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On the Spectrum Computation of Non-oscillatory and Highly Oscillatory Kernel with Weak Singularity

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Abstract We compute the spectra of integral compact operators with weak singularity. Jacobi-spectral collocation methods are applied for problems without high oscillation. A convergence rate is obtained for general non-oscillatory operators. Furthermore, if the bilinear form associated with the kernel is positive definite, the convergence rate is doubled. A spectral Galerkin method with modified Fourier expansion is developed to compute the spectra of highly oscillatory kernel. Numerical results are presented to demonstrate the effectiveness and accuracy of our algorithms and theorems.

1 Introduction

In this paper, we consider a class of non-oscillatory problem

$$Ty := \int_0^1 g(t,s)\frac{y(s)}{|t-s|^\nu}ds = \lambda y(t),\ t \in [0,1],\ 0 < \nu < 1, \tag{1.1}$$

where $g(t,s)$ is a smooth non-oscillatory function and a class of highly oscillatory problem

$$Ty := \int_{-1}^1 K(t,s)y(s)\frac{e^{i\omega|t-s|}}{|t-s|^\nu}ds = \lambda y(t), \quad -1 \le t < 1, \tag{1.2}$$

where $0 < \nu < 1$, and $\omega \gg 1$, which is a part of the open problem in [4].

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Clearly, T is compact from $C[0, 1]$ to itself for both cases [34]. Hence, its spectrum $\sigma(T)$ is a point one with the origin as its only possible cluster point. Weak singularity is an essential obstacle for the problems, which makes their usual numerical approximation inefficient. Hence, for (1.1), graded mesh is widely applied to obtain numerical schemes with an optimal order of convergence [1,11,23]. In [7], a spectral Jacobi-collocation method was proposed and analyzed to solve Volterra equations with weak singularity. The basic idea is to collocate equations at a certain type of Jacobi points and use a highly accurate quadrature to approximate the integration with weak singularity. Inspired by this idea, we investigated the eigenvalue problem (1.1) with $g(t, s) = 1$ in [15], where doubled convergence rates for eigenvalues are observed without a proof. This paper, however, will develop an efficient algorithm and prove the similar observation for any smooth g. For an extensive introduction of spectral collocation methods for Volterra equations, readers are referred to [3].

For highly oscillatory problems, a recent report [17] provides us several oscillators: regular oscillator $e^{i\omega x}$, irregular oscillator $e^{i\omega g(x)}$, singular oscillator $e^{i\omega x}/|x-y|^{\nu}$, Bessel oscillator $J_\nu(\omega x)$ and Airy oscillator $Ai(-\omega g(x))$. In particular, the numerical quadrature of irregular oscillator witnesses a blossom of its numerical algorithms. These methods can be categorized into Filon-type methods and Levin-type methods. A Filon-type method approximates the integrand $f(x)$ by a polynomial and computes the associated moments [18–20,29,30]. In particular, in [20], an algorithm based upon Hermite interpolation of integrand on both ending points and stationary points of $g(x)$ is proposed by virtue of an asymptotic expansion for the integral. It motivates many efforts in the field [28–30,36] and also this work. A Levin-type method, on the contrary, converts the integral problem into an equivalent ordinary differential equation and thus avoids the computation of moments [24,25]. Eigenvalue computation for such an oscillator is explored in [4,5]. Besides these two categories, steepest descent method [16] provides an alternative by constructing a path of integration in the complex plane so that the oscillations are removed. Efficient numerical quadratures for the Bessel oscillator $J_\nu(\omega x)$, $\nu \in R$ can be found in [25,28,36]. A Filon-type method for the generalized Bessel oscillator $J_\nu(\omega g(x))$ and Airy oscillator $Ai(-\omega g(x))$ is presented in [37].

Despite of extensive study of the subject, explicit numerical quadrature for singular oscillator in the report is rare and this paper tries to fill the gap. Obviously, the difficulty resides in the combination of weakly singular component and high oscillation component of the operator. Nevertheless, we obtain a similar result as that in [20] and it confirms the comment in the report that "*singularities play similar role to stationary points*". Naturally, we extend the idea to the Jacobi-spectral computation of (1.2). Unfortunately, it fails for the following reasons. Firstly, it requires a set of large number of Jacobi points to combat the oscillation. In this case, round-off error of the algorithm in Sect. 3.2 accumulates. Secondly, the iterative relation (3.3) implies that the (m, n)-th entry of the matrix whose eigenvalues we seek decays very slowly as $m, n \to \infty$. Finally, the iterative relation causes error propaganda if the degree of polynomial, which is an approximation to eigenfunction, is larger than ω. Without the relation, it is very time-consuming to compute such an abundance of hypergeometric functions. Therefore, we appeal to the spectral Galerkin method hinged on the modified Fourier expansion [21]. This expansion facilitates our work in two aspects. On the one hand, unlike traditional Fourier expansion, it can approximate analytic and non-periodic functions pointwisely; On the other hand, it simplifies the computation of matrix elements induced by spectral Galerkin method as we will see in Sect. 3.

This rest of this paper is organized as follows: In Sect. 2, we formulate our algorithm for (1.1) and provide a convergence analysis. Furthermore, we prove that if the associated bilinear form is positive definite, the convergence rate is doubled. One numerical experiment is included in this section to confirm the result. In Sect. 3, we first derive the numerical quadrature for the singular oscillator and provide a convergence analysis and two numerical experiments. Then, we illustrate an algorithm for the eigenvalue problem (1.2). Finally, some conclusions are presented in Sect. 4.

Throughout the paper, C stands for a generic constant that is independent of the number of collocation points $N+1$ and frequency ω but may depend on function g or K and singular index ν.

2 Eigenvalues of Weakly Singular Kernel Without Oscillation

This section explores Jacobi-collocation methods for (1.1).

2.1 Jacobi-Collocation Methods

Let $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, for $\alpha, \beta > -1, -1 < x < 1$ be the weight function for the associated Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$, which forms an orthogonal $L^2_{w^{(\alpha,\beta)}}(-1,1)$ system [31], which is a weighted space defined by

$$L^2_{w^{(\alpha,\beta)}}(-1,1) = \{v : (-1,1) \to R \big| \ v \text{ is measurable and } \|v\|_{w^{(\alpha,\beta)}} < \infty\},$$

where

$$\|v\|_{w^{(\alpha,\beta)}} = \left(\int_{-1}^{1} |v(x)|^2 w^{(\alpha,\beta)}(x)dx\right)^{\frac{1}{2}}$$

is the norm induced by the inner product

$$(u,v)_{w^{(\alpha,\beta)}} = \int_{-1}^{1} u(x)v(x)w^{(\alpha,\beta)}(x)dx.$$

For a given integer N, we denote the collocation points regarding to $w^{(\alpha,\beta)}(x)$ by $\{x_i\}_{i=0}^{N}$, the set of $(N+1)$ Jacobi Gauss, or Jacobi–Gauss–Radau, or Jacobi–Gauss–Lobatto points, and by $\{w_i\}_{i=0}^{N}$ the associated weights. Now let us change the form for (1.1) before collocation. By a linear transformation $t = (1+x)/2$, the equation can be written as

$$\int_{0}^{(1+x)/2} g\left(\frac{1+x}{2}, s\right)\left(\frac{1+x}{2} - s\right)^{-\nu} y(s)ds + \int_{(1+x)/2}^{1} g\left(\frac{1+x}{2}, s\right)\left(s - \frac{1+x}{2}\right)^{-\nu} y(s)ds = \lambda u(x),$$

where $x \in [-1,1]$ and $u(x) = y\left(\frac{1+x}{2}\right)$. We make a change of variable $s = (1+\tau)/2$ for the above equation,

$$\left(\frac{1}{2}\right)^{1-\nu} \int_{-1}^{x} \frac{\tilde{g}(x,\tau)}{(x-\tau)^{\nu}} u(\tau)d\tau + \left(\frac{1}{2}\right)^{1-\nu} \int_{x}^{1} \frac{\tilde{g}(x,\tau)}{(\tau-x)^{\nu}} u(\tau)d\tau = \lambda u(x), \tag{2.1}$$

where $\tilde{g}(x,\tau) = g((1+x)/2, (1+\tau)/2)$.

