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A Recovery Based Linear Finite Element Method For 1D Bi-Harmonic Problems

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Abstract We analyze a gradient recovery based linear finite element method to solve bi-harmonic equations and the corresponding eigenvalue problems. Our method uses only C^0 element, which avoids complicated construction of C^1 elements and nonconforming elements. Optimal error bounds under various Sobolev norms are established. Moreover, after a post-processing the recovered gradient is superconvergent to the exact one. Some numerical experiments are presented to validate our theoretical findings. As an application, the new method has been also used to solve 1-D fully nonlinear Monge–Ampère equation numerically.

Keywords Biharmonic equation · Fourth order eigenvalue problem · Gradient recovery · Linear finite element

Mathematics Subject Classification Primary 65N30 · Secondary 45N08

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1 Introduction

This is the first in a series of articles in which we study using gradient recovery technique as pre-processing tool to solve high-order (higher than 2) partial differential equations and associated eigenvalue problems. We start from the one dimensional problems in this paper. To fix the idea, we consider on $\Omega = (0, 1)$, the following string problem:

$$u'''' = f, \tag{1.1}$$

and the corresponding eigenvalue problem:

$$u'''' = \lambda u, \tag{1.2}$$

associated with one of the following two boundary conditions at $x = 0, 1$:

$$u = g, \quad u' = h, \tag{1.3}$$
$$u = g, \quad u'' = h. \tag{1.4}$$

The conforming finite element method requires C^1 space [3,4], which contains at least the Hermite basis functions. Although this requirement is not too much a restriction for the one-dimensional case, it is indeed a "heavy burden" for higher-dimensional situations. Therefore the C^1 element was almost abandoned from scientific and engineering computing since the 1980's. As an alternative, nonconforming finite elements find their market, c.f., [3,4,13]. The disadvantage of the nonconforming method lies in its delicate design of the finite element space in order to guarantee convergence.

In this paper, we analyze a recovery based linear finite element scheme to discrete (1.1) and (1.2) along with boundary conditions (1.3) or (1.4). Our scheme only uses values at nodal points as degrees of freedom, thus has much fewer global unknowns than nonconforming finite elements with the same convergence rate.

We notice a recent work [9] using gradient recovery operator in the finite element method to solve the bi-harmonic equation. The novelty of our approach lies in that (1) we do not need the orthogonal projection condition (R4), which is irrelevant for some popular recovery operators such as SPR and PPR; (2) our analysis does not follow the non-conforming element framework; and (3) we provide some convincing numerical examples to demonstrate the effectiveness of our methods (there is no numerical data in [9]).

Now we elaborate basic idea in some details. The variational formulation of (1.1) [or (1.2)] involves the term (u'', v''), which requires the second derivative from the discrete solution. Let u_h be a C^0 linear finite element solution. In general, u_h' is piecewise constants and discontinuous across each element, and further differentiation is out of question. Here we use a gradient (derivative in 1-D) recovery operator G_h to produce piecewise linear and globally continuous derivative $G_h u_h$ to replace u_h', so that further differentiation is possible. As a result, our proposed scheme is basically

$$((G_h u_h)', (G_h v_h)') = (f, v_h) \tag{1.5}$$

for (1.1) and

$$((G_h u_h)', (G_h v_h)') = \lambda_h (u_h, v_h) \tag{1.6}$$

for (1.2), respectively. More details involving different boundary conditions will be discussed later.

Moreover, we consider 1-D fully nonlinear Monge–Ampère equation:

$$(u''(x))^2 = f, \quad \text{in } [0, 1], \tag{1.7}$$
$$u = g, \quad \text{on } x = 0, 1. \tag{1.8}$$

We use the vanishing moment method to solve (1.7), the crux of which is that we approximate it by the following sequence fourth order quasilinear PDEs:

$$-\epsilon u''''(x) + (u''(x))^2 = f, \quad \text{in } [0, 1] \tag{1.9}$$
$$u = g, \quad u'' = \epsilon^2, \quad \text{on } x = 0, 1. \tag{1.10}$$

We use Newton method to yield a linearization of the PDE (1.9) before it is discretized. Given an approximation of the solution, u_k, we seek a perturbation δu such that

$$u_{k+1} = u_k + \delta u$$

fulfill the nonlinear equation (1.9). We insert u_{k+1} into nonlinear equation (1.9), and linearize the nonlinear term to get

$$-\epsilon \delta u'''' + 2(u_k''\delta u')' - 2u_k'''\delta u' = f - (u_k'')^2 + \epsilon u_k''''. \tag{1.11}$$

where we assume δu is so small that $(\delta u'')^2$ can be dropped.

Multiply (1.11) by a test function v and integrate on interval [0, 1], using integration by part to get the following weak form: Find $\delta u \in H^2 \cap H_0^1(0, 1)$

$$\begin{aligned}&-\epsilon(\delta u'', v'') - 2(u_k''\delta u', v') - 2(u_k'''\delta u', v)\\ &\quad = (f - (u_k'')^2, v) + (\epsilon u_k'', v'') - \epsilon^3(v'(1) - v'(0)), \quad \forall v \in H^2 \cap H_0^1(0, 1),\end{aligned} \tag{1.12}$$

where $(\cdot, \cdot)$ denote the inner product on interval. Note the perturbation δu should satisfy $\delta u = 0$ on $x = 0, 1$ since $u_k = g$ on $x = 0, 1$.

Then we use the gradient recovery operator G_h again to numerically solve (1.12) by finding $u \in V_h$ such that

$$\begin{aligned}&-\epsilon(G_h u', G_h v') - 2(G_h u_k' u', v') - 2(G_h G_h u_k' \delta u', v)\\ &\quad = (f - (G_h u_k')^2, v) + (\epsilon G_h u_k', G_h v') - \epsilon^3(G_h v(1) - G_h v(0)), \quad \forall v \in V_h,\end{aligned} \tag{1.13}$$

where V_h denotes a C^0 linear finite element space.

Note that generalization of the idea to higher dimensional setting is straightforward, i.e., $(\nabla \cdot G_h u_h, \nabla \cdot G_h v_h)$, even though the underlying theory is much more complicated and involved.

Gradient recovery technique has been applied in post-processing and achieved great success in scientific and engineering computation [1]. As an example, the celebrated Superconvergence Patch Recovery (SPR) by Zienkiewicz and Zhu [18] are widely used in commercial finite element packages such as Abaqus, DiffiPack, Nastran, etc. More recently the Polynomial Preserving Recovery (PPR) by Naga–Zhang has been adopted by COMSOL Multiphysics since 2008 [5,8]. For theoretical aspects of the gradient recovery technique, readers are referred to [16] and references therein.

As described above, here we use gradient recovery operator in pre-processing. As indicated in [16], at each node, $G_h u_h$ is basically a high accuracy local finite difference scheme to approximate the first derivative. As a consequence, at each node, $(G_h u_h)'$ could be viewed as a finite difference scheme for the second-order derivative with a larger stencil. An important feature is that this "larger stencil" is produced systematically by the gradient recovery, and it works for arbitrary meshes.

