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SUPERCONVERGENCE POINTS OF FRACTIONAL SPECTRAL INTERPOLATION*

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Abstract. We investigate superconvergence properties of the spectral interpolation involving fractional derivatives. Our interest in this superconvergence problem is, in fact, twofold: when interpolating function values, we identify the points at which fractional derivatives of the interpolant superconverge; when interpolating fractional derivatives, we locate those points where function values of the interpolant superconverge. For the former case, we apply various Legendre polynomials as basis functions and obtain the superconvergence points, which naturally unify the superconvergence points for the first order derivative presented in [Z. Zhang, *SIAM J. Numer. Anal.*, 50 (2012), pp. 2966–2985], depending on orders of the fractional derivatives. While for the latter case, we utilize the Petrov–Galerkin method based on generalized Jacobi functions [S. Chen, J. Shen, and L.-L. Wang, *Math. Comp.*, to appear] and locate the superconvergence points both for function values and fractional derivatives. Numerical examples are provided to verify the analysis of superconvergence points for each case.

Key words. superconvergence, fractional derivative, spectral collocation, Petrov–Galerkin, generalized Jacobi functions

AMS subject classifications. 65N35, 65M15, 26A33, 41A05, 41A10

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1. Introduction.

Superconvergence of numerical methods usually happens when the convergence rate at some special points is higher than the theoretical global rate [8, 12, 27, 35]. The investigation of superconvergence provides a fundamental insight for postprocessing and adaptive algorithm design, which leads to higher accuracy and more efficient numerical methods [1, 6].

For classical problems, the superconvergence phenomenon of the h-version finite element method is well understood and addressed in the literature; see, e.g., [6, 16, 27], and references therein. On the other hand, there has been only a limited study of the superconvergence phenomenon for spectral methods [35, 36, 37]. Specifically, in [35, 36], Zhang identified the derivative superconvergence points of the Legendre and Chebyshev spectral collocation methods for the two-point boundary value problem. In addition, Zhang [37] investigated various polynomial interpolations and located superconvergence points for the function value and the first derivative. Wang, Zhao, and Zhang [29] extended the study of superconvergence properties of Chebyshev–Gauss-type spectral interpolation in [37] and addressed the approximations to functions with limited regularity.

In this paper, we conduct the first superconvergence study of fractional operators. Since fractional calculus generalizes the classical (integer-order) differentiation and

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integration to any order, the study of superconvergent fractional derivatives unifies the theoretical investigation in [37] for the first-order derivative.

Over the last two decades, fractional differential equations (FDEs) have been demonstrated to be more effective, when modeling some complex systems in physics, finance, etc., [17, 18, 19, 21, 22, 24], compared to classical models. These new models are usually derived by replacing integer-order derivatives with fractional derivatives in classical models. Meanwhile, there has been a growing need for the development of high-order numerical algorithms for solving FDEs. Due to the nonlocal definition of fractional derivative, existing methods including finite difference and finite element methods [20, 23, 25, 26, 28, 31, 38, 40] mostly lead to low-order schemes.

Spectral methods are promising candidates for solving FDEs since their global nature fits well with the nonlocal definition of fractional operators. Using integer-order orthogonal polynomials as basis functions, spectral methods [4, 14, 15, 30, 34, 39] really help with the alleviation of the memory cost for discretization of fractional derivatives. Furthermore, to deal with singularities, which usually appear in fractional problems, in the works [5, 10, 32, 33] the authors design suitable bases. Recently, Fatone and Funaro [9] provided the optimal distributions of collocation nodes of different bases for the approximations of fractional Riemann–Liouville derivatives.

A major difficulty in the investigation of superconvergence of spectral methods for fractional problems, compared with integer-order derivatives, is the nonlocality of the fractional operator and the complicated form of fractional derivatives. The second challenge is the construction of a good basis for a spectral scheme. Given a suitable basis, one can then begin the analysis of the approximation error in order to locate the superconvergence points.

In this paper, we interpolate function values using Legendre polynomials, and use the error equation to obtain superconvergent points for Riemann–Liouville fractional derivatives. It turns out that these superconvergence points are zeros of the fractional derivatives of the corresponding Legendre polynomials. We also consider the interpolation of fractional Riemann–Liouville derivatives, and apply a generalized Jacobi function (GJF) spectral Petrov–Galerkin scheme [5] to solve an equivalent fractional initial value problem. We have found that superconvergence points for numerical function values and fractional Riemann–Liouville derivatives are zeros of corresponding basis functions and Gauss points, respectively.

