(u_N^{e-1})^R}{(1-\nu)\Gamma(1-\nu)} - H_{e,k}, \end{aligned}$$

in which

$$H_{e,k} = F_e\big(t_{e+1/2}^-\big) + (-1)^{k+1} F_e\big(t_{e-1/2}^+\big) - \left(F_e(t), \frac{d}{dt} P_k^{0,\nu/2}\big(x_e(t)\big) \right). \tag{15}$$

All integrals in the algorithm are computed by the corresponding Gauss–Jacobi quadrature.

One easily observes that Algorithm 2 gains its accuracy by increasing its degree of basis as well as decreasing h_e. Hence, it is an hp-version method. We expect to obtain an hp-version rate of convergence. Here, we skip the (DSEM: Single-Element) in [10] because it is a special case of Algorithm 2. As for the Fractional Final Value Problem (FFVP), a similar analysis can be carried out and yields the same result.

Next, we will prove the optimal convergence rate of the p-version and hp-version method and explain why exponential rate of convergence can be observed in the numerical examples [10].

2. Convergence rate of the spectral methods in [10]

Denote by $x^*(x) := \langle x, x^* \rangle$ the value of bounded linear operator $x^* \subseteq X^*$ on x in a Hilbert space X. Choose two finite-dimensional spaces $X_n \subseteq X$ and $Y_n \subseteq X^*$ satisfying condition (H) [2]: for each $x \in X$, $y \in X^*$, there exist $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ and for any $y \in X^*$, we have $\|y_n - y\| \to 0$ as $n \to \infty$, and

$$\dim X_n = \dim Y_n, \quad n = 1, 2, \cdots. \tag{16}$$

Furthermore, we call $\{X_n, Y_n\}$ a *regular pair* [2] if there exists a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ and satisfy the conditions

$$(a) \quad \|x_n\| \le C_1 \langle x_n, \Pi_n x_n \rangle^{1/2} \quad \text{for all } x_n \in X_n; \tag{17a}$$

$$(b) \quad \|\Pi_n x_n\| \le C_2 \|X_n\| \quad \text{for all } x_n \in X_n. \tag{17b}$$

For any $x \in X$, $P_n x$ is a *generalized best approximation* from X_n to x with respect to Y_n if we have the identity

$$(x - P_n x, y_n) = 0, \quad \text{for any } y_n \in Y_n. \tag{18}$$

Then, we have the following two lemmas.

Lemma 2.1. *(See [2].) For each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if*

$$Y_n \cap X_n^{\perp} = \{0\}.$$

Under this condition, P_n is a projection.

Lemma 2.2. *(See [2].) Assume that $\{X_n, Y_n\}$ satisfies condition (H) and is a regular pair. Then, the following statements hold.*

(i) $\|P_n x - x\| \to 0$, *as $n \to \infty$ for all $x \in X$;*
(ii) *There exists a constant $C > 0$ such that $\|P_n\| \le C$ for all $n = 1, 2, \cdots$;*
(iii) $\|P_n x - x\| \le C\|Q_n x - x\|$ *for some constant $C > 0$ independent of n,*

where $Q_n x$ is the usual best approximation from X_n to x; that is, $Q_n x \in X_n$ satisfies the equation

$$\|x - Q_n x\| = \inf_{x_n \in X_n} \|x - x_n\|.$$

Let the interval $I = [a, b]$ with $h = b - a$ and define the shifted Jacobi weight function on I

$$w^{\alpha,\beta}(x) = \left(\frac{2}{h}\right)^{\alpha+\beta} (x - a)^{\beta} (b - x)^{\alpha}, \quad \alpha, \beta > -1. \tag{19}$$

Then for $r \ge 0$, we can define a non-uniformly weighted Sobolev space $H_*^r(I)$ with norm

$$\|v\|_{r,*} = \left(\left\| \sum_{k=0}^{r} \partial_x^k v \right\|_{w^{k,k}}^2 \right)^{1/2}. \tag{20}$$