Let $u_N(x) = \sum_{j=0}^{N} c_j P_j^{(\alpha,\beta)}(x)$ be the approximation of $u(x)$. Obviously, c_j's satisfy the equation

$$\left(\frac{1}{2}\right)^{1-\nu} \sum_{j=0}^{N} c_j \int_{-1}^{x_i} \frac{\tilde{g}(x_i,\tau) P_j^{(\alpha,\beta)}(\tau)}{(x_i-\tau)^{\nu}} d\tau + \left(\frac{1}{2}\right)^{1-\nu} \sum_{j=0}^{N} c_j \int_{x_i}^{1} \frac{\tilde{g}(x_i,\tau) P_j^{(\alpha,\beta)}(\tau)}{(\tau-x_i)^{\nu}} d\tau$$
$$= \lambda_N \sum_{j=0}^{N} c_j P_j^{(\alpha,\beta)}(x_i). \tag{2.2}$$

To proceed, we transfer the integral interval into $[-1, 1]$. As a consequence, we write

$$\int_{-1}^{x_i} \frac{\tilde{g}(x_i,\tau) P_j^{(\alpha,\beta)}(\tau)}{(x_i-\tau)^{\nu}} d\tau = \left(\frac{1+x_i}{2}\right)^{1-\nu} \int_{-1}^{1} \frac{\tilde{g}(x_i,\tau_i(\theta)) P_j^{(\alpha,\beta)}(\tau_i(\theta))}{(1-\theta)^{\nu}} d\theta, \tag{2.3}$$

where $\tau_i(\theta) = \frac{1+x_i}{2}\theta + \frac{x_i-1}{2}$, $\theta \in [-1, 1]$. Next, we use $(N+1)$-Jacobi quadrature to approximate the integral in (2.3).

$$\int_{-1}^{1} \frac{\tilde{g}(x_i,\tau_i(\theta)) P_j^{(\alpha,\beta)}(\tau_i(\theta))}{(1-\theta)^{\nu}} d\theta \approx \sum_{k=0}^{N} w_k \tilde{g}(x_i,\tau_i(\theta_k)) P_j^{(\alpha,\beta)}(\tau_i(\theta_k)) := K_{ij},$$

where $\{\theta_k\}_{k=0}^{N}$ and $\{w_k\}_{k=0}^{N}$ associate with weight $w^{(-\nu,0)}(x)$. Similarly,

$$\int_{xi}^{1} \frac{\tilde{g}(x_i,\tau) P_j^{(\alpha,\beta)}(\tau)}{(\tau-x_i)^{\nu}} d\tau \approx \left(\frac{1-x_i}{2}\right)^{1-\nu} \sum_{k=0}^{N} \bar{w}_k \tilde{g}(x_i,\bar{\tau}_i(\bar{\theta}_k)) P_j^{(\alpha,\beta)}(\bar{\tau}_i(\bar{\theta}_k))$$
$$:= \left(\frac{1-x_i}{2}\right)^{1-\nu} \bar{K}_{ij},$$

where $\{\bar{\theta}_k\}_{k=0}^{N}$ and $\{\bar{w}_k\}_{k=0}^{N}$ associate with weight $w^{(0,-\nu)}(x)$, and

$$\bar{\tau}_i(\bar{\theta}) = \frac{1-x_i}{2}\bar{\theta} + \frac{x_i+1}{2}, \quad \bar{\theta} \in [-1, 1].$$

Hence, we obtain the generalized eigenvalue problem. For $i = 0, \cdots, N$,

$$\sum_{j=0}^{N} c_j \left[\left(\frac{1+x_i}{4}\right)^{1-\nu} K_{ij} + \left(\frac{1-x_i}{4}\right)^{1-\nu} \bar{K}_{ij} \right] = \lambda_N \sum_{j=0}^{N} c_j P_j^{(\alpha,\beta)}(x_i). \tag{2.4}$$

2.2 Preliminaries

Let $T : X \to X$ be a compact linear operator on a Banach space X and $\sigma(T)$ and $\rho(T)$ be the spectrum and resolvent of T respectively. Let λ be a nonzero eigenvalue of T with multiplicity m and let Γ be a circle centered at λ which lies in $\rho(T)$ and which encloses no other points in $\sigma(T)$. Then, the spectral projection associated with T and λ is defined by

$$E = -\frac{1}{2\pi i} \int_{\Gamma} (T - zI)^{-1} dz$$

and $\max_{z\in\nu} \|(T-zI)^{-1}\| \le C$, where $z \in \rho(T)$.

Let $\{T_n\}$ be a sequence of operators in Banach space $\mathcal{B}(X)$ that converges to T in a collectively way, i.e., the set $\{T_n x : \|x\| \le 1, n = 1, 2, \cdots\}$ is sequentially compact. For n large enough, $z \in \rho(T_n)$ and the associated projection,

$$E_n = -\frac{1}{2\pi i}\int_{\Gamma} (T_n - zI)^{-1} dz$$

exists and $\max_{z\in\Gamma} \|(T_n - zI)^{-1}\| \le C$. Clearly, $\dim(E) = \dim(E_n) = m$ and $T_n E_n = E_n T_n$ [27]. Furthermore, the spectrum of T_n inside Γ, contains m approximations of λ, i.e. $\lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,m}$, counted according to their algebraic multiplicities [2,6,27]. Let

$$\hat{\lambda}_n = \frac{\lambda_{n,1} + \lambda_{n,2} + \cdots + \lambda_{n,m}}{m}.$$

Then we have the following lemma.

Lemma 2.1 [27] *For all n sufficiently large,*

$$|\lambda - \hat{\lambda}_n| \le C\|(T - T_n)|_{R(E)}\|, \tag{2.5}$$

where $R(E)$ is the range of the projection E, and $(T - T_n)|_{R(E)}$ means the restriction of operator $T - T_n$ on $R(E)$.

The lemma gives us a starting point for approximation of eigenvalues of compact operators. However, the result can be refined if the kernel is positive definite. Let

$$a(u, v) = \int_0^1\int_0^1 k(t,s)u(s)v(t)w^{(\alpha,\beta)}(t)dsdt, \quad b(u, v) = \int_0^1 u(t)v(t)w^{(\alpha,\beta)}(t)dt,$$

where v is a test function in the L^2 space. If the bilinear operator $a(u, v)$ is coercive, then we can list eigenvalues of T by

$$\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge 0$$

with zero the only possible cluster point.

Consider a numerical approximation of the first eigen-pair (λ, u). Let (λ_p, v_p) be their Galerkin approximation, and let u_p be the Jacobi expansion of u. We have

$$\lambda = \frac{a(u,u)}{b(u,u)} = \sup_{v\in V}\frac{a(v,v)}{b(v,v)}, \quad \lambda_p = \frac{a(v_p,v_p)}{b(v_p,v_p)} = \max_{v\in\mathcal{P}_p}\frac{a(v,v)}{b(v,v)}.$$

Here $\mathcal{P}_p$ is the polynomial space with degree no more than p. Denote $\tilde{\lambda}_p = \dfrac{a(u_p,u_p)}{b(u_p,u_p)}$, then the following lemma follows.

Lemma 2.2 [15] *Let λ, λ_p, u_p and $\tilde{\lambda}_p$ be defined as above and $a(u, v)$ be coercive, then*

$$0 \le \lambda - \lambda_p \le \lambda - \tilde{\lambda}_p = \lambda\frac{\|u - u_p\|_b^2}{\|u\|_b^2} - \frac{\|u - u_p\|_a^2}{\|u\|_b^2}. \tag{2.6}$$

We now introduce some Hilbert spaces for future use [12]. For any $m \in N$, define

$$H^m_{w^{(\alpha,\beta)}} = \{v \mid v|v \text{ is measurable and } \|v\|_{m,w^{(\alpha,\beta)}} \leq \infty\},$$

where

$$\|v\|_{m,w^{(\alpha,\beta)}} = \left(\sum_{k=0}^{[(r-1)/2]} \|(1-x^2)^{m/2-k}\partial_x^{m-k} v\|^2_{w^{(\alpha,\beta)}} + \|v\|^2_{[m/2],w^{(\alpha,\beta)}} \right)^{1/2}.$$

In particular, $H^0_{w^{(\alpha,\beta)}} = L^2_{w^{(\alpha,\beta)}}$. Another space introduced in [12] is

$$H^m_{w^{(\alpha,\beta)},*} = \{v \mid \partial_x v \in H^{m-1}_{w^{(\alpha,\beta)}}\}$$

with norm

$$\|v\|_{m,w^{(\alpha,\beta,*)}} = \|\partial_x v\|_{m-1,w^{(\alpha,\beta)}}.$$

For real $r > 0$, the spaces $H^r_{w^{(\alpha,\beta)}}$ and $H^r_{w^{(\alpha,\beta)},*}$ are defined by space interpolation [26].