Some basic properties of various gradient recovery operators have been established by previous works, e.g., [10–12,14,17]. The most important properties include the polynomial preserving and boundedness

$$\|G_h u_h\|_0 \lesssim |u_h|_1. \tag{1.14}$$

However, in order to invest stability and convergence of the recovery operator in pre-processing stage, some further properties needs to be developed, e.g., if the inverse of (1.14) is valid. It turns out that

$$|u_h|_1 \lesssim h^{-1/2}\|G_h u_h\|_0, \tag{1.15}$$

and a counter-example demonstrates that the factor $h^{-1/2}$ is the best we can expect. Therefore, we need to re-examine the recovery operator and establish some further basic properties before pursuing stability and convergence of our proposed schemes (1.5) and (1.6). In Sects. 3 and 4 our scheme will be proved to produce optimal convergence rates in the L^2, H^1, and broken H^2 norms. Moreover, using the same gradient recovery to post-process u_h, the approximation solution $G_h u_h$ is superconvergence u' under the L^2 norm.

The remaining parts of this paper are organized as follows. Section 2 introduces a gradient recovery operator and its new properties. Section 3 is devoted to the discretization of the string problem. We present a recovery based linear finite element method and derive error estimates in various norms. Section 4 applies the new scheme to the corresponding 4th order eigenvalue problem. We prove optimal convergence rates for discrete eigenpairs. Some numerical experiments are presented in Sect. 5. Finally, we draw some concluding and remarks in Sect. 6.

Throughout the paper, the letter C denotes a generic positive constant, which may be different at different occurrences. For convenience, the symbol $\lesssim$ will be used: $x \lesssim y$ means $x \le Cy$ for some constants C independent of the mesh size. Then $x \sim y$ means both $x \lesssim y$ and $y \lesssim x$ hold.

2 A Gradient Recovery Operator and Its Properties

Let $\mathcal{T}_h = \{[x_{i-1}, x_i] : i = 1, \ldots, N\}$ be a partition of the domain Ω with mesh-size h. Let V_h be the standard linear finite element space corresponding to $\mathcal{T}_h$ of Ω with the following approximation property

$$\inf_{v_h \in V_h} \|u - v_h\|_0 + h|u - v_h|_1 \lesssim h^2\|u\|_2, \quad u \in H^2(\Omega). \tag{2.1}$$

We recall the gradient recovery operator $G_h : V_h \longrightarrow V_h$ defined by [18]

$$G_h v_h(x_i) = \frac{1}{|\omega_i|}\int_{\omega_i} v_h'(x)dx,$$

where $\omega_i = (x_{i-1}, x_{i+1}) \cap \Omega$, $i = 1, \ldots, N$. Denoting the nodal basis corresponding to the vertex x_i by ϕ_i, we have

$$G_h v_h(x) = \sum_{i=0}^{N} G_h v_h(x_i)\phi_i(x), \quad x \in \Omega.$$

By this definition, it is easy to obtain that

$$\|G_h v_h\|_0 \lesssim |v_h|_1, \quad v_h \in V_h \tag{2.2}$$

and when the mesh is *sufficient uniform*, we have [12],

$$\|u' - G_h u_I\|_j \lesssim h^{2-j}|u|_{3,\infty} \quad u \in W^{3,\infty}(\Omega), \quad j = 0, 1, \tag{2.3}$$

where u_I is the interpolation of u in V_h.

In the following, we study the other properties of the operator $G_h v_h$.

Theorem 2.1 *If $c_i \in \mathbb{R}, i = 0, \dots, N$ and $d_i : i = 1, \dots, N$ satisfies the conditions:*

$$c_{i-1} + c_i = 2d_i, \quad i = 1, \dots, N,$$

then for all $v_h \in V_h$,

$$\sum_{i=0}^{N} \int_{\omega_i} c_i G_h v_h(x_i)\phi_i(x)dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} d_i v_h'(x)dx. \tag{2.4}$$

Proof In fact,

$$\begin{aligned} \sum_{i=0}^{N} \int_{\omega_i} c_i G_h v_h(x_i)\phi_i &= \sum_{i=0}^{N} c_i G_h v_h)(x_i) \int_{\omega_i} \phi_i \\ &= \sum_{i=0}^{N} c_i G_h v_h(x_i) \frac{|\omega_i|}{2} \\ &= \sum_{i=0}^{N} \frac{1}{2} c_i \int_{\omega_i} v_h'(x)dx \\ &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} d_i v_h'(x)dx. \end{aligned}$$

□

Let

$$V_h^0 = \{v_h \in V_h : v_h = 0 \text{ on } \partial\Omega\}$$

and

$$V_h^{00} = \{v_h \in V_h : v_h = G_h v_h = 0 \text{ on } \partial\Omega\}.$$

Theorem 2.2 *If $\mathcal{T}_h$ is a quasi-uniform mesh, then*

$$|v_h|_1 \lesssim h^{-\frac{1}{2}} \|G_h v_h\|_0, \quad \forall v_h \in V_h, \tag{2.5}$$

$$\|v_h\|_0 \lesssim \|G_h v_h\|_0, \quad \forall v_h \in V_h^0, \tag{2.6}$$

and

$$|v_h|_1 \lesssim |G_h v_h|_1, \quad \forall v_h \in V_h^0. \tag{2.7}$$

Proof Choosing $d_i = v_h'(x_{i-\frac{1}{2}}), i = 1, \dots, N$ in (2.4), we obtain

$$|v_h|_1^2 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} d_i v_h'(x)dx = \sum_{i=0}^{N} c_i G_h v_h(x_i)|\omega_i|. \tag{2.8}$$

We now let $c_0 = 0$, $c_i = 2d_i - c_{i-1}$. By the fact that

$$|d_i| = h_i^{-\frac{1}{2}} |v_h|_{1,[x_{i-1},x_i]},$$

we have

$$|c_i| \lesssim h^{-\frac{1}{2}} |v_h|_1, \quad \forall i = 1, \dots, N.$$

Then by the fact that

$$\|G_h v_h\|_0 \sim \left(\sum_{i=0}^{N} |w_i| G_h v_h(x_i)^2 \right)^{\frac{1}{2}}, \tag{2.9}$$

we obtain (2.5).