Let $\pi_{N,I} : H_*^r(I) \to P_N(I)$ be the classical L^2 projection defined by

$$(\pi_{N,I} v - v, \phi) = 0, \quad \forall \phi \in P_N(I). \tag{21}$$

Then, we have [7] for $r \le N + 1$

$$\|v - \pi_{N,I} v\|_{L^2(I)} \le C \left(\frac{h}{N}\right)^r \left\|\partial_x^r v\right\|_{w^{r,r},I}. \tag{22}$$

Next, we construct a projection operator $\Pi_{N,I} : H_*^r(I) \to P_N(I)$

$$\Pi_{N,I} v = \int_a^t \pi_{N-1} v'(\tau) d\tau + v(a). \tag{23}$$

Clearly, we have $\Pi_{N,I} v(a) = v(a)$. Furthermore, the following lemma holds.

Lemma 2.3. *Let $v \in L^2(I)$, $\partial_x^r v \in L^2_{w^{r,r}}(I)$, $0 \le r \le N + 1$ and $\Pi_{N,I}$ be defined as in (23). Then*

$$\|\Pi_{N,I} v - v\|_{\mu,I} \le C \left(\frac{h}{N}\right)^{r-\mu} \left\|\partial_x^r v\right\|_{w^{r,r},I}. \tag{24}$$

Proof. See Appendix A. □

Let $H^s([0,T])$ be the Hilbert space with real $0<s<1$ defined by standard space interpolation [1]: $H^s([0,T]) = [H^1([0,T]), L^2([0,T])]_{1-s}$. Its norm and seminorm are denoted by $\|\cdot\|_{s,[0,T]}$ and $|\cdot|_{s,[0,T]}$, respectively. Moreover, for any $v \in H^1([0,T])$:

$$\|v\|_{s,[0,T]} \le \|v\|^s_{1,[0,T]} \|v\|^{1-s}_{L^2([0,T])}.$$

Denote spaces

$$X_N = \text{span}\left\{ t^\mu P_n^{-\mu,\mu}\left(\frac{2}{T}t-1\right) : n=0,\cdots,N-1\right\}, \tag{25}$$

$$Y_N = \text{span}\left\{ P_n\left(\frac{2}{T}t-1\right) : n=0,\cdots,N-1\right\}, \tag{26}$$

$$Z_N = \text{span}\left\{ (T-t)^\mu P_n^{\mu,-\mu}\left(\frac{2}{T}t-1\right) : n=0,\cdots,N-1\right\}. \tag{27}$$

It turns out that the p-version method (Algorithm 1) is a generalized best approximation problem in Y_N, which can be concluded in the following theorem.

Theorem 2.4. *Let $u(t)$ and $u_N(t)$ be the true solution of (1) and its approximation by the spectral method in [10] (Algorithm 1), respectively. Assume $\partial_x^{m+\nu/2}u(t) \in L^2_{w^{m,m}}([0,T])$. Then*

$$\left|(u-u_N)\right|_{\nu/2}\big([0,T]\big) \le CN^{-m}\left\|\partial_x^{m+\nu/2}u\right\|_{w^{m,m},([0,T])}, \tag{28}$$

where ν and f are the fractional index and the source term in (1).

Furthermore, if $u(t)$ satisfies condition R (e.g. analytic function): $\|u^{(k+\nu/2)}\|_{L^\infty[0,T]} \le ck!R^{-k}$, we have

$$\|u-u_N\|_{\nu/2}\big([0,T]\big) \le C\left(\frac{T}{4R}\right)^{N+1}; \tag{29}$$

if $u(t)$ satisfies condition M (e.g. a class of entire function): $\|u^{(k+\nu/2)}\|_{L^\infty[0,T]} \le cM^k$, $M>1$, then

$$\|u-u_N\|_{\nu/2}\big([0,T]\big) \le C\left(\frac{eMT}{4(N+1)}\right)^{N+1}. \tag{30}$$

Proof. Denote $\mu=\nu/2$ and $I=[0,T]$. Obviously, the spectral method is equivalent to finding $u_N \in X_N$ such that

$$\big({}_0D_t^\nu u_N, z\big) = (f,z), \quad z \in Z_N, \tag{31}$$

i.e. $({}_0D_t^\nu(u-u_N), z) = ({}_0D_t^\mu u - {}_0D_t^\mu u_N, {}_tD_T^\mu z) = 0$, which together with (5), indicates

$$\big({}_0D_t^\mu u - \tilde{u}_N, y\big) = 0, \quad y \in Y_N, \tag{32}$$

where $\tilde{u}_N = {}_0D_t^\mu u_N = \sum_{n=1}^N a_n(\frac{2}{T})^\mu \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}(x(t)) \in Y_N$. Hence, the spectral algorithm is the same as seeking an orthogonal projection in Y_N for ${}_0D_t^\mu u$.