From [35], eigenfunctions y of (1.1) have the property $y \in H^{(3+\min(\alpha,\beta)/2-\nu-\epsilon}_{w^{(\alpha,\beta)}}$, where $\epsilon > 0$ is arbitrarily small. In particular, $y \in H^{3/2-\nu-\epsilon} \hookrightarrow C^0[0,1]$. Hence we can define an interpolatory projection $\pi_N : R(E) \to R(E_p)$ by $\pi_N x(t) := \sum_{k=0}^N \xi_k T_k(t_i) = x(t_i)$, where $\{t_i\}_{i=0}^N$ are Jacobi Gauss points (G), or Jacobi Gauss-Radau points (R), or Jacobi Gauss-Lobatto points (L). Clearly, π_N is equivalent to the interpolation operator $I_{Z,N,\alpha,\beta}$, where $Z = G, R, L$ respectively for these three cases.

Lemma 2.3 [32] *For any function* $v \in H^1_{w^{(\alpha,\beta)},*}(-1,1)$,

$$\begin{aligned}
\|I_{G,N,\alpha,\beta}v\|_{w^{(\alpha,\beta)}} &\leq C(\|v\|_{w^{(\alpha,\beta)}} + N^{-1}|v|_{1,w^{(\alpha+1,\beta+1)}}), \\
\|I_{R,N,\alpha,\beta}v\|_{w^{(\alpha,\beta)}} &\leq C(N^{-\beta-1}|u(-1)| + \|v\|_{w^{(\alpha,\beta)}} + N^{-1}|v|_{1,w^{(\alpha+1,\beta+1)}}), \\
\|I_{L,N,\alpha,\beta}v\|_{w^{(\alpha,\beta)}} &\leq C(N^{-\alpha-1}|u(1)| + N^{-\beta-1}|u(-1)| + \|v\|_{w^{(\alpha,\beta)}} \\
&\quad + N^{-1}|v|_{1,w^{(\alpha+1,\beta+1)}}),
\end{aligned} \tag{2.7}$$

Lemma 2.4 [12] *For any* $v \in H^r_{w^{(\alpha,\beta)},*}$, $0 \leq \mu \leq 1 \leq r$, *and* $-1 < \alpha, \beta < 1$,

$$\|I_{Z,N,\alpha,\beta}v - v\|_{\mu,w^{(\alpha,\beta)}} \leq CN^{2\mu-r}\|v\|_{r,w^{(\alpha,\beta)},*}, \tag{2.8}$$

where $Z = G$ *or* R;

For any $-1 < \alpha, \beta \leq 0$ *or* $0 < \alpha, \beta < 1$, *any* $v \in H^r_{w^{(\alpha,\beta)},*}$, *and* $0 \leq \mu \leq 1 \leq r$

$$\|I_{L,N,\alpha,\beta}v - v\|_{\mu,w^{(\alpha,\beta)}} \leq CN^{2\mu-r}\|v\|_{r,w^{(\alpha,\beta)},*}, \tag{2.9}$$

2.3 Error Analysis

This subsection is devoted to error estimate of (2.4). Clearly, the algorithm can be written as $\pi_N T\hat{\pi}_N u = \lambda_N \pi_p u$, where $T\hat{\pi}_N$ is a type of Jacobi quadrature for $T\pi_N$ on some function. Furthermore, $\pi_N T\hat{\pi}_N$ is of finite-rank, and therefore, compact.

Theorem 2.5 *Let* $\{x_i\}_{i=0}^N$ *be a set of collocation points with respect to Jacobi weight* $w^{\alpha,\beta}$. *Let* $u(x) = y(t)$ *be the eigenfunction associated with the eigenvalue of largest modulus* λ *of* (1.1) *and algorithm* (2.4) *is applied for the problem, then*

$$|\lambda - \hat{\lambda}_N| \leq CN^{\nu-(\frac{3+\min(\alpha,\beta)}{2})}\|u\|_{H^{\frac{(3+\min(\alpha,\beta))}{2}-\nu-\epsilon}_{w^{(\alpha,\beta)},*}}, \tag{2.10}$$

where $-1 < \alpha, \beta < 1$ *for Jacobi–Gauss points;* $-1 < \alpha < 1, -1 < \beta < 1-2\nu$ *for Jacobi–Gauss–Radau points; and* $-1 < \alpha, \beta < 1-2\nu$ *for Jacobi–Gauss–Lobatto points.*

Proof Denote the identity operator by I. Clearly,

$$\begin{aligned}\|(T-\pi_N T\hat{\pi}_N)u\|_{w^{(\alpha,\beta)}} &\le \|(I-\pi_N)Tu\|_{w^{(\alpha,\beta)}} + \|\pi_N T(I-\pi_N)u\|_{w^{(\alpha,\beta)}}\\ &\quad + \|\pi_N T\hat{\pi}_N u - \pi_N T\pi_N u\|_{w^{(\alpha,\beta)}}\\ &:= I_1 + I_2 + I_3.\end{aligned}$$

Since Tu is smoother than u itself [7,14], one easily has through Lemma 2.4

$$I_1 \le CN^{\nu-(3+\min(\alpha,\beta))/2}\|u\|_{H^{(3+\min(\alpha,\beta))/2-\nu-\epsilon}_{w^{(\alpha,\beta)},*}}, \tag{2.11}$$

for $-1 < \alpha, \beta < 1$ regarding to Jacobi–Gauss points and Jacobi–Gauss–Radau points and $-1 < \alpha, \beta \le 0$ or $0 < \alpha, \beta < 1$ regarding to Jacobi–Gauss–Lobatto points.

On the other hand, Lemma 2.3 implies

$$I_2 \le \begin{cases} CN^{\nu-(3+\min(\alpha,\beta))/2}\|u\|_{H^{(3+\min(\alpha,\beta))/2-\nu-\epsilon}_{w^{(\alpha,\beta)},*}}, & \text{(Jacobi–Gauss points)}\\ C(N^{-\beta-1}|(Tu-T\pi_N u)(-1)| + N^{\nu-(3+\min(\alpha,\beta))/2}\|u\|_{H^{(3+\min(\alpha,\beta))/2-\nu-\epsilon}_{w^{(\alpha,\beta)},*}}), & \\ \qquad\qquad \text{(Jacobi–Gauss–Radau points)} & \\ C(N^{-\beta-1}|(Tu-T\pi_N u)(-1)| + N^{\nu-(3+\min(\alpha,\beta))/2}\|u\|_{H^{(3+\min(\alpha,\beta))/2-\nu-\epsilon}_{w^{(\alpha,\beta)},*}}), & \\ \qquad +N^{-\alpha-1}|(Tu-T\pi_N u)(1)| \quad \text{(Jacobi–Gauss–Lobatto points)}. & \end{cases}$$

For Jacobi–Gauss–Radau points, we specify $\pi_N = \pi_{R,N,\alpha,\beta}$,

$$\begin{aligned}|(Tu-T\pi_{R,N,\alpha,\beta}u)(-1)| &= \left|\int_{-1}^{1} g(x,-1)(1+x)^{-\nu}(u-\pi_{R,N,\alpha,\beta}u)dx\right|,\\ &\le C\|u-\pi_{R,N,\alpha,\beta}u\|_{w^{(\alpha,\beta)}}\int_{-1}^{1}\frac{g^2(x)}{(1-x)^{\alpha}(1+x)^{2\nu+\beta}}dx,\\ &\le CN^{\nu-(3+\min(\alpha,\beta))/2}\|u\|_{H^{(3+\min(\alpha,\beta))/2-\nu-\epsilon}_{w^{(\alpha,\beta)},*}} \end{aligned} \tag{2.12}$$

if $\beta < 1-2\nu$.

Similarly, for Jacobi–Gauss–Lobatto points, $|(Tu - T\pi_{L,N,\alpha,\beta}u)(-1)|$ and $|(Tu - T\pi_{L,N,\alpha,\beta}u)(1)|$ have the same bounds if $-1 < \alpha, \beta < 1-2\nu$. Therefore,

$$I_2 \le CN^{\nu-(\frac{3+\min(\alpha,\beta)}{2})}\|u\|_{H^{\frac{(3+\min(\alpha,\beta))}{2}-\nu-\epsilon}_{w^{(\alpha,\beta)},*}} \begin{cases} -1<\alpha,\beta<1, \quad \text{(Jacobi–Gauss points)};\\ -1<\alpha<1, -1<\beta<1-2\nu,\\ \qquad \text{(Jacobi–Gauss–Radau points)};\\ -1<\alpha,\beta<1-2\nu,\\ \qquad \text{(Jacobi–Gauss–Lobatto points)}.\end{cases}$$

Since both g and T_j are smooth functions, all three types of Jacobi quadrature yields high order term I_3 for large N. The result follows.