If i is even,

$$\begin{aligned} v_h(x_{i+1}) = v_h(x_{i-1}) + |w_i| G_h v_h(x_i) &= v_h(x_{i-3}) + |w_{i-2}| G_h v_h(x_{i-2}) + |w_i| G_h v_h(x_i) \\ &= \cdots = v_h(x_1) + \sum_{j=1}^{i/2} |w_{2j}| G_h v_h(x_{2j}). \end{aligned} \tag{2.10}$$

Since for $v_h \in V_h^0$, we have

$$v_h(x_1) = |w_0| G_h v_h(x_0),$$

together with (2.10), we get

$$|v_h(x_{i+1})| \lesssim |w_0| \sum_{j=0}^{i/2} |w_{2j}| |G_h v_h(x_{2j})|. \tag{2.11}$$

If i is odd,

$$\begin{aligned} v_h(x_{i+1}) = v_h(x_{i-1}) + |w_i| G_h v_h(x_i) &= v_h(x_{i-3}) + |w_{i-2}| G_h v_h(x_{i-2}) + |w_i| G_h v_h(x_i) \\ &= \cdots = \sum_{j=0}^{\frac{i-1}{2}} |w_{2j+1}| G_h v_h(x_{2j+1}). \end{aligned} \tag{2.12}$$

In summary, for all $1 \le i \le N-1$, we have

$$\begin{aligned} |v_h(x_i)| &\le \sum_{j=0}^{N} |w_j| |G_h v_h(x_j)| \\ &\lesssim \left(\sum_{j=0}^{N} |w_j| \right)^{\frac{1}{2}} \left(\sum_{j=0}^{N} |w_j| |G_h v_h(x_j)|^2 \right)^{\frac{1}{2}} \\ &\lesssim \|G_h v_h\|_0. \end{aligned}$$

Consequently,

$$\|v_h\|_0 \lesssim \left(\sum_{i=0}^{N} |w_i| v_h(x_i)^2 \right)^{\frac{1}{2}} \lesssim \|G_h v_h\|_0.$$

The desired result (2.6) is proved.

Next we show (2.7). Denoting $m_i = (G_h v_h)'(x_{i-\frac{1}{2}}), i = 1, \ldots, N$ and $h_0 = 1, m_0 = G_h v_h(x_0)$ for convenience, we have

$$G_h v_h(x_i) = \sum_{j=0}^{i} h_j m_j.$$

Then

$$\begin{aligned}
|v_h|_1^2 &= \sum_{i=0}^{N} c_i |\omega_i| \left(\sum_{j=0}^{i} h_j m_j \right) \\
&= \sum_{j=0}^{N} h_j m_j \sum_{i=j}^{N} c_i |\omega_i| \\
&= 2 \sum_{j=0}^{N} h_j m_j \left(\sum_{i=j}^{N} d_i h_i \right) \\
&\le 2 \sum_{j=0}^{N} |h_j m_j| \sum_{i=1}^{N} |d_i h_i| \\
&\lesssim |v_h|_1 |G_h v_h|_1,
\end{aligned}$$

from which the statement (2.7) follows. □

Remark The following inverse inequality

$$|v_h|_1 \lesssim |G_h v_h|_0$$

is Not valid in general. In fact, let N be even, $x_i = ih, h = 1/N, i = 0, 1 \ldots, N$ and

$$v_h(x_i) = \begin{cases} h & i = 3, 5, \ldots, N-3, \\ 0 & otherwise \end{cases}$$

Then one easily verifies that

$$|v_h|_1^2 = 1 - 4h, \quad |G_h v_h|_0^2 = h/3, \quad |G_h v_h|_1^2 = 2h^{-1}.$$

Theorem 2.3 *There holds*

$$\left| \int_\Omega g(G_h v_h - v_h') \right| \lesssim h|g|_1 |v_h|_1, \quad g \in H^1(\Omega), \quad v_h \in V_h. \tag{2.13}$$

Proof For all $i = 0, 1, \ldots, N$, we choose

$$c_i = \frac{1}{|\omega_i|} \int_{\omega_i} g(x)dx.$$

Then

$$\|g - c_i\|_{0,\omega_i} \lesssim |\omega_i||g|_{1,\omega_i}.$$

By (2.4)

$$\int_\Omega g(x)(G_h v_h - v_h')(x)dx = \sum_{i=0}^N \int_{\omega_i} (G_h v_h)(x_i) g(x)\phi_i(x)dx - \sum_{i=1}^N \int_{x_{i-1}}^{x_i} g(x)v_h'(x)dx$$
$$= \sum_{i=0}^N \int_{\omega_i} (g-c_i)(G_h v_h)(x_i)\phi_i(x)dx - \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (g-d_i)v_h'(x)dx.$$

On one side,

$$\left|\int_{\omega_i} (g-c_i)(G_h v_h)(x_i)\phi_i dx\right| \lesssim |\omega_i|^{\frac{1}{2}} |G_h v_h(x_i)| \cdot \|g-c_i\|_{0,\omega_i}$$
$$\lesssim |v_h|_{1,\omega_i} \cdot |\omega_i||g|_{1,\omega_i}$$
$$\lesssim h|v_h|_{1,\omega_i}|g|_{1,\omega_i}.$$

On the other side,

$$\left|\int_{x_{i-1}}^{x_i} (g-d_i)v_h'(x)dx\right| \lesssim \|g-d_i\|_{0,[x_{i-1},x_i]}|v_h|_{1,[x_{i-1},x_i]}$$
$$\lesssim h|g|_{1,\bar{\omega}_i}|v_h|_{1,[x_{i-1},x_i]},$$

where $\bar{\omega}_i = \omega_{i-1} \cup \omega_i$. Therefore, by Cauchy-Schwartz inequality, we obtain (2.13). □

Moreover, if g has more regularity, we can get an improved result than Theorem 2.3.

Theorem 2.4 *There holds*

$$\left|\int_\Omega g(G_h v_h - v_h')\right| \lesssim h^2|g|_2|G_h v_h|_1, \quad g \in H_0^1(\Omega) \cap H^2(\Omega), \quad v_h \in V_h^0. \tag{2.14}$$

Proof For any $w \in H^1(\Omega)$, let $w_I = \sum_{i=0}^N w(x_i)\phi_i \in V_h$ be the Lagrange interpolation of w. For all $v_h \in V_h^0$,

$$\int_\Omega (gG_h v_h)_I = \sum_{i=0}^N \int_{\omega_i} g(x_i)G_h v_h(x_i)\phi_i$$
$$= \sum_{i=0}^N \frac{1}{2} g(x_i) \int_{\omega_i} v_h'$$
$$= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \frac{1}{2}(g(x_{i-1}) + g(x_i))v_h'$$
$$= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} g_I v_h' = \int_\Omega g_I v_h'.$$

Therefore,

$$\left|\int_\Omega g(G_h v_h - v_h')\right| = \left|\int_\Omega [gG_h v_h - (gG_h v_h)_I] - \int (g-g_I)v_h'\right|$$
$$\lesssim h^2(|gG_h v_h|_2 + |g|_2|v_h|_1)$$
$$\lesssim h^2|g|_2|G_h v_h|_1).$$

The declaration (2.14) is proved. □

3 Recovery Based FEM for String Problems

This section is devoted to the presentation of a recovery based linear element scheme for solving (1.1) and (1.3), which reads as: Find $u_h \in V_h$ such that $u_h = g$, $G_h u_h = h$ and

$$a_h(u_h, v_h) := ((G_h u_h)', (G_h v_h)') = (f, v_h), \quad v_h \in V_h^{00}. \tag{3.1}$$

Taking in account the different boundary condition (1.4), the recovery based linear element scheme for solving (1.1) and (1.4) is to find $u_h \in V_h$ such that $u_h = g$ and

$$a_h(u_h, v_h) = (f, v_h) + (h(1)G_h v_h(1) - h(0)G_h v_h(0)), \quad v_h \in V_h^0. \tag{3.2}$$

Note that by (2.7), $|G_h v_h|_1$ is a norm in V_h^0. Then by the Lax–Milagram theorem, (3.1) and (3.2) both have a unique solution.