Therefore, by the standard error of L^2 projection (cf. [6, Theorem 3.35]), we have

$$|u-u_N|_{\mu,[0,T]} \le CN^{-m}\left\|\partial_x^{\mu+m}u\right\|_{w^{m,m},[0,T]}. \tag{33}$$

Let $I_N w$ be the interpolation of w on $N+1$ Gauss–Lobatto points. From Lemma 2.2,

$$|u-u_N|_{\mu,[0,T]} \le C\left\|{}_0D_t^\mu u - I_N\, {}_0D_t^\mu u\right\|_{L^2(I)}. \tag{34}$$

Eqs. (29) and (30) are indicated by (34) and the following two inequalities [3,8]

$$\left\|{}_0D_t^\mu u - I_N\, {}_0D_t^\mu u\right\|_{L^\infty} \le C\left(\frac{T}{4R}\right)^{N+1} \quad \text{(Condition R)}; \tag{35}$$

$$\left\|{}_0D_t^\mu u - I_N\, {}_0D_t^\mu u\right\|_{L^\infty} \le C\left(\frac{eMT}{4(N+1)}\right)^{N+1} \quad \text{(Condition M)}. \quad \square \tag{36}$$

Now let us carry on the analysis of Algorithm 2. It is clearly a Galerkin method instead of a Petrov–Galerkin one.

Theorem 2.5. *Let $u(t)$ and $u_{h,N}(t)$ be the true solution of (1) and its approximation by the hp-version method in [10] (Algorithm 2), respectively. Assume $\partial_x^m u \in L^2_{w^{m,m}}([0,T])$ and $h = \max_e h_e$. Then*

$$\|u - u_{h,N}\|_{H^{\nu/2}([0,T])} \le C\left(\frac{h}{N}\right)^{m-\nu/2} \|\partial_x^m u\|_{m,w^{m,m},[0,T]}. \tag{37}$$

Proof. We start by rewriting Algorithm 2 in a bilinear form. The process can be found in the derivation of Algorithm 2 in [10]. For the completeness of the paper, we elaborate it here. For any $v \in V_N^h$,

$$\begin{aligned} B(u_N, v)_{I_e} &= \left({}_0D_t^\nu u_N, v(t)\right)_{I_e} \\ &= \left({}_{t^+_{e-1/2}}D_t^\nu u_N^e(t), v^e(t)\right)_{I_e} - \frac{v^e(t^-_{e+1/2})h_e^{1-\nu}}{(1-\nu)\Gamma(1-\nu)}[\![u_N^e(t_{e-1/2})]\!] \\ &\quad + \left(\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^{t^-_{e-1/2}} \frac{u(s)ds}{(t-s)^\nu}, v^e(t)\right)_{I_e} \\ &= \left({}_{t^+_{e-1/2}}D_t^{\nu/2} u_N^e(t), {}_tD^{\nu/2}_{t^-_{e+1/2}} v^e(t)\right)_{I_e} - \frac{v^e(t^-_{e+1/2})h_e^{1-\nu}}{(1-\nu)\Gamma(1-\nu)}[\![u_N^e(t_{e-1/2})]\!] \\ &\quad + H_e, \end{aligned} \tag{38}$$