Theorem 2.6 *Let* $\{x_i\}_{i=0}^{N}$ *be a set of collocation points with respect to Jacobi weight* $w^{\alpha,\beta}$ *and* λ *and* $\hat{\lambda}_N$ *be the exact and numerical approximation of our weakly singular operator* T

by Jacobi–Gauss collocation of our algorithm. If the bilinear form $a(u, v)$ is positive definite, then

$$|\lambda - \hat{\lambda}_N| \le CN^{2\nu-(3+\min(\alpha,\beta)}\|y\|_{H^{\frac{(3+\min(\alpha,\beta))}{2}-\nu-\epsilon}_{w^{(\alpha,\beta)},*}}.$$

Proof We divide the proof into three steps.

Step 1. By [13, Page 334], the numerical quadrature error of both

$$\int_{-1}^{x_i} \frac{\tilde{g}(x_i,\tau)P_j^{(\alpha,\beta)}(\tau)}{(x_i-\tau)^\nu}d\tau \quad \text{and} \quad \int_{x_i}^{1} \frac{\tilde{g}(x_i,\tau)P_j^{(\alpha,\beta)}(\tau)}{(\tau-x_i)^\nu}d\tau$$

in our algorithm is approximately $\frac{\pi}{2^{2N-\nu}}\frac{(gP_j^{(\alpha,\beta)})^{(2N)}}{(2N)!}$ for N large enough. Furthermore, $P_j^{(\alpha,\beta)}(x)$ has the property

$$\frac{d^k}{dx^k}P_j^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+j+1+k)}{2^k\Gamma(\alpha+\beta+j+1)}P_{j-k}^{(\alpha+k,\beta+k)}(x),\ k \le j.$$

Noting that

$$|P_n^{(\alpha+k,\beta+k)}(x)| \sim n^{\max(k+q,-1/2)}, \tag{2.13}$$

where $q = \max(\alpha, \beta)$ [32, P. 78], we have

$$\begin{aligned}
\left|\frac{\pi}{2^{2N-\nu}}\frac{(gP_j^{(\alpha,\beta)})^{(2N)}}{(2N)!}\right| &\le \frac{C\pi}{2^{2N-\nu}(2N)!}\left[j^{\max(-1/2,q)} + \sum_{k=1}^{j}\frac{\Gamma(\alpha+\beta+1+j+k)}{2^k\Gamma(\alpha+\beta+1+j)}\binom{j+q}{j-k}\right]\\
&\le \frac{C\pi N^{1/2}}{2^{2N-\nu}(2N)!}\sum_{k=0}^{N}\frac{\Gamma(\alpha+\beta+1+N+k)}{2^k\Gamma(\alpha+\beta+1+N)}\binom{N+q}{N-k}\\
&< \frac{C\pi N^{1/2}}{2^{2N-\nu}(2N)!}\sum_{k=0}^{N}\frac{\Gamma(N+k+3)}{2^k\Gamma(N+3)}\frac{(q+k+1)\cdots(q+N)}{(N-k)!}\\
&\le \frac{C\pi N^{1/2}}{2^{2N-\nu}(2N)!}\sum_{k=0}^{N}\frac{(N+k+2)!}{2^k(k+1)!(N-k)!(N+2)}\\
&\le \frac{Ce\pi N^{3/2}}{2^{2N-\nu}(2N)!}\sqrt{\frac{2}{\pi}}K_N(1),
\end{aligned} \tag{2.14}$$

where $K_v(z)$ is the modified Bessel function of the second kind and $\binom{n}{r} = \frac{(-1)^r(-n)_r}{r!}$. Moreover, when $0 < z \ll \sqrt{v+1}$,

$$K_v(z) \sim \frac{\nu(v)}{2}\left(\frac{2}{z}\right)^v,\ v > 0.$$

Hence, by Stirling's formula,

$$\begin{aligned}
\left|\frac{\pi}{2^{2N-\nu}}\frac{(gP_j^{(\alpha,\beta)})^{(2N)}}{(2N)!}\right| &\le \frac{Ce\pi N^{3/2}}{2^{2N-\nu}(2N)!}\sqrt{\frac{2}{\pi}}\frac{\nu(N)}{2}2^N\\
&\le \frac{CN^{1/2}}{8^N}\left(\frac{e}{N}\right)^N.
\end{aligned} \tag{2.15}$$

Step 2. Denote the kernel by $k(t,s)=\dfrac{g(t,s)}{|t-s|^{\nu}}$. By virtue of our algorithm,

$$\tilde{\int_0^1} k(t_i,s)y_N(s)ds=\lambda_N y_N(t_i),$$

where ˜ means the integration is approximated by numerical quadrature.

We multiply both sides by $P_j^{(\alpha,\beta)}(t_i)w_i$ and sum up from 0 to N,

$$\sum_{j=0}^{N}\tilde{\int_0^1} k(t_i,s)y_p(s)P_j^{(\alpha,\beta)}(t_i)w_i ds=\lambda_N\sum_{j=0}^{N} y_N(t_i)P_j^{(\alpha,\beta)}(t_i)w_i. \tag{2.16}$$

Here, t_i and w_i are Gauss-Jacobi quadrature points and weights on [0, 1].

Hence, the system (2.16) can be written as

$$\tilde{A}=\left(\tilde{\int_0^1}\tilde{\int_0^1} k(t,s)P_j^{(\alpha,\beta)}(s)P_i^{(\alpha,\beta)}(t)w^{(\alpha,\beta)}(t)dsdt\right)_{ij},$$

$$\tilde{B}=\left(\tilde{\int_0^1} P_j^{(\alpha,\beta)}(t)P_i^{(\alpha,\beta)}(t)w^{(\alpha,\beta)}(t)dt\right)_{ij}.$$

We recall that $y_N(x)=\sum_{i=0}^{N} y_i P_i^{(\alpha,\beta)}(x)$, then generalized eigenvalue problem we solve is

$$\tilde{A}y=\lambda_N\tilde{B}y,$$

where $y=[y_1,\cdots,y_N]^T$.

Next, we write

$$A=\left(\int_0^1\int_0^1 k(t,s)P_j^{(\alpha,\beta)}(s)P_i^{(\alpha,\beta)}(t)w^{(\alpha,\beta)}(t)dsdt\right)_{ij},$$

$$B=\left(\int_0^1 P_j^{(\alpha,\beta)}(t)P_i^{(\alpha,\beta)}(t)w^{(\alpha,\beta)}(t)dt\right)_{ij}.$$

and $Ay=\tilde{\lambda}_N By$. Then,

$$|\lambda-\lambda_N|\le|\lambda-\tilde{\lambda}_N|+|\tilde{\lambda}_N-\lambda_N|. \tag{2.17}$$

Lemma 2.2 indicates

$$|\lambda-\tilde{\lambda}_N|\le CN^{2\nu-(3+\min(\alpha,\beta)}\|y\|_{H_{w^{(\alpha,\beta)},*}^{\frac{(3+\min(\alpha,\beta))}{2}-\nu-\epsilon}}. \tag{2.18}$$

Therefore, we only need to estimate the second term in the right of (2.17).