We next analyze the convergence properties of the scheme (3.1). Similar results can also be obtained for the scheme (3.2). Since the classic Lagrange interpolating function u_I of u is not necessary in V_h^{00}, we introduce a slightly modified interpolation function $\hat{u}_I \in V_h^{00}$ defined by

$$\hat{u}_I(x_i) = u(x_i), \quad i \in \{0, \dots, N\} \setminus \{1, N-1\}$$

and by letting $\hat{u}_I(x_1), \hat{u}_I(x_{N-1})$ satisfy

$$G_h \hat{u}_I(x_0) = G_h \hat{u}_I(x_N) = 0.$$

By this definition, it is easy to verify that the inequality

$$\|u' - G_h \hat{u}_I\|_j \lesssim h^{2-j}|u|_{3,\infty}, \quad j = 0, 1 \tag{3.3}$$

also holds for $u \in W^{3,\infty}(\Omega)$ satisfying $u = u' = 0$ on $\partial\Omega$.

Theorem 3.1 *Assume u is the solutions of* (1.1) *and* (1.3)*, u_h is the finite element solution of* (3.1)*. Let $\hat{u}_I$ be the interpolating of u in V_h^{00}, we have*

$$|G_h(u_h - \hat{u}_I)|_1 \lesssim h(\|u\|_{3,\infty} + \|u\|_4). \tag{3.4}$$

As a direct consequence,

$$\|u'' - (G_h u_h)'\|_0 \lesssim h(\|u\|_{3,\infty} + \|u\|_4). \tag{3.5}$$

Proof We first show (3.4). For all $v_h, w_h \in V_h^{00}$,

$$\begin{aligned}((G_h u_h)', (G_h v_h)') &= (f, v_h) = (u^{(4)}, v_h) = -(u^{(3)}, v_h') \\ &= (u^{(3)}, G_h v_h - v_h') + (u'', (G_h v_h)') \\ &= (u^{(3)}, G_h v_h - v_h') + (u'' - (G_h \hat{u}_I)', (G_h v_h)') + ((G_h \hat{u}_I)', (G_h v_h)').\end{aligned}$$

Therefore

$$((G_h(u_h - \hat{u}_I))', (G_h v_h)') = E_1 + E_2 \tag{3.6}$$

with

$$\begin{aligned}E_1 &= E_1(v_h) = (u^{(3)}, G_h v_h - v_h'), \\ E_2 &= E_2(v_h) = (u'' - (G_h \hat{u}_I)', (G_h v_h)').\end{aligned}$$

We next estimate E_1, E_2 separately. First, by (2.13) and (2.7),

$$|E_1| \lesssim h\|u\|_4 |v_h|_1 \lesssim h\|u\|_4 |G_h v_h|_1 \tag{3.7}$$

Secondly, by (3.3)

$$|E_2| \lesssim h\|u\|_{3,\infty}|G_h v_h|_1. \tag{3.8}$$

Substituting (3.7)–(3.8) into (3.6), we obtain that

$$|((G_h(u_h - \hat{u}_I))', (G_h v_h)')|_1 \lesssim h(\|u\|_{3,\infty} + \|u\|_4)|G_h v_h|_1. \tag{3.9}$$

Letting $v_h = u_h - u_I$, we immediately obtain (3.4). The statement (3.5) is a direct consequence of (3.4) and (3.3). □

In the following, we estimate the error $\|u' - G_h u_h\|_0$ by using the Aubin–Nitsche techniques and the improved result in Theorem 2.4.

Theorem 3.2 *Assume u is the solutions of* (1.1) *and* (1.3)*, u_h is the finite element solution of* (3.1)*. There holds*

$$\|u' - G_h u_h\|_0 \lesssim h^2\|u\|_5. \tag{3.10}$$

Proof Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of

$$(w', v') = (G_h u_h - u', v), \quad \forall v \in H_0^1(\Omega). \tag{3.11}$$

Then

$$\|w\|_2 \lesssim \|G_h u_h - u'\|_0 \tag{3.12}$$

and

$$\|G_h u_h - u'\|_0^2 = (w', (G_h u_h - u')'). \tag{3.13}$$

Moreover it is known that

$$\|w - \hat{w}_I\|_i \lesssim h^{2-i}\|w\|_2, \quad i = 0, 1.$$

Then by choosing $w_h \in V_h^{00}$ such that

$$G_h w_h = \hat{w}_I,$$

we have

$$\|w - G_h w_h\|_i \lesssim h^{2-i}\|w\|_2, \quad i = 0, 1. \tag{3.14}$$

We are ready to estimate $\|u' - G_h u_h\|_0$.

By (3.13), for all $w_h \in V_h^0$,

$$\begin{aligned}\|G_h u_h - u'\|_0^2 &= (w' - (G_h w_h)', (G_h u_h)' - u'') + ((G_h w_h)', (G_h u_h)' - u'')\\ &= (w' - (G_h w_h)', (G_h u_h)' - u'') + (f, w_h) + (u^{(3)}, G_h w_h))\\ &= (w' - (G_h w_h)', (G_h u_h)' - u'') + (u^{(3)}, G_h w_h - w_h'). \end{aligned} \tag{3.15}$$

By (3.14), (3.12) and (3.5), we have

$$|(w' - (G_h w_h)', (G_h u_h)' - u'')| \lesssim h^2\|G_h u_h - u'\|_0\|u\|_4. \tag{3.16}$$

On the other hand, by (2.14), (3.14) and (3.12)

$$|(u^{(3)}, G_h w_h - w_h')| \lesssim h^2\|u\|_5|G_h w_h|_1 \lesssim h^2\|u\|_5\|w\|_2 \lesssim h^2\|u\|_5\|G_h u_h - u'\|_0. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we obtain (3.10). □

At the end of this section, we prove the usual H^1, L^2 error estimates as follows.

Theorem 3.3 *Under the same assumptions as Theorem 3.2. There holds*

$$\|u - u_h\|_1 \lesssim h\|u\|_4. \tag{3.18}$$

$$\|u - u_h\|_0 \lesssim h^2\|u\|_5. \tag{3.19}$$

Proof First by (2.7), (2.3) and (3.5),

$$\begin{aligned}\|u_I - u_h\|_1 &\lesssim \|G_h(u_I - u_h)\|_1\\ &\lesssim \|u' - G_h u_I\|_1 + \|u' - G_h u_h\|_1\\ &\lesssim h\|u\|_4,\end{aligned}$$

then

$$\begin{aligned}\|u - u_h\|_1 &\lesssim \|u - u_I\|_1 + \|u_I - u_h\|_1\\ &\lesssim h\|u\|_4,\end{aligned}$$

which is the desired result (3.18).