in which

$$\begin{aligned} H_e &= \left(\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^{t^-_{e-1/2}} \frac{u_N(s)ds}{(t-s)^\nu}, v^e(t)\right)_{I_e} \\ &= \left(\sum_{i=1}^{e-1}\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_{I_i} \frac{u_N^i(s)ds}{(t-s)^\nu}, v^e(t)\right)_{I_e} \\ &= \left(v^e(t)\frac{1}{\Gamma(1-\nu)}\sum_{i=1}^{e-1}\int_{I_i}\frac{u_N^i(s)ds}{(t-s)^\nu}\right)\Bigg|_{t=t^+_{e-1/2}}^{t=t^-_{e+1/2}} \\ &\quad - \left(\frac{1}{\Gamma(1-\nu)}\sum_{i=1}^{e-1}\int_{I_i}\frac{u_N^i(s)ds}{(t-s)^\nu}, \frac{d}{dt}v^e(t)\right)_{I_e}, \end{aligned} \tag{39}$$

where u_N^i is the known solution that we solved on I_i. To reduce the double integral, M. Zayernouri and G.E. Karniadakis [10] perform integration by parts N times

$$\begin{aligned} \sum_{i=1}^{e-1}\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_{I_i}\frac{u_N^i(s)ds}{(t-s)^\nu} &= \sum_{i=1}^{e-1}\sum_{\delta=0}^{N}\tau_\delta(t-s)^{\delta+1-\nu}u_N^{(\delta)i}(s)\Big|_{s=t^+_{i-1/2}}^{s=t^-_{i+1/2}} \\ &= \sum_{i=1}^{e-1}F_e^i(t) \\ &= F_e(t). \end{aligned} \tag{40}$$

Hence, by choosing $v^e(t) = P_k^{0,\chi}(x^e(t))$, we can exactly obtain Algorithm 2. Furthermore, by collecting all $B(u_N, v)_{I_e}$, we obtain the expression of $B(u_N, v)$ on $[0,T]$.

Now, let us prove the boundedness and coercivity of $B(u,v)$ on $[0,T]$. For any $u, v \in H^{\nu/2}([0,T])$, by the equivalence of the norm [5]

$$B(u,v) = \left({}_0D_t^\nu u, v\right) = \left({}_0D_t^{\nu/2}u, {}_tD_T^{\nu/2}v\right) \le C\|u\|_{\nu/2,([0,T])}\|v\|_{\nu/2,([0,T])}. \tag{41}$$

On the other hand, choose $v = u$,

$$B(u,u) = \left({}_0D_t^{\nu/2}u, {}_tD_T^{\nu/2}u\right) \ge C\|u\|^2_{\nu/2,([0,T])}. \tag{42}$$

Clearly,

$$\|u\|_{\nu/2,[0,T]} \le C\|f\|_{L^2}. \tag{43}$$

Next, we construct a projection operator Π_N on I_e.

$$\Pi_N u = \int_{t_{e-1/2}}^{t} \pi_{N-1} u'(\tau) d\tau + u(t_{e-1/2}), \tag{44}$$

where π is the classical $L^2(I_e)$ orthogonal projection operator in I_e. Clearly, $\Pi_N u(t_{e-1/2}) = u(t_{e-1/2})$. Then, Lemma 2.3 implies

$$\|\Pi_N u - u\|_{\nu/2,I_e} \le C\left(\frac{h_e}{N}\right)^{m-\nu/2} \left\|\partial_x^m u\right\|_{W^{m,m},I_e}. \tag{45}$$

Hence, summing up I_e on both sides of the above equation, Lemma 2.2 yields

$$\|u - u_{h,N}\|_{\nu/2,([0,T])} \le \|u - \Pi_N u\|_{\nu/2([0,T])} \le C\left(\frac{h}{N}\right)^{m-\nu/2} \left\|\partial_x^m u\right\|_{W^{m,m},[0,T]}. \quad \square \tag{46}$$

Remark 1. In Algorithm 1, ${}_0D_t^{\mu} u$ is approximated by polynomials spanned by $\{P_n(x)\}_{n=0}^N$, whereas in Algorithm 2, the true solution u is approximated by piecewise polynomials.