The organization of this paper is as follows. In section 2, we define notation and give properties of fractional derivatives and of GJFs. The interpolations of the function values using various Legendre polynomials are presented in section 3. Moreover, we analyze the error to identify superconvergence points for fractional derivatives of the interpolant. In section 4, we show the locations of the superconvergence points both for the numerical solution and the fractional derivatives using a GJF spectral Petrov–Galerkin scheme. The numerical tests for both cases are displayed at the end of sections 3 and 4, respectively. We conclude with a discussion of our results and their applications in section 5.

2. Preliminaries.

In this section, we present some notations and lemmas which will be used in the following sections.

DEFINITION 2.1. *The fractional integral of order $\mu \in (0,1)$ for function $f(x)$ is defined as*

$$({}_{x_L}\mathcal{I}_x^\mu f)(x) = \frac{1}{\Gamma(\mu)} \int_{x_L}^x \frac{f(s)}{(x-s)^{1-\mu}} ds, \quad x > x_L. \tag{2.1}$$

DEFINITION 2.2. *The Caputo fractional derivative of order $\mu \in (0,1)$ for function $f(x)$ is defined as*

$$({}_{x_L}^{C}\mathcal{D}_x^{\mu} f)(x) = {}_{x_L}\mathcal{I}_x^{1-\mu}\left[\frac{d}{dx}f(x)\right] = \frac{1}{\Gamma(1-\mu)}\int_{x_L}^{x}\frac{f'(s)}{(x-s)^{\mu}}ds, \quad x > x_L. \tag{2.2}$$

DEFINITION 2.3. *The left Riemann–Liouville fractional derivative of order $\mu \in (0,1)$ for function $f(x)$ is defined as*

$$({}_{x_L}^{RL}\mathcal{D}_x^{\mu} f)(x) = \frac{d}{dx}\left[({}_{x_L}\mathcal{I}_x^{1-\mu} f)(x)\right] = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{x_L}^{x}\frac{f(s)}{(x-s)^{\mu}}ds, \quad x > x_L. \tag{2.3}$$

DEFINITION 2.4. *The right Riemann–Liouville fractional derivative of order $\mu \in (0,1)$ for function $f(x)$ is defined as*

$$({}_{x}^{RL}\mathcal{D}_{x_R}^{\mu} f)(x) = \frac{1}{\Gamma(1-\mu)}\left(-\frac{d}{dx}\right)\int_{x}^{x_R}\frac{f(s)}{(s-x)^{\mu}}ds, \quad x < x_L. \tag{2.4}$$

For convenience, if $\mu < 0$, we denote $({}_{x_L}^{RL}\mathcal{D}_x^{\mu} f)(x) = ({}_{x_L}\mathcal{I}_x^{-\mu} f)(x)$ throughout the paper.

LEMMA 2.5 (see [11]). *Let $p \in \mathbb{R}$ and $n-1 < p < n \in \mathbb{R}$. If $f(t)$ and $\varphi(t)$ along with all its derivatives are continuous in $[a,t]$ then under this condition the Leibniz rule for Caputo fractional differentiation takes the form*

$${}_{a}^{C}\mathcal{D}_t^{p}\left[\varphi(t)f(t)\right] = \sum_{k=0}^{\infty}\binom{p}{k}\varphi^{(k)}(t)\,{}_{a}^{C}\mathcal{D}_t^{p-k}f(t) - \sum_{k=0}^{n-1}\frac{t^{k-p}}{\Gamma(k+1-p)}[\varphi(t)f(t)]^{(k)}(-1).$$

LEMMA 2.6 (see [21, Chap. 2]). *Let $p \in \mathbb{R}$ and $n-1 < p < n \in \mathbb{R}$. If $f(t)$ and $\varphi(t)$ along with all its derivatives are continuous in $[a,t]$ then under this condition the Leibniz rule for Riemann–Liouville fractional differentiation takes the form*

$${}_{a}^{RL}\mathcal{D}_t^{p}\left[\varphi(t)f(t)\right] = \sum_{k=0}^{\infty}\binom{p}{k}\varphi^{(k)}(t)\,{}_{a}^{RL}\mathcal{D}_t^{p-k}f(t).$$

We recall the definition of GJFs, which extend the parameters of classical Jacobi polynomials to a wider range.