Next, since from (2.6), (2.3) and (3.10),

$$\begin{aligned}\|u_I - u_h\|_0 &\lesssim \|G_h(u_I - u_h)\|_0\\ &\lesssim \|u' - G_h u_I\|_0 + \|u' - G_h u_h\|_0\\ &\lesssim h^2\|u\|_5,\end{aligned}$$

we have

$$\begin{aligned}\|u - u_h\|_0 &\lesssim \|u - u_I\|_0 + \|u_I - u_h\|_0\\ &\lesssim h^2\|u\|_5.\end{aligned}$$

Therefore, we obtain (3.19) and complete the proof. □

Remark Compare Theorem 3.2 and (3.18), then we find the approximation solution is superconvergence to the exact one in H^1 norm after recovery.

Remark If we replace the scheme (3.2) by

$$a_h(u_h, v_h) = (f, v_h) + (h(1)v_h'(1) - h(0)v_h'(0)), \quad v_h \in V_h^0, \tag{3.20}$$

the convergence rates for $\|u' - G_h u_h\|_0$ and $\|u' - G_h u_h\|_1$ will be one half order lower than the ones of the scheme (3.2), which can be seen from the numerical experiment in Sect. 5.

4 Recovery Based FEM for 4th Order Eigenvalue Problems

In this section we consider to discrete 4th order eigenvalue problem (1.2). The corresponding weak form for (1.2) and (1.3) with $g = h = 0$ is: Find $(\lambda, u) \in \mathbb{R} \times V$ such that $\|u\|_0 = 1$ and

$$a(u, v) := (u'', v'') = (\lambda u, v), \quad \forall v \in V, \tag{4.1}$$

where $V := H_0^2(\Omega) = \{v \in H^2(\Omega) \mid v = v' = 0, \text{on } 0, 1\}$. And the corresponding weak form for (1.2) and (1.4) with $g = h = 0$ is: Find $(\lambda, u) \in \mathbb{R} \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)$ such that $\|u\|_0 = 1$ and

$$a(u, v) = (\lambda u, v), \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4.2}$$

The discrete eigenvalue problem for (4.1) seeks eigenpairs $(\lambda_h, u_h) \in \mathbb{R} \times V_h^{00}$ with $\|u_h\|_0 = 1$ and

$$a_h(u_h, v) = \lambda_h(u_h, v), \quad \forall v \in V_h^{00}. \tag{4.3}$$

And the discrete eigenvalue problem for (4.2) seeks eigenpairs $(\lambda_h, u_h) \in \mathbb{R} \times V_h^0$ with $\|u_h\|_0 = 1$ and

$$a_h(u_h, v) = \lambda_h(u_h, v), \quad \forall v \in V_h^0. \tag{4.4}$$

Next, we only consider the problem (4.1) and its discretization (4.3), and similar results also can be obtained for (4.2) and its discretization (4.4).

It is well known from the spectral theory of selfadjoint compact operators [2,6] that the eigenvalue problem (4.1) has countably many eigenvalues, which are real and positive with $+\infty$ as only accumulation point. Suppose that the eigenvalues are enumerated as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

and let $(u_1, u_2, u_3, \ldots)$ be some L^2-orthonormal system of corresponding eigenfunctions. For any $j \in \mathbb{N}$, the eigenspace corresponding to λ_j is defined as

$$V_{\lambda_j} := \{u \in H_0^2(\Omega) \mid (\lambda_j, u) \text{ satisfies } (4.1)\} = \text{span}\{u_k \mid k \in \mathbb{N} \text{ and } \lambda_k = \lambda_j\}.$$

In the present case of the 4th order eigenvalue problem which is the inverse of a compact operator, the spaces V_{λ_j} have finite dimension. From (2.7) it is known that $\|v\|_1 \lesssim |G_h v|_1$ for any $v \in V_h^0$. Therefore, the discrete eigenvalues for (4.3) can be enumerated

$$0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \lambda_{h,3} \leq \cdots$$

with corresponding L^2-orthonormal eigenfunctions $(u_{h,1}, u_{h,2}, u_{h,3}, \ldots)$. The discrete eigenspace corresponding to $\lambda_{h,j}$ is defined as

$$V_{\lambda_{h,j}} := \{u \in V_h^{00} \mid (\lambda_{h,j}, u_h) \text{ satisfies } (4.3)\} = \text{span}\{u_{h,k} \mid k \in \mathbb{N} \text{ and } \lambda_{h,k} = \lambda_{h,j}\}.$$

Given $f \in V$, let $u \in V$ denote the unique solution to the linear problem

$$a(u, v) = (f, v), \quad \forall v \in V.$$

Then define $u = Tf$. Therefore (4.1) has an equivalent formulation

$$Tu = \lambda^{-1} u = \mu u,$$

where $\mu := \lambda^{-1}$ and $V_\mu := V_\lambda$. Moreover, T is selfadjoint, which can be seen from the following equality: for any $u, v \in L^2$,

$$(Tu, v) = (v, Tu) = a(Tv, Tu) = a(Tu, Tv) = (u, Tv).$$

Similarly, define the discrete operator $T_h : L^2 \to V_h^{00} \subset L^2$,

$$a_h(T_h f, v) = (f, v), \quad \forall v \in V_h^{00}. \tag{4.5}$$

Therefore, (4.3) has an equivalent formulation

$$T_h u_h = \lambda_h^{-1} u_h = \mu_h u_h,$$

where $\mu_h := \lambda_h^{-1}$ and $V_{\mu_h} := V_{\lambda_h}$. Moreover, T_h is also selfadjoint, since we have for any $u, v \in L^2$

$$(T_h u, v) = (v, T_h u) = a_h(T_h v, T_h u) = a_h(T_h u, T_h v) = (u, T_h v).$$

Combining the error estimate (3.19) for the discretization of source problem with the regularity result for the solution of source problem, we have

$$\|Tf - T_h f\|_0 = \|u - u_h\|_0 \le Ch^r \|f\|_0,$$

where $r > 0$. Then it immediately follows that

$$\|T - T_h\| \to 0, \quad \text{if } h \to 0,$$

which induces $\mu_{h,k} \to \mu_k$ and $\lambda_{h,k} \to \lambda_k$ [15, Theorem1.4.5].

Assume $\mu, \mu_h \neq 0$ are eigenvalues of T and T_h respectively and $\mu_h \to \mu$. Since μ is isolated, there exists a constant $d(\mu) > 0$, such that provided h is sufficiently small it holds that

$$\min_{\mu_j \neq \mu} |\mu_h - \mu_j| \ge d(\mu).$$

Lemma 4.1 [15, Theorem1.4.6] *Assume* $\|T - T_h\| \to 0 \quad (h \to 0)$, (μ_h, u_h) *are the k-th eigenpair of* T_h *and* $\|u_h\|_0 = 1$.μ *is the k-th eigenvalue of* T. *Then* $\mu_h \to \mu$, *and there exists* $u \in V_\mu(\|u\|_0 = 1)$ *such that*

$$\mu - \mu_h = \frac{1}{(u, u_h)}(Tu_h - T_h u_h, u), \tag{4.6}$$

$$\|u_h - u\| \le \frac{\|Tu_h - T_h u_h\|_0}{d(\mu)} \left(1 + \frac{\|Tu_h - T_h u_h\|_0^2}{d(\mu)^2}\right)^{\frac{1}{2}}. \tag{4.7}$$

Using Lemma 4.1, we can estimate the error for the discrete eigenpairs as follows.