3. Numerical examples

Both p-version and hp-version convergence rates can be easily observed from examples of [10] by applying Algorithm 1 or Algorithm 2. In this section, we will first test the convergence rate if the true solution of (1) satisfies condition R or condition M for Algorithm 1. One example, whose true solution satisfies condition M, is selected from [10], the other one with true solution satisfying condition R, is proposed by ourselves; Secondly, we also consider an example for Algorithm 2 on the improvement of accuracy for h refinement and p refinement, respectively. In the following examples, a general hypergeometric series with p upper parameters and q lower parameters is used and defined as follows:

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where $(a)_k$ is the Pochhammer symbol

$$(a)_k = \begin{cases} 1, & k = 0, \\ a(a+1)\cdots(a+k-1), & k = 1, 2, \cdots. \end{cases}$$

Example 1. Recall that the equation we consider is of the form,

$$\begin{cases} {}_0D_t^{\nu} u(t) = f(t), & t \in (0, T], \\ u(0) = u_0. \end{cases} \tag{47}$$

As in [10], we choose $u(t) = t^6 \sin(\pi t)$ as the exact solution on [0, 1]. Clearly, it is an entire function satisfying condition M with $M = 2\pi$. Theorem 2.4 implies the quotient of the obtained $H^{\nu/2}$-error and $(\frac{2e\pi}{2(N+1)})^{N+1}$ approaches a constant for different N. Indeed, we observe such a pattern in the numerical experiment, see the left part of Fig. 1.

Example 2. In this example, we test the above equation with $f(t) = 2/\Gamma(8/3)t^{5/3}\,{}_2F_1(-32/3, 3; 3-\nu; -t)$ such that the true solution $u(t) = t^2(1+t)^{32/3}$. Obviously, $u(t)$ satisfies condition R with $R = 1$. Hence, in this case, we plot the quotient of $H^{\nu/2}$-error of the algorithm and $(\frac{1}{4})^{N+1}$, which is depicted in the right part of Fig. 1.

Example 3. This example is designed to test the accuracy of Algorithm 2. In (1), we choose a proper $f(t)$ such that the true solution is $u(t) = t^{8+5/11}$. According to Theorem 2.5, the approximation is refined as the length of time interval h decreases when the polynomial order N is chosen and as N increases when h is fixed. Since a complicated history term is rooted in ${}_0D^{\nu/2}u_{h,N}$, we consider the L^2 norm of error for simplicity. It is noteworthy that errors in [10] are measured in L^2 norm too. One can easily observe from the log-log plot of the error estimate that when the degree of polynomial is 1, the error decreases with an algebraic rate of $\mathcal{O}(h^{1.5})$ for $\nu = 9/10$ and $\mathcal{O}(h^2)$ for $\nu = 1/10$, respectively. When we set the degree of polynomial on each interval to be 3, the algebraic rate is refined to $\mathcal{O}(h^{3.2})$ and $\mathcal{O}(h^4)$ corresponding to $\nu = 9/10$ and 1/10 respectively, see Fig. 2. However, if h is fixed, we observe a more rapidly decreased error as the polynomial order increases from Fig. 3.