Step 3. Clearly, $\tilde{B} = B$ since the Jacobi–Gauss quadrature is exact for all polynomial $p \in P_{2N+1}$.

$$
\begin{aligned}
|A_{ij} - \tilde{A}_{ij}| &= \Bigg| \int_0^1 \int_0^1 k(t,s) P_j^{(\alpha,\beta)}(s) ds P_i^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t) dt \\
&\quad - \int_0^1 \tilde{\int_0^1} k(t,s) P_j^{(\alpha,\beta)}(s) ds P_i^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t) dt \Bigg| \\
&\quad + \Bigg| \int_0^1 \tilde{\int_0^1} k(t,s) P_j^{(\alpha,\beta)}(s) ds P_i^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t) dt \\
&\quad - \tilde{\int_0^1} \tilde{\int_0^1} k(t,s) P_j^{(\alpha,\beta)}(s) ds P_i^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t) dt \Bigg| \\
&:= I_1 + I_2.
\end{aligned} \tag{2.19}
$$

By the error estimate in **step 1**,

$$
\begin{aligned}
I_1 &\le \Bigg| \int_0^1 \Bigg| \int_0^1 k(t,s) P_j^{(\alpha,\beta)}(s) w^{(\alpha,\beta)}(t) ds - \tilde{\int_0^1} k(t,s) P_j^{(\alpha,\beta)}(s) ds \Bigg| |P_i^{(\alpha,\beta)}(t)| w^{(\alpha,\beta)}(t) dt \Bigg| \\
&\le \frac{CN^{3/2}}{8^N} \left(\frac{e}{N}\right)^N.
\end{aligned}
$$

Recall from (2.4) that

$$
\tilde{\int_0^1} k(t,s) P_j^{(\alpha,\beta)}(s) ds = \left(\frac{1+x}{4}\right)^{1-\nu} K_{\cdot j} + \left(\frac{1-x}{4}\right)^{1-\nu} \bar{K}_{\cdot j},
$$

where $t = (1+x)/2$ and both $K_{\cdot j}$ and $\bar{K}_{\cdot j}$ are smooth function of x indicated by (2.4). However, the numerical integral function as a whole is non-smooth. Fortunately, we can approximate the integral in the following way.

$$
\begin{aligned}
&\int_{-1}^1 \left[\left(\frac{1+x}{4}\right)^{1-\nu} K_{\cdot j}(x) + \left(\frac{1-x}{4}\right)^{1-\nu} \bar{K}_{\cdot j}(x) \right] P_i^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx \\
&= \left(\frac{1}{4}\right)^{1-\nu} \Bigg(\int_{-1}^1 K_{\cdot j}(x) P_i^{(\alpha,\beta)}(x) w^{(\alpha,\beta+1-\nu)}(x) dx \\
&\quad + \left(\frac{1}{4}\right)^{1-\nu} \int_{-1}^1 \bar{K}_{\cdot j}(x) P_i^{(\alpha,\beta)}(x) w^{(\alpha+1-\nu,\beta)}(x) dx \Bigg).
\end{aligned}
$$

For the first term, we use Jacobi–Gauss quadrature associated with $w^{(\alpha,\beta+1-\nu)}(x)$ and for the second one, we use Jacobi–Gauss quadrature associated with weight $w^{(\alpha+1-\nu,\beta)}(x)$. Therefore, $I_2 \le \frac{CN^{1/2}}{8^N}\left(\frac{e}{N}\right)^N$, which together with (2.20), yields that

Table 1 The largest eigenvalue computed for Example 1

N	5	15	25	35
	3.1786613834	3.1753595621	3.1752133961	3.1751869763
N	40	45	50	55
	3.1751819966	3.1751790661	3.1751771892	3.1751759654

$$\|A - \tilde{A}\|_n \leq \frac{CN^{5/2}}{8^N}\left(\frac{e}{N}\right)^N, \quad n = 1 \text{ or } \infty.$$

Finally, our generalized problem is equivalent to

$$\tilde{B}^{-1}\tilde{A}y_N = \lambda_N y_N.$$

Since

$$\int_{-1}^{1} (P_n^{(\alpha,\beta)}(x))^2 w^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!},$$

we have $\|B^{-1}\| \leq CN$. Thereby,

$$\begin{aligned}\|\tilde{B}^{-1}\tilde{A} - B^{-1}A\|_n &\leq \|B^{-1}\|_n\|\tilde{A} - A\|_n \\ &\leq \frac{CN^{7/2}}{8^N}\left(\frac{e}{N}\right)^N.\end{aligned}$$

Therefore, by a perturbation theory [6, page 30],

$$|\lambda_N - \tilde{\lambda}_N| \leq \frac{CN^{7/2}}{8^N}\left(\frac{e}{N}\right)^N, \tag{2.20}$$

which is a high order term compared with (2.18). The results follows by (2.17), (2.18) and (2.20).

2.4 Numerical Experiment

We present one example here to illustrate the effectiveness of our algorithm. The specific case $g(t,s) = 1$ for the problem is intensively explored in [15].

Example 1 Consider the largest eigenvalue of the problem

$$\int_0^1 \frac{e^{t-s}}{|t-s|^\nu}y(s)ds = \lambda y(t)$$

with $\nu = 1/3$. From Theorem 2.6, the possible largest convergence rate is $\mathcal{O}(N^{-10/3})$ if $\alpha \to 1$ and $\beta \to 1$. In this example, we choose $\alpha = \beta = 0.99$. Eigenvalues obtained are listed in Table 1. To obtain the convergence rate, we froze the result for $N = 80$ as our "exact" eigenvalue and we choose reference curve exactly as $N^{-10/3}$. Figure 1 demonstrates the accuracy of our algorithm and theorem.

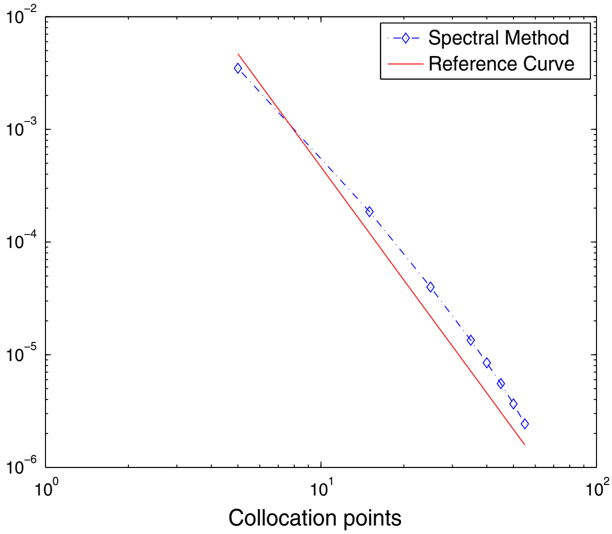

Fig. 1 Convergence rate for Example 1

3 Eigenvalue of Weakly Singular Kernel with Oscillation

In this section, we consider the highly oscillatory problem (1.2). First, we concern a numerical quadrature for

$$K[g] = \int_0^1 g(x) \frac{e^{i\omega|x-y|}}{|x-y|^{\nu}} dx, \quad 0 \le y < 1, \tag{3.1}$$

where $0 < \nu < 1$, $\omega \gg 1$ and $g(x)$ is a smooth function. Before proceeding, we need to introduce some basic knowledge on hypergeometric series.

3.1 Preliminaries

A general hypergeometric series with p upper parameters and q lower parameters is defined as follows:

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where $(a)_k$ is the Pochhammer symbol

$$(a)_k = \begin{cases} 1, & k = 0, \\ a(a+1)\cdots(a+k-1), & k = 1, 2, \ldots \end{cases}$$

and $b_i \neq 0, -1, -2, \ldots$ for all $i = 1, \ldots q$. Clearly, the orders of the parameters are not essential and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

When the series is infinite, its radius of convergence ρ is

$$\rho = \begin{cases} \infty, & p < q+1, \\ 1, & p = q+1, \\ 0, & p > q+1. \end{cases}$$

As we will see, all hypergeometric series in our algorithms are convergent for all real numbers.

The following identity acts as an invaluable tool for the analysis of our algorithm.

Lemma 3.1 [8,9] *If $\phi(t)$ is N times continuously differentiable for $\alpha \le t \le \beta$, and $0 < \lambda \le 1$, $0 < \mu \le 1$, then,*

$$\int_\alpha^\beta e^{ixt}(t-\alpha)^{\lambda-1}(\beta-1)^{\mu-1}\phi(t)dt = B_N(x) - A_N(x) + \mathcal{O}(x^{-N}) \text{ as } x \to \infty,$$

where

$$A_N(x) = \sum_{k=0}^{N-1} \frac{\nu(k+\lambda)}{k!} e^{\pi i(k+\lambda-2)/2} x^{-k-\lambda} e^{ix\alpha} \frac{d^k}{d\alpha^k}[(\beta-\alpha)^{\mu-1}\phi(\alpha)],$$

$$B_N(x) = \sum_{k=0}^{N-1} \frac{\nu(k+\mu)}{k!} e^{\pi i(k-\mu)/2} x^{-k-\mu} e^{ix\beta} \frac{d^k}{d\beta^k}[(\beta-\alpha)^{\lambda-1}\phi(\beta)]. \tag{3.2}$$