Theorem 4.2 *Under the same assumptions as Lemma 4.1, we have* $\lambda_h \to \lambda$ *and there exists* $u \in V_\lambda$, $\|u\|_0 = 1$ *such that*

$$\lambda_h - \lambda = \frac{\lambda\lambda_h}{(u, u_h)}((T - T_h)u, u) + R_1, \tag{4.8}$$

$$\|u_h - u\|_0 \lesssim \|(T - T_h)u\|_0, \tag{4.9}$$

$$|u - u_h|_1 = \lambda|(T - T_h)u|_1 + R_2, \tag{4.10}$$

$$|G_h u_h - u'|_0 \lesssim \lambda\|(Tu)' - G_h(T_h u)\|_0 + \|(T - T_h)u\|_0, \tag{4.11}$$

$$|G_h u_h - u'|_1 = \lambda|(Tu)' - G_h(T_h u)|_1 + R_3, \tag{4.12}$$

where $|R_1| \lesssim \|(T - T_h)u\|_0^2$, $|R_2| \lesssim \|(T - T_h)u\|_0, $, $|R_3| \lesssim \|(T - T_h)u\|_0$.

Proof Equations (4.8) and (4.9) can be proved by using Lemma 4.1. From (4.3) and the definition of T_h, it follows that

$$\begin{aligned}|G_h(u_h - \lambda T_h u)|_1^2 &= a_h(u_h - \lambda T_h u, u_h - \lambda T_h u) = (\lambda_h u_h - \lambda u, u_h - \lambda T_h u)\\ &\le \|\lambda_h u_h - \lambda u\|_0 \|u_h - \lambda T_h u\|_0 \le (\|\lambda_h u_h - \lambda u\|_0 + \|u_h - \lambda T_h u\|_0)^2.\end{aligned}$$

Together with (4.8) and (4.9), we obtain

$$|G_h(u_h - \lambda T_h u)|_1 \lesssim \|(T - T_h)u\|_0. \tag{4.13}$$

Therefore,

$$|u' - G_h u_h|_1 = \lambda|(Tu)' - G_h(T_h u)|_1 + R_3,$$

where from (4.13) it induces that

$$|R_3| = ||u' - G_h u_h|_1 - \lambda|(Tu)' - G_h(T_h u)|_1| = ||u' - G_h u_h|_1 - |u' - \lambda G_h(T_h u)|_1| \\ \leq |G_h u_h - \lambda G_h(T_h u)|_1 \lesssim \|(T - T_h)u\|_0.$$

Therefore, we obtain (4.12).

It follows from (2.7) and (4.13) that

$$|u_h - \lambda T_h u|_1 \lesssim |G_h(u_h - \lambda T_h u)|_1 \lesssim \|(T - T_h)u\|_0. \tag{4.14}$$

Hence,

$$|u - u_h|_1 = \lambda|(T - T_h)u|_1 + R_2,$$

where from (4.14)

$$|R_2| = ||u - u_h|_1 - \lambda|(T - T_h)u|_1| \leq |u_h - \lambda T_h u|_1 \lesssim \|(T - T_h)u\|_0.$$

This is the desired result (4.10).

Finally, from (4.13)

$$\|u' - G_h u_h\|_0 \leq \|(\lambda T u)' - \lambda G_h T_h u\|_0 + \|\lambda G_h T_h u - G_h u_h\|_0 \\ \lesssim \|(\lambda T u)' - \lambda G_h T_h u\|_0 + \|(T - T_h)u\|_0,$$

which is the desired result (4.11). Consequently, we complete the proof. □

Theorem 4.2 transfers the error estimates of eigenvalue problem into the ones of the corresponding source problem. Therefore, from the known error estimates of source problem in Sect. 3 it immediately induces the following result.

Theorem 4.3 *Assume that* (λ_h, u_h) *is the kth eigenpair of* (4.3), $\|u_h\|_0 = 1$ *and* λ *is the kth eigenvalue of* (4.1)*, we have* $\lambda_h \to \lambda$ *and there exists* $u \in V_\lambda$ *with* $\|u\|_0 = 1$ *such that*

$$|\lambda - \lambda_h| \lesssim h^2\|u\|_5,$$
$$\|u_h - u\|_0 \lesssim h^2\|u\|_5,$$
$$|u - u_h|_1 \lesssim h\|u\|_4,$$
$$|G_h u_h - u'|_0 \lesssim h^2\|u\|_5,$$
$$|G_h u_h - u'|_1 \lesssim h\|u\|_4.$$

5 Numerical Experiments

In this section, we perform a sequence of numerical tests to study the convergence behavior and show the effectiveness of our algorithm.

5.1 Numerical Experiments for String Problems

We now consider the numerical experiments for string problem (1.1) with two boundary conditions (1.3) and (1.4). The results in Sect. 3 indicate that we can obtain optimal convergence rates under L^2, H^1 and discrete H^2 norms. Moreover, after a post-processing the recovered gradient is superconvergent to the exact one.

Test 1 In (1.1) and (1.3) choose $f = \pi^4 \sin(\pi x)$, $g = 0$ and $h = \pi \cos(\pi x)$ such that the exact solution is $u = \sin(\pi x)$. The convergence rates under each norm shown in Table 1 is expected by the above theory.

Table 1 Test 1

Mesh size	$\|u-u_h\|_0$	Rate	$\|u-u_h\|_1$	Rate	$\|u'-G_hu_h\|_0$	Rate	$\|u'-G_hu_h\|_1$	Rate
1.56e−2	6.36e−4		3.17e−3		1.14e−3		1.01e−1	
7.81e−3	1.62e−4	1.98	1.58e−2	1.01	2.89e−4	1.98	5.00e−2	1.02
3.91e−3	4.10e−5	1.99	7.87e−3	1.00	7.31e−5	1.99	2.49e−2	1.01
1.95e−3	1.01e−5	1.99	3.94e−3	1.00	1.81e−5	2.00	1.24e−2	1.00
9.77e−4	2.51e−6	2.00	1.97e−3	1.00	4.52e−6	1.99	6.19e−3	1.00

Table 2 Test 2

Mesh size	$\|u-u_h\|_0$	Rate	$\|u-u_h\|_{H^1}$	Rate	$\|u'-G_hu_h\|_0$	Rate	$\|u'-G_hu_h\|_{H^1}$	Rate
1.56e−2	4.63e−4		3.01e−2		1.39e−3		9.38e−2	
7.81e−3	1.18e−4	1.97	1.50e−2	1.01	3.57e−4	1.97	4.61e−2	1.03
3.91e−3	2.95e−5	1.98	7.49e−3	1.00	9.01e−5	1.98	2.28e−2	1.01
1.95e−3	7.51e−6	1.99	3.74e−3	1.00	2.25e−5	1.99	1.13e−2	1.01
9.77e−4	1.88e−6	1.98	1.87e−3	1.00	5.63e−6	1.98	5.66e−3	1.00