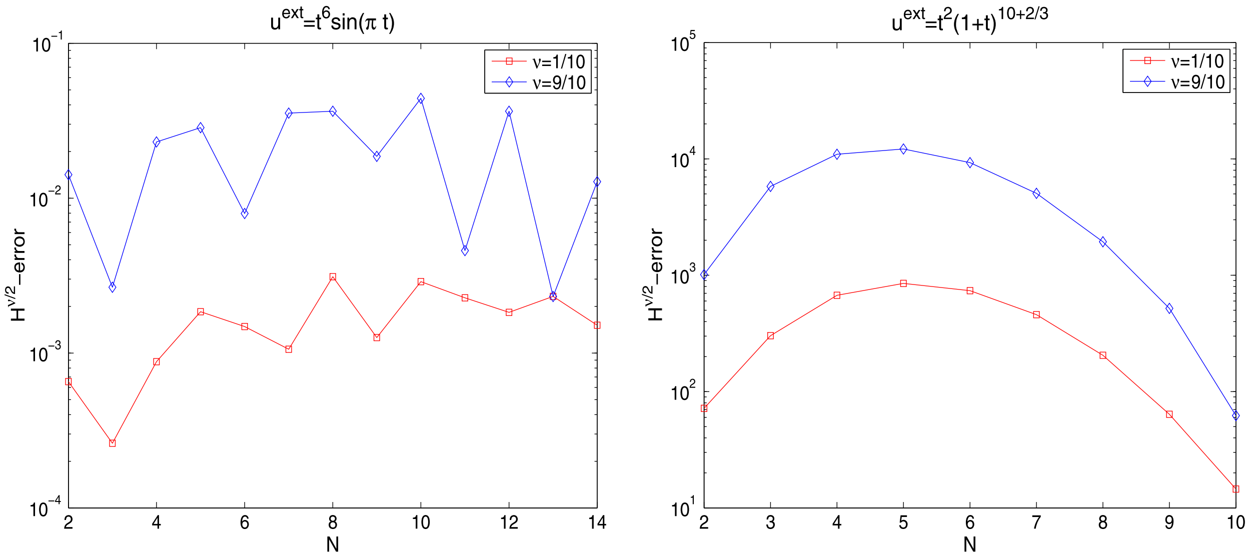

Fig. 1. Left: Quotient of $H^{\nu/2}$-error and the error bound $(\frac{2e\pi}{2(N+1)})^{N+1}$ for $u(t)=t^6\sin(\pi t)$; Right: Quotient of $H^{\nu/2}$-error and the error bound $(\frac{1}{4})^{N+1}$ for $u(t)=t^2(1+t)^{10+2/3}$.

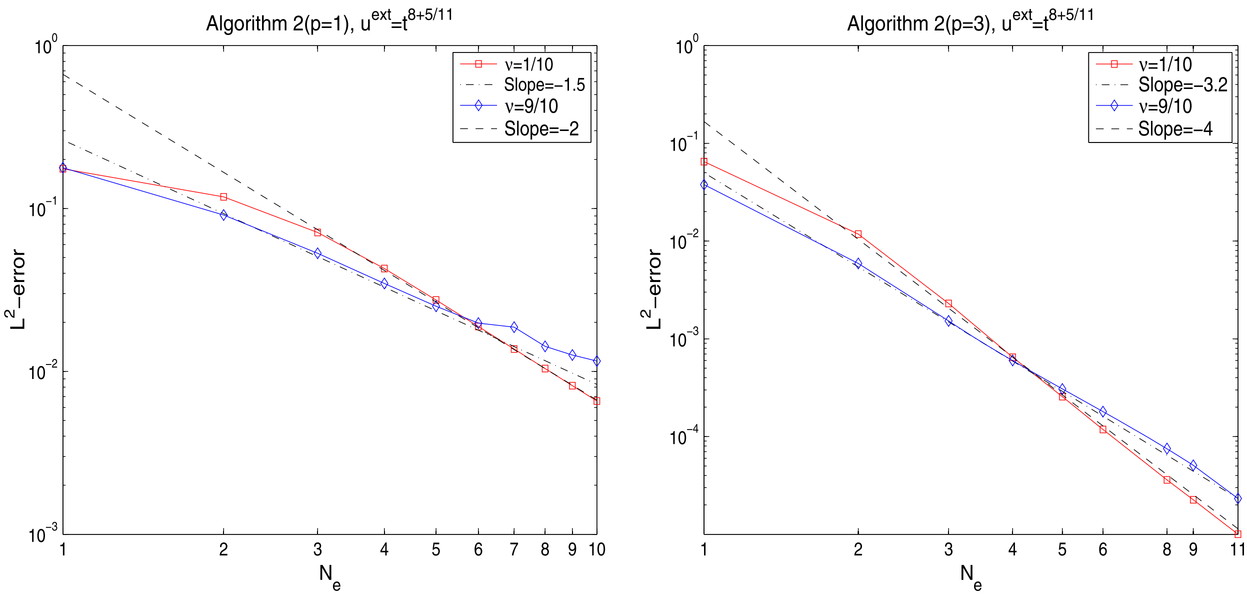

Fig. 2. L^2-error of Algorithm 2 for ${}_0D_t^{\nu}u(t)=f(t)$, $t\in[0,1]$, corresponding to $\nu=1/10$ and $\nu=9/10$ versus the number of elements. Left: $p=1$; Right: $p=3$.