3.2 Algorithm

If $g(x)$ is smooth, we approximate it by its Hermite interpolation $I_H[g] = \sum_{j=0}^n g_j x^j$ at nodes 0, y and 1 satisfying

$$I_H[g](\theta) = g(\theta),\ I_H[g]'(\theta) = g'(\theta), \cdots, I_H[g]^{(m)}(\theta) = g^{(m)}(\theta),$$

where $\theta = 0$, y or 1. Substituting it into (3.1), we have

$$K[g] \approx \sum_{j=0}^n g_j \left[\int_0^y x^j (y-x)^{-\nu} e^{i\omega(y-x)} dx + \int_y^1 x^j (x-y)^{-\nu} e^{i\omega(x-y)} dx \right]$$

Define

$$I_1^{j,\zeta}(y) := \int_0^y t^{j+\zeta} \cos \omega t dt, \qquad I_2^{j,\zeta}(y) := \int_0^y t^{j+\zeta} \sin \omega t dt.$$

Simple calculation shows that

$$\begin{aligned} K[g] \approx \sum_{j=0}^n g_j \Bigg[& \sum_{k=0}^j \binom{j}{k} (-1)^{j-k} y^k I_1^{j-k,-\nu}(y) + \sum_{k=0}^j \binom{j}{k} y^k I_1^{j-k,-\nu}(1-y) \\ & + i \left(\sum_{k=0}^j \binom{j}{k} (-1)^{j-k} y^k I_2^{j-k,-\nu}(y) + \sum_{k=0}^j \binom{j}{k} y^k I_2^{j-k,-\nu}(1-y) \right) \Bigg] \\ := Q_m[g]. \end{aligned}$$

By integration by parts, we immediately obtain the following relation

$$\begin{cases} I_2^{j,\zeta} = -\dfrac{y^{j+\zeta}\cos\omega y}{\omega} + \dfrac{j+\zeta}{\omega} I_1^{j-1,\zeta}, \\ I_1^{j,\zeta} = \dfrac{y^{j+\zeta}\sin\omega y}{\omega} - \dfrac{j+\zeta}{\omega} I_2^{j-1,\zeta}, \quad j \geq 1. \end{cases} \tag{3.3}$$

To start the iteration, we find

$$I_1^{0,\zeta} = \int_0^y t^\zeta \cos\omega t dt = \frac{{}_1F_2(\frac{1+\zeta}{2}; \frac{1}{2}, \frac{\zeta}{2}+\frac{3}{2}; -\frac{1}{4}\omega^2 y^2) y^{\zeta+1}}{\zeta+1},$$

$$I_2^{0,\zeta} = \int_0^y t^\zeta \sin\omega t dt = \frac{{}_1F_2(\frac{\zeta}{2}+1; \frac{3}{2}, \frac{\zeta}{2}+2; -\frac{1}{4}\zeta^2 y^2) w y^{\zeta+2}}{\zeta+2}.$$

Remark 1 The iteration matrix for the case of I_1^j and I_2^j is A, where

$$A = \begin{bmatrix} 0 & \frac{j+\nu}{\omega} \\ -\frac{j+\nu}{\omega} & 0 \end{bmatrix}$$

with eigenvalues $\pm\frac{j+\nu}{\omega} i$. The fact $\omega \gg 1$ implies that there does not exist error propaganda in the iteration.

Theorem 3.2 *Let $I_H[g]$ be the Hermite interpolation of smooth function g at end points* 0, 1 *and y of order m. Then,*

$$K[g] - Q_m[g] \sim \mathcal{O}(\omega^{-m-1}). \tag{3.4}$$

Proof The order of convergence follows directly from Lemma 3.1 as $\omega \to \infty$, since

$$K[g] - Q_m[g] = K[g] - K[I_H[g]] = K[g - I_H[g]],$$

and $g - I_H[g]$ and its first m derivative vanishes at points 0, y and 1. Hence, both A_k and B_k are 0 for $K[g - I_H[g]]$, $k = 0, 1, \cdots, m$.

3.3 Numerical Experiments

In this subsection, we provide two examples for an illustration of Theorem 3.2.

Example 2 Consider the asymptotic quadrature for approximating

$$\int_0^1 \sin(x) \frac{e^{i\omega|x-3/7|}}{|x-3/7|^{1/3}} dx.$$

We take the approximation for $N = 4$ in Lemma 3.1 as true value.

Figure 2 shows the errors with nodes {0, 3/7, 1} scaled by ω^2 and ω^3 with order $m = 1$ and $m = 2$, respectively. Clearly, the approximate accuracy enhances as the multiplicity increases.

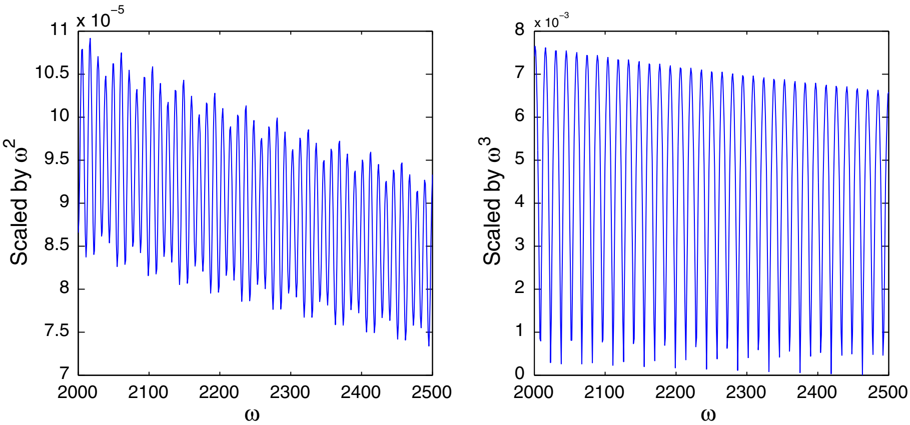

Fig. 2 The modulus error for $Q_1[\sin(x)]$ (*the left*) and $Q_2[\sin(x)]$ (*the right*) scaled by ω^2 and ω^3, respectively

Example 3 Consider the numerical quadrature to

$$\int_0^1 \frac{1}{1+x}\frac{e^{i\omega|x-y|}}{|x-y|^{\nu}} \text{ with } y=\frac{4}{5} \text{ and } \nu=\frac{\sqrt{2}}{2}.$$

This example seems more complicated to implement since ν is irrational and the function is not very smooth. However, Table 3 and Fig. 3 exhibit similar behavior as the previous example and confirm the effectiveness of Theorem 3.2.

3.4 Spectrum Computation

In [4,5], the spectral problem for the kernel $e^{i\omega|t-s|}$ is intensively explored and leave the computation of spectra of the kernel $K(t,s)=k_0(t,s)\frac{e^{i\omega|t-s|}}{|t-s|^{\nu}}$ with $k_0(t,s)=k_0(s,t)$ as an open problem. For simplicity, in this paper, we consider $k_0(t,s)=h(s)$ and we start with the case $k_0(t,s)=1$.

3.4.1 Algorithm

For the computation of spectra, reference [5] recommends spectral Galerkin method instead of spectral collocation one. Hence, in this paper, we apply the Galerkin method and adopt the *Modified Fourier Expansions* [21] and express the eigenfunction for (1.2) by

$$u(s)=\sum_{n=0}^{\infty}\hat{u}_n^C\cos\pi ns+\sum_{n=1}^{\infty}\hat{u}_n^S\sin\pi\left(n-\frac{1}{2}\right)s, \tag{3.5}$$

where $\hat{u}_n^C$ and $\hat{u}_n^S$ are unknown Modifier Fourier coefficients. Modifier Fourier expansion enjoys two advantages for the problem. Firstly, it owns rapid convergence for non-periodic functions in $L^2[-1,1]$; Secondly, it facilities the implementation of our algorithm thanks to the Euler formula (Table 2).