Table 3 Test 3 for the scheme (3.20)

Mesh size	$\|u-u_h\|_0$	Rate	$\|u-u_h\|_{H^1}$	Rate	$\|u'-G_hu_h\|_0$	Rate	$\|u'-G_hu_h\|_{H^1}$	Rate
1.56e−2	7.37e−4		4.73e−2		9.30e−3		1.00e+0	
7.81e−3	1.87e−4	1.98	2.42e−2	0.97	3.28e−3	1.50	0.72e+0	0.48
3.91e−3	4.71e−5	1.99	1.22e−2	0.98	1.16e−3	1.50	0.51e+0	0.49
1.95e−3	1.18e−5	1.99	6.14e−3	0.99	4.10e−4	1.50	0.36e+0	0.50
9.77e−4	2.93e−6	2.01	3.08e−3	1.00	1.45e−4	1.50	0.26e+0	0.50

Test 2 In (1.1) and (1.3) we take $f = e^{x^2}(12+48x^2+16x^4)$, $g = e^{x^2}$ and $h = 2xe^{x^2}$ such that the exact solution is $u = e^{x^2}$. We observe that the convergence rates shown in Table 2 is expected by the above theory.

Test 3 We compute (1.1) with natural boundary condition (1.4). Choose $f = e^{x^2}(12 + 48x^2 + 16x^4)$, $g = e^{x^2}$ and $h = 2(1 + 2x^2)e^{x^2}$ such that the exact solution is $u = e^{x^2}$. Table 3 shows the convergence rates under various norms for the scheme (3.20), which indicates the ones under the norms $\|u' - G_hu_h\|_0$, $|u' - G_hu_h|_1$ is one half order lower than optimal. As a comparison, we list in Table 4 the results obtained by the formulation (3.2). We observe that the numerical solutions converges with optimal rates in various norms as mesh size decreases. These results coincide with the above theory.

5.2 Numerical Experiments for 4th Order Eigenvalue Problems

We now consider the numerical experiments for 4th order eigenvalue problem (1.2) with two boundary conditions (1.3) and (1.4). As indicated in Sect. 4, we observe once again the discrete eigenpairs achieve optimal convergence rates in various norms from the examples below.

Table 4 Test 3 for the scheme (3.2)

Mesh size	$\|u-u_h\|_0$	Rate	$\|u-u_h\|_1$	Rate	$\|u'-G_hu_h\|_0$	Rate	$\|u'-G_hu_h\|_1$	Rate
1.56e−02	4.34e−04		3.01e−02		1.07e−03		9.38e−02	
7.81e−03	1.11e−04	1.96	1.49e−02	1.00	2.72e−04	1.97	4.60e−02	1.02
3.90e−03	2.80e−05	1.98	7.48e−03	1.00	6.85e−05	1.98	2.28e−02	1.01
1.95e−03	7.05e−06	1.99	3.74e−03	1.00	1.72e−05	1.99	1.13e−02	1.00
9.76e−04	1.73e−06	2.02	1.87e−03	1.00	4.21e−06	2.03	5.65e−03	1.00

Table 5 Test 4: error estimate for eigenvalue

Mesh size	$\lambda-\lambda_h$	Rate
1.56e−02	5.30e−01	
7.81e−03	1.34e−01	1.97
3.90e−03	3.39e−02	1.98
1.95e−03	8.52e−03	1.99
9.76e−04	2.14e−03	1.99

Table 6 Test 5 for $j=1$: error estimate for eigenvalue

Mesh size	$\lambda-\lambda_h$	Rate
1.56e−02	5.59e−02	
7.81e−03	1.43e−02	1.96
3.90e−03	3.62e−03	1.98
1.95e−03	9.11e−04	1.99

Test 4 Consider the eigenvalue problem (1.2) with coupled boundary conditions $u(0)=u'(0)=u(1)=u'(1)=0$. Since the exact eigenpair for this problem is unknown, choose 237.72106753 as the accurate approximation of the exact smallest eigenvalue which was computed in [7]. The convergence rate for the eigenvalue error is shown at Table 5, which is as expected by the above theory. Also it is found that the numerical eigenvalue approximates the exact eigenvalue from below.

Test 5 It is easy to find the eigenvalue of the eigenvalue problem (1.2) and (1.4) is

$$\lambda_j=(\pi j)^4,\quad u_j=\sqrt{2}\sin\pi jx.$$

The convergence rates for the smallest eigenpair of the formulation (4.4) are shown at Tables 6 and 7. Also it is found that the numerical eigenvalue approximates the exact eigenvalue from below, which is also seen from Table 8 for the second small eigenvalue.

The numerical results above indicate that the accuracy of the recovery based linear element method is comparable with quadratic element method, since the proposed method achieves 1st order convergence rate in discrete H^2 norm and 2nd order rate in H^1 norm after a post-processing. Moreover, the proposed method using linear element only has one half degrees of freedom than the known C^1 element, e.g., cubic Hermite element.

Table 7 Test 5 for $j = 1$: error estimate for eigenvector

Mesh size	$\|u - u_h\|_0$	Rate	$\|u - u_h\|_1$	Rate	$\|u' - G_h u_h\|_0$	Rate	$\|u' - G_h u_h\|_1$	Rate
1.56e−02	9.26e−05	0	4.47e−02	0	5.30e−04	0	1.43e−01	0
7.81e−03	2.26e−05	2.03	2.22e−02	1.00	1.45e−04	1.87	7.07e−02	1.01
3.90e−03	5.62e−06	2.00	1.11e−02	1.00	3.78e−05	1.93	3.51e−02	1.00
1.95e−03	1.40e−06	2.00	5.56e−03	1.00	9.65e−06	1.97	1.75e−02	1.00

Table 8 Test 5 for $j = 2$: error estimate for eigenvalue

Mesh size	$\lambda - \lambda_h$	Rate
1.56e−02	3.57e+00	
7.81e−03	9.16e−01	1.96
3.90e−03	2.31e−01	1.98
1.95e−03	5.83e−02	1.99

5.3 Numerical Experiments for 1-D Monge–Ampère Equation

Now we solve 1-D Monge Ampère equation (1.7) and (1.8) numerically. We provide several numerical experiments to gauge the efficiency of the scheme (1.13). We show rates of convergence for both $\|u - u^\epsilon\|$ and $\|u^\epsilon - u_h^\epsilon\|$.

Test 6 For this test, we calculate $\|u - u_h^\epsilon\|$ for fixed $h = 0.001$ but varying ϵ. Since h is so small that u_h^ϵ can be viewed as a good approximation of u^ϵ. Then $\|u - u_h^\epsilon\|$ approximate the error $\|u - u^\epsilon\|$ well. We set to solve the Monge–Ampère problem with the following test functions:

$$u = (1 + x^2)e^{x^2}, \ g = (1 + x^2)e^{x^2},$$
$$f = (16 + 112x^2 + 228x^4 + 112x^6 + 16x^8)e^{2x^2}.$$

In this case, our experiments tell us the initial guess can be any convex function satisfying boundary condition, where the convex property is necessary. Moreover, the convex property can be maintained at each Newton iteration by our algorithm. After having computed the errors, we estimate the rate of convergence with respect to ϵ in various norms. Table 9 clearly shows $\|u - u^\epsilon\|_0 \approx \|u - u_h^\epsilon\|_0 = O(\epsilon)$. Similarly, we see from Table 9 that $|u - u^\epsilon|_{H^1} \approx O(\epsilon^{3/4})$, $|u' - G_h u^\epsilon|_0 \approx O(\epsilon^{3/4})$ and $|u' - G_h u_h^\epsilon|_{H^1} \approx O(\epsilon^{1/4})$.