DEFINITION 2.7 (see [5, generalized Jacobi functions]). *Define*

$${}^{+}J_n^{(-\alpha,\beta)}(x) := (1-x)^{\alpha}P_n^{(\alpha,\beta)}(x) \quad \textit{for } \alpha > -1,\ \beta \in \mathbb{R}, \tag{2.5}$$

and

$${}^{-}J_n^{(\alpha,-\beta)}(x) := (1+x)^{\beta}P_n^{(\alpha,\beta)}(x) \quad \textit{for } \alpha \in \mathbb{R},\ \beta > -1, \tag{2.6}$$

for $x \in \Lambda = (-1,1)$ and $n \geqslant 0$.

In what follows, we present the selected two special cases for the fractional derivatives of GJFs:

- Let $\alpha > 0,\ \beta \in \mathbb{R}$, and $n \in \mathbb{N}_0$,

$${}_{x}^{RL}\mathcal{D}_1^{\alpha}\left\{{}^{+}J_n^{(-\alpha,\beta)}(x)\right\} = \frac{\Gamma(n+\alpha+1)}{n!}P_n^{(0,\alpha+\beta)}(x). \tag{2.7}$$

- Let $\beta > 0$, $\alpha \in \mathbb{R}$, and $n \in \mathbb{N}_0$,

$$ {}_{-1}^{RL}\mathcal{D}_x^{\alpha}\left\{ {}^{-}J_n^{(\alpha,-\beta)}(x)\right\} = \frac{\Gamma(n+\beta+1)}{n!} P_n^{(\alpha+\beta,0)}(x). \tag{2.8}$$

For convenience, we introduce two sets of parameters for GJFs:

$$ {}^{+}\Upsilon_1^{\alpha,\beta} := \{(\alpha,\beta) : \alpha > 0,\ \beta > -1\}, \tag{2.9}$$

$$ {}^{+}\Upsilon_2^{\alpha,\beta} := \{(\alpha,\beta) : \alpha > 0,\ -\alpha-1 < \beta = -k \le -1,\ k \in \mathbb{N}\}. \tag{2.10}$$

For $(\alpha,\beta) \in {}^{+}\Upsilon_1^{\alpha,\beta}$, we define the finite-dimensional fractional-polynomial space

$$ {}^{+}\mathcal{F}_N^{(-\alpha,\beta)}(\Lambda) := \operatorname{span}\{{}^{+}J_n^{(-\alpha,\beta)}, 0 \le n \le N\}. \tag{2.11}$$

3. Legendre interpolant. In this section, we interpolate a smooth function u at a set of $N+1$ special points, which are zeros of some orthogonal polynomials on $[-1,1]$. In particular, we want to find $u_N \in P_N$ such that

$$ u_N(x_k) = u(x_k), \quad -1 \le x_0 < x_1 < \cdots < x_N \le 1, \tag{3.1}$$

where the interpolation points $\{x_k,\ k = 0,1,\ldots,N\}$ are zeros of, say, the Legendre polynomials. Our goal is to identify those points $y_j^{(\beta)}$, where ${}_{-1}^{RL}\mathcal{D}_x^{\beta} u_N$, $\beta \in (0,1)$, the fractional Riemann–Liouville derivative of the interpolant, superconverges to ${}_{-1}^{RL}\mathcal{D}_x^{\beta} u$ in the sense that

$$ N^{\alpha} |\, {}_{-1}^{RL}\mathcal{D}_x^{\beta}(u-u_n)(y_j^{(\beta)})| \le C \max_{x\in[-1,1]} |\, {}_{-1}^{RL}\mathcal{D}_x^{\beta}(u-u_n)(x)|, \quad \alpha > 0. \tag{3.2}$$

According to the description (3.2) of superconvergence points originated from [37], if $\beta = 1$ and $\{y_j'\}$ are independent of the particular choice of u, we say that the $\{y_j'\}$ are the superconvergence points for the first derivative of the interpolant. To step further, here we want to seek out the points $y_j^{(\beta)}$ independent of the function u, at which the corresponding βth Riemann–Liouville derivative ${}_{-1}^{RL}\mathcal{D}_x^{\beta} u_N$ superconverges to ${}_{-1}^{RL}\mathcal{D}_x^{\beta} u$. In other words, for different order of fractional derivative, different superconvergence points are needed.

3.1. Analysis. Considering polynomial interpolation for (3.1), the key rule for locating the superconvergece points is to analyze the interpolation error. We are now in the position to show the theoretical analysis.