Since the basis $\{\cos\pi ns, \sin\pi(n-1/2)s\}$ are orthonomal, we have an eigenvalue problem

$$AU=\lambda U,$$

Table 2 Relative error of $K[\sin(x)]$ with different orders

ω	$m=1$	$m=2$
100	3.8240e−06	1.0666e−06
500	1.0094e−07	4.7644e−09
1,000	2.7560e−08	1.6118e−09
10,000	2.6730e−10	6.4817e−13

Table 3 Relative error for $K[\frac{1}{1+x}](4/5)$ with different orders

ω	$m=1$	$m=2$
100	9.4832e−06	3.1047e−06
500	2.9920e−07	4.1994e−09
1,000	7.8332e−08	3.7956e−10
10,000	7.6209e−10	2.6167e−13

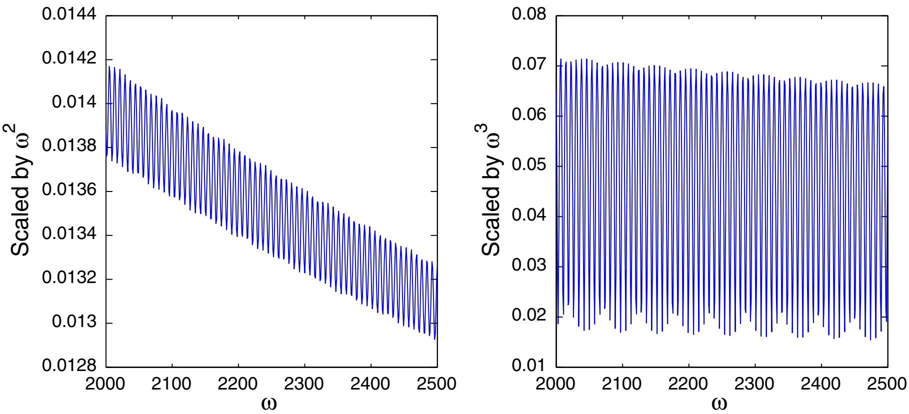

Fig. 3 The modulus error for $Q_1[\frac{1}{1+x}]$ (*the left*) and $Q_2[\frac{1}{1+x}]$ (*the right*) scaled by ω^2 and ω^3, respectively

where

$$A_{2m,2n} = \int_{-1}^{1}\int_{-1}^{1} \cos \pi m x \cos \pi n y K_\omega(x, y) dx dy, \tag{3.6}$$

$$A_{2m+1,2n} = \int_{-1}^{1}\int_{-1}^{1} \sin \pi (m - 1/2) x \cos \pi n y K_\omega(x, y) dx dy,$$

$$A_{2m,2n+1} = \int_{-1}^{1}\int_{-1}^{1} \cos \pi m x \sin \pi (n - 1/2) y K_\omega(x, y) dx dy,$$

$$A_{2m+1,2n+1} = \int_{-1}^{1}\int_{-1}^{1} \sin \pi (m - 1/2) x \sin \pi (n - 1/2) y K_\omega(x, y) dx dy,$$

where $K_\omega(x, y) = \dfrac{e^{i\omega|x-y|}}{|x-y|^\nu}$ for $m, n = 0, 1, \cdots, N$. Furthermore, $A_{2m+1,2n} = A_{2m,2n+1} = 0$ by a symmetry argument in [5].

To find the explicit expression of $A_{2m,2n}$ and $A_{2m+1,2n+1}$, we need to the following lemmas.

Lemma 3.3 *For any* $\mu > 0$ *and* $a \in R$,

$$\int_{-1}^{1} (1 \pm y)^{\mu-1} e^{ia(1\pm y)} dy = \frac{2^\mu}{\mu}\, {}_1F_1(\mu; \mu+1; 2ai).$$

Proof The result follows directly from the two identities below [10, page 430] by noting that for $a \in R$.

$$\int_0^1 x^{\mu-1} \sin(ax) dx = \frac{-i}{2\mu}[\,{}_1F_1(\mu; \mu+1; ai) - {}_1F_1(\mu; \mu+1; -ai)];$$

$$\int_0^1 x^{\mu-1} \cos(ax) dx = \frac{1}{2\mu}[\,{}_1F_1(\mu; \mu+1; ai) + {}_1F_1(\mu; \mu+1; -ai)].$$

Lemma 3.4 *Let*

$$F(a, b) = \int_{-1}^{1}\int_{-1}^{y} \frac{e^{iax+iby}}{(y-x)^\nu} dx dy,$$

where $a, b \in R$ *and* $0 \le \nu < 1$. *Then*

$$F(a,b) = \begin{cases} \frac{2^{2-\nu}}{(1-\nu)(2-\nu)}\, {}_1F_1(1-\nu; 3-\nu; -2ai), & a+b=0; \\ \frac{2^{1-\nu}}{(a+b)(1-\nu)i}[e^{(a+b)i} - e^{-(a+b)i}]\, {}_1F_1(1-\nu; 2-\nu; -2ai) & \\ -\frac{2^{1-\nu}e^{-(a+b)i}}{(a+b)(1-\nu)i}[\,{}_1F_1(1-\nu; 2-\nu; 2bi) - {}_1F_1(1-\nu; 2-\nu; -2ai)], & \\ \qquad a+b \neq 0. & \end{cases}$$

Proof By a simple change of variable and Lemma 3.3,

$$\begin{aligned} F(a,b) &= \int_{-1}^{1} e^{i(a+b)y}\left(\frac{y+1}{2}\right)^{1-\nu} \int_{-1}^{1} (1-\xi)^{-\nu} e^{-ia\frac{y+1}{2}(1-\xi)} d\xi dy \\ &= \int_{-1}^{1} e^{i(a+b)y}\left(\frac{y+1}{2}\right)^{1-\nu} \frac{2^{2-\nu}}{1-\nu}\, {}_1F_1(1-\nu; 2-\nu; -ia(y+1))dy \\ &= e^{-i(a+b)} \sum_{k=0}^{\infty} \frac{(-ia)^k}{(k+1-\nu)k!} \int_{-1}^{1} (1+y)^{k+1-\nu} e^{i(a+b)(1+y)} dy \\ &= e^{-i(a+b)} \sum_{k=0}^{\infty} \frac{(-ia)^k}{(k+1-\nu)k!} \frac{2^{k+2-\nu}}{(k+2-\nu)}\, {}_1F_1(k+2-\nu; k+3-\nu; 2(a+b)i). \end{aligned} \tag{3.7}$$

For the case $a+b=0$,

$$ {}_1F_1(k+1-\nu;k+2-\nu;0)=1 $$

implies the expression for $F(a,b)$ in the lemma directly.

If $a+b\neq 0$,

$$ \begin{aligned} F(a,b) &= e^{-i(a+b)}2^{2-\nu}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-2ia)^k[2(a+b)i]^j}{(k+1-\nu)(k+j+2-\nu)k!j!} \\ &= e^{-i(a+b)}2^{2-\nu}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-2ia)^k[2(a+b)i]^j}{(k+1-\nu)k!(j+1)!} \\ &\quad -e^{-i(a+b)}2^{2-\nu}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-2ia)^k[2(a+b)i]^j}{(k+2+j-\nu)k!(j+1)!} \\ &:= I_1-I_2. \end{aligned} $$

On the one hand,

$$ \begin{aligned} I_1 &= e^{-i(a+b)}2^{2-\nu}\sum_{k=0}^{\infty}\frac{(-2ai)^k}{k!(k+1-\nu)}[e^{2(a+b)i}-1] \\ &= e^{-i(a+b)}2^{2-\nu}[e^{2(a+b)i}-1]\frac{1}{1-\nu}\,{}_1F_1(1-\nu;2-\nu;-2ai); \end{aligned} \tag{3.8} $$

and

$$ \begin{aligned} I_2 &= e^{-i(a+b)}2^{2-\nu}\sum_{k=0}^{\infty}\frac{(-2ai)^k}{k!}\sum_{j=0}^{\infty}\frac{[2(a+b)i]^j}{(j+1)!}\int_0^1 u^{k+1+j-\nu}du \\ &= e^{-i(a+b)}2^{2-\nu}\int_0^1\sum_{k=0}^{\infty}\frac{(-2aui)^k}{k!}\sum_{j=0}^{\infty}\frac{[2(a+b)ui]^j}{(j+1)!}u^{1-\nu}du \\ &= e^{-i(a+b)}\frac{2^{2-\nu}}{2(a+b)i}\int_0^1 e^{-2aui}[e^{2(a+b)ui}-1]u^{-\nu}du \\ &= e^{-i(a+b)}\frac{2^{2-\nu}}{2(a+b)i}\left[\int_0^1 e^{2bui}u^{-\nu}du-\int_0^1 e^{-2aui}u^{-\nu}du\right] \\ &= e^{-i(a+b)}\frac{2^{2-\nu}}{2(a+b)(1-\nu)i}\left[{}_1F_1(1-\nu;2-\nu;2bi)-{}_1F_1(1-\nu;2-\nu;-2ai)\right]. \end{aligned} \tag{3.9} $$

Thus, (3.8) and (3.9) imply the result.