We also note that in this case, we can compute the problem for the case $\epsilon = 0$ since the exact solution is smooth. Note that the added boundary condition $u''(x) = \epsilon^2$ can be changed to other boundary conditions $u''(x) = \epsilon^\alpha$ where α can be any positive real number or even $u''(x) = 0$.

Test 7 As Test 6, we calculate $\|u - u_h^\epsilon\|$ for fixed $h = 0.001$ but varying ϵ in order to approximate the error $\|u - u^\epsilon\|$. However, in this test we set to solve the Monge–Ampère problem with the following test functions:

$$u = -(1 + x^2)e^{x^2}, \ g = -(1 + x^2)e^{x^2},$$
$$f = (16 + 112x^2 + 228x^4 + 112x^6 + 16x^8)e^{2x^2},$$

Table 9 Test 6

ϵ	$\|u - u_h^\epsilon\|_0$	Rate	$\|u' - G_h u_h^\epsilon\|_0$	Rate	$\|u' - G_h u_h^\epsilon\|_{H^1}$	Rate
1.00	1.81e−01	0	1.05e+00	0	1.35e+01	0
0.75	1.35e−01	1.02	8.88e−01	0.59	1.28e+01	0.18
0.50	8.56e−02	1.12	6.87e−01	0.63	1.17e+01	0.21
0.25	4.02e−02	1.08	4.35e−01	0.65	9.99e+00	0.23
0.10	1.66e−02	0.96	2.32e−01	0.68	7.93e+00	0.25
0.075	1.27e−02	0.92	1.90e−01	0.69	7.36e+00	0.25
0.05	8.78e−03	0.91	1.42e−01	0.70	6.64e+00	0.25
0.025	4.67e−03	0.91	8.71e−02	0.71	5.56e+00	0.25
0.0125	2.47e−03	0.91	5.28e−02	0.72	4.66e+00	0.25
0.005	1.05e−03	0.93	2.71e−02	0.72	3.70e+00	0.25
0.0025	5.46e−04	0.95	1.62e−02	0.73	3.12e+00	0.24
0.00125	2.79e−04	0.96	9.76e−03	0.73	2.64e+00	0.24
0	2.26e−06		1.15e−05		2.88e−02	

Table 10 Test 7

ϵ	$\|u - u_h^\epsilon\|_0$	Rate	$\|u' - G_h u_h^\epsilon\|_0$	Rate	$\|u' - G_h u_h^\epsilon\|_{H^1}$	Rate
7.5e−1	1.56e−1		9.33e−1		1.31e+1	
5.0e−1	9.09e−2	1.33	6.99e−1	0.71	1.19e+1	0.24
1.0e−1	1.66e−2	1.06	2.33e−1	0.68	7.93e+0	0.25
7.5e−2	1.27e−2	0.93	1.90e−1	0.70	7.37e+0	0.26
5.0e−2	8.78e−3	0.92	1.43e−1	0.71	6.64e+0	0.26
2.5e−2	4.67e−3	0.91	8.71e−2	0.71	5.56e+0	0.26
1.25e−2	2.48e−3	0.91	5.29e−2	0.72	4.66e+0	0.25
5.0e−3	1.06e−3	0.93	2.71e−2	0.73	3.70e+0	0.25
2.5e−3	5.46e−4	0.95	1.63e−2	0.73	3.12e+0	0.25
1.25e−3	2.79e−4	0.97	9.77e−3	0.74	2.64e+0	0.25
0	2.29e−6		1.08e−5		2.76e−2	

where the exact solution only change the sign of the one in Test 6 so it becomes concave function. In this case the scheme should be changed to

$$\epsilon u''''(x) + (u''(x))^2 = f.$$

Now the initial guess should be changed to any concave function satisfying boundary condition, where the concave property is necessary. Moreover, the concave property can be maintained at each Newton iteration by our algorithm.

After having computed the errors, we get the same rate of convergence with respect to ϵ in various norms as Test 6 shown in Table 10. We also note that in this case, we can compute the problem for the case $\epsilon = 0$ since the exact solution is smooth.

Test 8 The purpose of this test is to calculate the rate of convergence of $\|u^\epsilon - u_h^\epsilon\|$ for fixed ϵ in various norms. We solve the problem with boundary condition $(u^\epsilon)'' = \epsilon^2$ being replaced by $(u^\epsilon)'' = \phi$ on $x = 0, 1$. We use the following test functions:

Table 11 Test 8: $\epsilon = 0.01$

Mesh size	$\|u - u_h^\epsilon\|_0$	Rate	$\|u - u_h^\epsilon\|_{H^1}$	Rate
1.5625e−02	2.5598e−04		4.5257e−02	
7.8125e−03	6.3079e−05	2.02	2.2608e−02	1.00
3.9063e−03	1.5918e−05	1.98	1.1294e−02	1.00
1.9531e−03	4.0272e−06	1.98	5.6410e−03	1.00
9.7656e−04	1.0123e−06	1.99	2.8195e−03	1.00

Table 12 Test 8: $\epsilon = 0.01$

Mesh size	$\|u' - G_h u_h^\epsilon\|_0$	Rate	$\|u' - G_h u_h^\epsilon\|_{H^1}$	Rate
1.5625e−02	1.3594e−03		2.2298e−01	
7.8125e−03	3.4871e−04	1.96	1.1225e−01	0.99
3.9063e−03	8.6976e−05	2.00	5.5326e−02	1.02
1.9531e−03	2.1030e−05	2.04	2.6396e−02	1.06
9.7656e−04	5.1765e−06	2.02	1.2918e−02	1.03

$$u^\epsilon = x^6, \quad f^\epsilon = 900x^8 - 360\epsilon x^2, \quad g^\epsilon = x^6, \quad \phi^\epsilon = 30x^4.$$

After recording the errors, we estimate the rate of convergence with respect to h, and get optimal convergence rate from Tables 11 and 12.

6 Conclusion and Final Remarks

In this paper, we have presented and analyzed a recovery based linear element method for the one dimensional bi-harmonic problem and corresponding eigenvalue problem. Moreover, we also apply the method to numerically solve 1-D fully nonlinear Monge Ampère equation. The method circumvents the use of C^1 conforming elements, simplifies numerical schemes, and reduced the computation cost. In addition, it is easy to see that the proposed methods can be generalized to time dependent problems such as $u_t = u_{xxxx}$, without any essential difficulty.

In the subsequence works, we will study higher dimensional bi-harmonic problem, the corresponding eigenvalue problem, as well as related nonlinear problems.

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