4. Conclusion

In this paper, we have proved the convergence rate of the Petrov–Galerkin type p-version method and the Galerkin type hp-version method proposed in [10] for fractional differential equation. Numerical experiments confirm the validity of the convergence rate. In the future, we are going to apply these algorithms to solve fractional partial differential equations with solution of limited regularity, such as time fractional differential equations and fractional Laplace equations.

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Appendix A. Proof of Lemma 2.3

We need to introduce some preliminary results first. Recall that $I=[a,b]$ with interval length h.

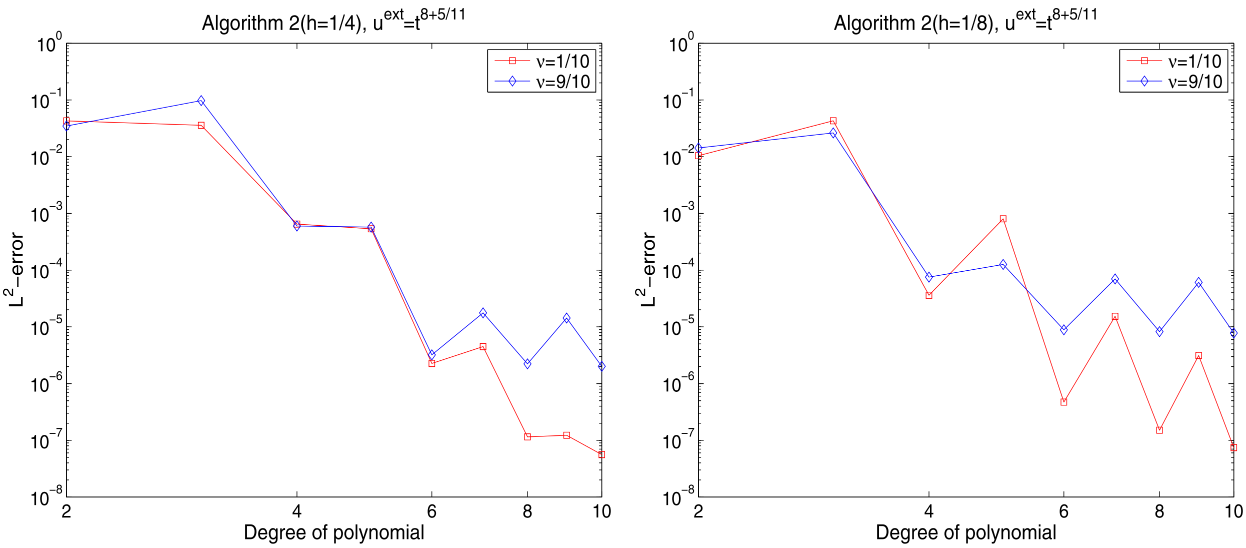

Fig. 3. L^2-error of Algorithm 2 for ${}_0D_t^{\nu}u(t) = f(t)$, $t \in [0, 1]$, corresponding to $\nu = 1/10$ and $\nu = 9/10$ versus the polynomial order. Left: $h = 1/4$; Right: $h = 1/8$.

Lemma A.1. *(See [4, Lemma 2.4].) If*

$$\alpha \le \gamma + 2, \quad \beta \le 0, \quad \delta \ge 0, \tag{A.1}$$

or

$$\alpha \le \gamma + 1, \quad \beta \le \delta + 2, \quad 0 < \alpha < 1, \quad \beta < 1, \tag{A.2}$$

then for any $v \in H^1_{w^{\alpha,\beta}}(-1, 1)$ *with* $v(-1) = 0$,

$$\|v\|_{w^{\gamma,\delta}} \le \eta_{\alpha,\beta,\gamma,\delta}|v|_{1,w^{\alpha,\beta}}, \tag{A.3}$$

where $\eta_{\alpha,\beta,\gamma,\delta}$ *is a constant that depends only on* α, β, γ, δ.