Similarly, we define

$$ G(a,b)=\int_{-1}^{1}\int_{y}^{1}\frac{e^{iax+iby}}{(x-y)^{\nu}}dxdy. $$

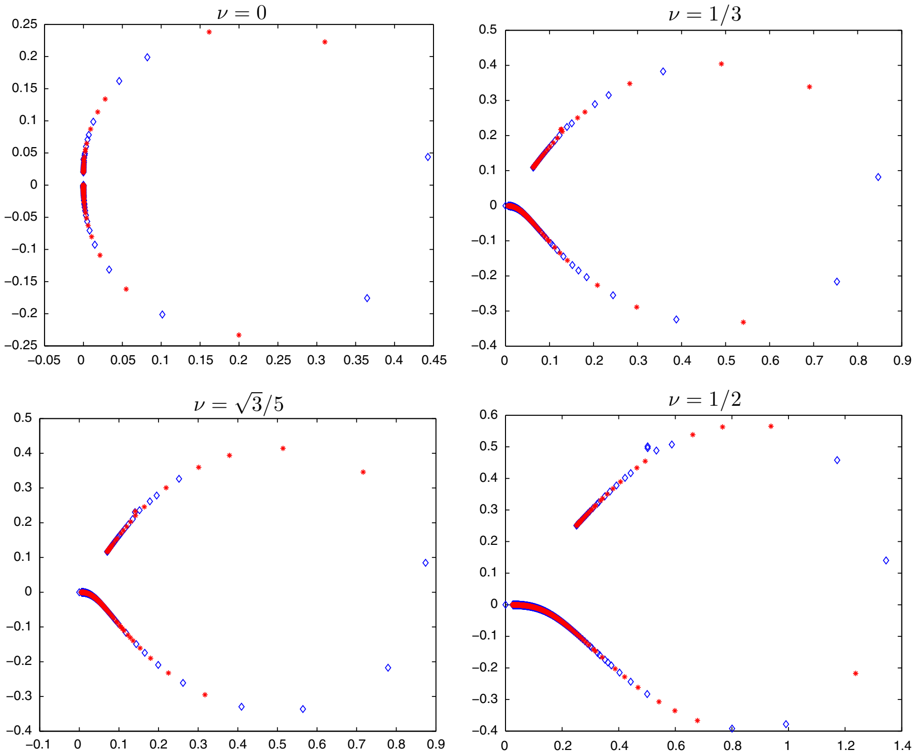

Fig. 4 Spectrum of highly oscillatory problem with $k_0(t, s) = 1$

Then, we easily obtain

$$G(a, b) = e^{i(a+b)} \sum_{k=0}^{\infty} \frac{(ia)^k}{(k+1-\nu)k!} \frac{2^{k+2-\nu}}{(k+2-\nu)} \, {}_1F_1(k+2-\nu; k+3-\nu; -2(a+b)i).$$

Hence, $G(-a, -b) = F(a, b)$. Then, one can easily establish the following result.

Proposition 3.5 *For* $m, n = 0, \cdots, N$,

$$\begin{aligned} A_{2m,2n} = & \frac{1}{2} [F(\pi m - \omega, \pi n + \omega) + F(-\pi m - \omega, \pi n + \omega) \\ & + F(\pi m - \omega, -\pi n + \omega) + F(-\pi m - \omega, -\pi n + \omega)]; \end{aligned} \tag{3.10}$$

$$\begin{aligned} A_{2m+1,2n+1} = & -\frac{1}{2}\Big[F\left(\pi\left(m - \frac{1}{2}\right) - \omega, \pi\left(n - \frac{1}{2}\right) + \omega\right) \\ & - F\left(-\pi\left(m - \frac{1}{2}\right) - \omega, \pi\left(n - \frac{1}{2}\right) + \omega\right) \\ & - F\left(\pi\left(m - \frac{1}{2}\right) - \omega, -\pi\left(n - \frac{1}{2}\right) + \omega\right) \\ & + F\left(-\pi\left(m - \frac{1}{2}\right) - \omega, -\pi\left(n - \frac{1}{2}\right) + \omega\right)\Big]. \end{aligned} \tag{3.11}$$

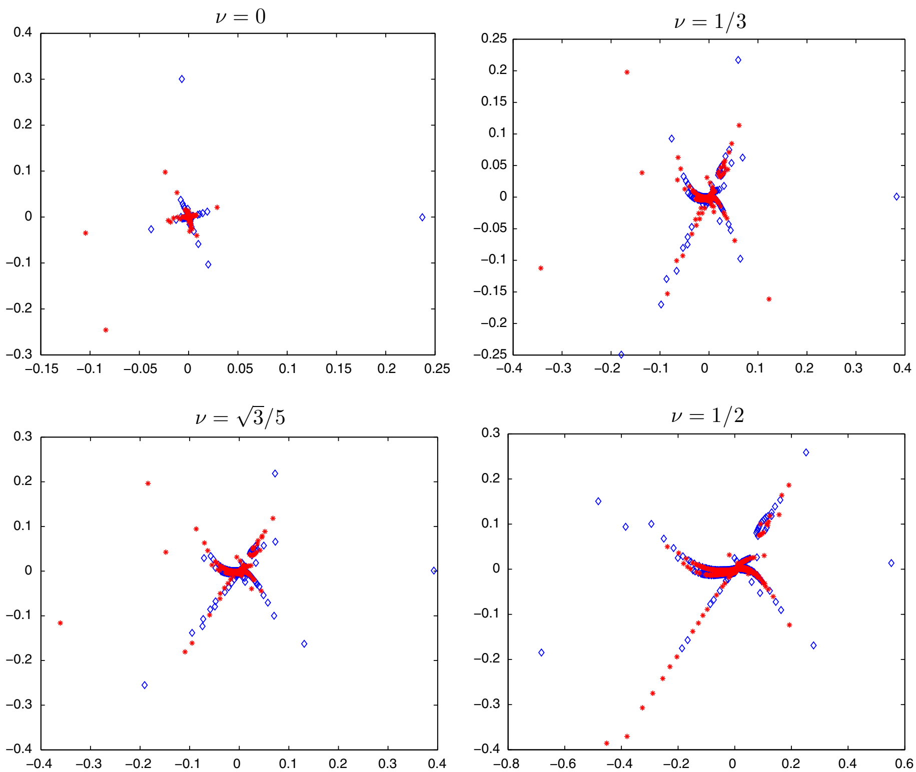

Fig. 5 Spectrum of highly oscillatory problem with $k_0(t, s) = \sin(\pi s)$

Therefore, both $A_{2m,2n}$ and $A_{2m+1,2n+1}$ decay at a rate $\mathcal{O}(\min(m, n)^{\nu-1})$ since for large z, the hypergeometric function has asymptotic expansion [33]

$$ {}_1F_1(1-\nu, 2-\nu, z) \approx \nu(2-\nu)e^{\epsilon i\pi(1-\nu)}z^{\nu-1} + (1-\nu)e^z z^{-1}\left(1+\frac{1}{z}\right)^{-\nu} $$

and

$$ {}_1F_1(1-\nu, 3-\nu, z) = -\frac{(\nu-2)(1-\nu+z)}{z}\,{}_1F_1(1-\nu, 2-\nu, z) - \frac{(\nu-2)(\nu-1)}{z}e^z. $$

For $N = 1{,}000$, we obtain eigenvalues for $\nu = 0, 1/3, \sqrt{3}/5$ and $1/2$. Clearly, when $\nu = 0$, all eigenvalues are located on the segment of a complex circle [5] and approach 0 from the lower half plane. For $\nu > 0$, they are not on a circle any more. Furthermore, there exists a gap between the origin and the commencing point in the upper complex plane and the larger ν is, the larger the gap is Fig. 4.

If $k_0(t, s) = h(s) \in C^2[-1, 1]$ and with bounded second order derivatives, we approximate the function by Modified Fourier expansion. Truncating the series at $n = M$, we can have the point wise error $\mathcal{O}(M^{-1})$ [22]. Then, we follow the previous algorithm to obtain the result.

Example 4 Consider the eigenvalue problem (1.2) with $h(x) = \sin(\pi x)$. We choose $M = 100$ and $N = 1{,}000$ for the Modified Fourier expansion of $h(x)$ and eigenfunction, respectively. For $\nu = 0, 1/3, \sqrt{3}/5$ and $1/2$, the plots of spectrum is presented in Fig. 5.

4 Conclusion

We have investigated the spectra computation of a class of compact operators with weak singularity. For non-oscillatory operators, we have obtained a convergence rate for general kernels. Furthermore, if the bilinear form associated with the operator is positive definite, the convergence rate is doubled. For highly oscillatory operators, we have developed an efficient algorithm to find its spectrum. Moreover, a numerical quadrature to highly oscillatory integral with weak singularity is also provided and its optimal convergence rate is obtained.

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