Lemma A.2. *Let* $v \in L^2(I)$, $\partial_x^r v \in L^2_{w^{r,r}}(I)$, $0 \le r \le N + 1$ *and* $\Pi_{N,I}$ *be defined as in* (23). *Then*

$$\|\Pi_N v - v\|_{\mu,I} \le C\left(\frac{h}{N}\right)^{r-\mu} \left\|\partial_x^r v\right\|_{w^{r,r},I}. \tag{A.4}$$

Proof. The proof follows the lines of Theorem 3.2 and Theorem 3.1 in [4]. For completeness of the paper, we present it here.

Recall that

$$\Pi_N v = \int_a^t \pi_{N-1} v'(\tau)d\tau + v(a). \tag{A.5}$$

Hence, by (22) and Lemma A.1,

$$\begin{aligned}\|\Pi_N v - v\|_{1,I} &\le (1 + \eta_{0,0,0,0})|\Pi_{N-1} v - v|_{1,I} \\ &\le (1 + \eta_{0,0,0,0})\left\|\pi_{N-1} v' - v'\right\|_I \\ &\le C\left(\frac{h}{N}\right)^{r-1} |v|_{r,w^{r,r},I}.\end{aligned} \tag{A.6}$$

Now, we consider the error in L^2 norm. For $g \in L^2(I)$, construct an auxiliary problem

$$(g, z) = a(w, z) := \left(w', z'\right) + (w, z), \quad \forall z \in H^1(I). \tag{A.7}$$

Choosing $z = w$ leads to

$$\|w\|_{1,I} \le \|g\|_I. \tag{A.8}$$

Furthermore, in the sense of distribution, we have

$$-\partial_x^2 w(x) = g(x) - w(x). \tag{A.9}$$

Therefore, by (A.8)

$$\begin{aligned}\left\|\partial_x^2 w(x-a)(b-x)\right\|_I^2 &= \int_I \big(g(x)-w(x)\big)^2 (x-a)^2(b-x)^2 dx \\ &\le C\big(\|g\|_I^2 + \|w\|_I^2\big) \\ &\le C\|g\|_I^2. \end{aligned} \tag{A.10}$$

Then, (A.6) and (A.10) imply

$$\begin{aligned}\|\Pi_N w - w\|_{1,I} &\le C\left(\frac{h}{N}\right)\left\|\partial_x^2 w\right\|_{w^{2,2},I} \\ &\le C\left(\frac{h}{N}\right)\|g\|_I. \end{aligned} \tag{A.11}$$

Next, we take $z = \Pi_N v - v$ in the auxiliary problem,

$$\begin{aligned}(\Pi_N v - v, g) &\le \left|a(w - \Pi_N w, \Pi_N v - v)\right| \\ &\le C\left(\frac{h}{N}\right)\|g\|_I \left(\frac{h}{N}\right)^{r-1} |v|_{r,w^{r,r},I} \\ &\le C\left(\frac{h}{N}\right)^{r} \|g\|_I |v|_{r,w^{r,r},I}. \end{aligned} \tag{A.12}$$

It immediately leads to

$$\begin{aligned}\|\Pi_N v - v\|_I &= \sup_{g\in L^2(I)} \frac{|(\Pi_N v - v, g)|}{\|g\|_I} \\ &\le C\left(\frac{h}{N}\right)^{r} |v|_{r,w^{r,r},I}. \end{aligned} \tag{A.13}$$

By the definition of space interpolation, $H^\mu(I) = [H^1(I), L^2(I)]_{1-\mu}$ with the bound

$$\|z\|_{\mu,I} \le \|z\|_{1,I}^{\mu} \|z\|_I^{1-\mu}, \quad \forall z \in H^1(I). \tag{A.14}$$

Let $z = \Pi_{N,I} v - v$,

$$\begin{aligned}\|\Pi_N v - v\|_{\mu,I} &\le \|\Pi_N v - v\|_{1,I}^{\mu} \|\Pi_N v - v\|_I^{1-\mu} \\ &\le C\left(\frac{h}{N}\right)^{(r-1)\mu + r(1-\mu)} |v|_{r,w^{r,r},I} \\ &\le C\left(\frac{h}{N}\right)^{r-\mu} |v|_{r,w^{r,r},I}, \end{aligned} \tag{A.15}$$

which is the desired result. □

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