PROPOSITION 3.1. *For the interpolation of* (3.1) *using collocation points as the zeros of Legendre polynomials* $L_{N+1}(x)$*, Legendre–Lobatto polynomials* $(L_{N+1}-L_{N-1})(x)$*, and Legendre–Radau (right and left) polynomials* $(L_{N+1} \pm L_N)(x)$*, the* β*th Riemann–Liouville fractional derivative superconverges at* $\{\xi_k^{(\beta)}\}$*, which satisfy*

$$ {}_{-1}^{RL}\mathcal{D}_x^{\beta} w_{N+1}(\xi_k^{(\beta)}) = 0, k = 0,1,2,\ldots,N, \tag{3.3}$$

where $w_{N+1}(x)$ *denotes the aforementioned four sets of polynomials, respectively.*

Proof. Let u be analytic on $I = [-1,1]$. According to [3, 37], u can be analytically extended to B_ρ, which is enclosed by an ellipse E_ρ with ± 1 as foci and $\rho > 1$ as the sum of its semimajor and semiminor axes:

$$ E_\rho : \quad z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1} e^{-i\theta}), \quad 0 \le \theta \le 2\pi. \tag{3.4}$$

We consider polynomial $u_N \in P_N$ which interpolates u at $N+1$ points $-1 \le x_0 < x_1 < \cdots < x_N \le 1$. The error equation is, according to [7], expressed as

$$u(x) - u_N(x) = \frac{1}{2\pi i}\int_{E_\rho} \frac{w_{N+1}(x)}{w_{N+1}(z)}\frac{u(z)}{z-x}dz, \tag{3.5}$$

where $w_{N+1}(x) = c\prod_{j=0}^{N}(x - x_j)$.

Noting Lemma 2.6, taking the fractional derivative of the above equation, we have

$$ {}_{-1}^{RL}\mathcal{D}_x^{\beta}[u(x) - u_N(x)] = \frac{1}{2\pi i}\int_{E_\rho}\left[\sum_{k=0}^{\infty}\binom{\beta}{k}\frac{{}_{-1}^{RL}\mathcal{D}_x^{\beta-k}w_{N+1}(x)}{(z-x)^{k+1}}\right]\frac{u(z)}{w_{N+1}(z)}dz. \tag{3.6}$$

Let us examine the error equation (3.6). According to the analysis in [37], the exponential decay of the error is dominated by the integration of $\frac{{}_{-1}^{RL}\mathcal{D}_x^{\beta}w_{N+1}(x)}{z-x}\frac{u(z)}{w_{N+1}(z)}$ on E_ρ. At the N special points $\{\xi_k^{(\beta)}, k = 0,\dots,N\}$, which satisfy ${}_{-1}^{RL}\mathcal{D}_x^{\beta}w_{N+1}(\xi_k^{(\beta)}) = 0$, we have the remaining terms, which are usually smaller than the first term in magnitude at least by a factor N^α. Here we interpolate the function using Legendre polynomials, Legendre–Lobatto polynomials, and Legendre–Radau (right and left) polynomials, thus the corresponding superconvergence points are zeros of ${}_{-1}^{RL}\mathcal{D}_x^{\beta}w_{N+1}(x)$, where $w_{N+1}(x)$ denotes the aforementioned four sets of polynomials, respectively. ∎

From Lemma 2.5, the superconvergence points for the Caputo fractional derivative are easily drawn.

Remark 3.2. Similarly to the proof in Propsition 3.1, for the interpolation of (3.1) using collocation points as the zeros of the aforementioned four kinds of Legendre polynomials, the βth Caputo fractional derivative superconverges at $\{z_k^{(\beta)}\}$, which satisfy

$$ {}_{-1}^{C}\mathcal{D}_x^{\beta}w_{N+1}(z_k^{(\beta)}) = 0, k = 0,1,2,\dots,N, \tag{3.7}$$

where $w_{N+1}(x)$ denotes the aforementioned four sets of polynomials, respectively.

As an extension, superconvergence results are derived similarly for Chebyshev polynomials in the following, since (3.5) is valid for general Jacobi–Gauss-type points.

PROPOSITION 3.3. *For the interpolation of* (3.1) *using collocation points as the zeros of Chebyshev polynomials of the first kind* $T_{N+1}(x)$*, Chebyshev–Lobatto polynomials* $(T_{N+1} - T_{N-1})(x)$*, and Chebyshev–Radau (right and left) polynomials* $(T_{N+1} \pm T_N)(x)$*, the* β*th Riemann–Liouville fractional derivative superconverges at* $\{y_k^{(\beta)}\}$*, which satisfy*

$$ {}_{-1}^{RL}\mathcal{D}_x^{\beta}w_{N+1}(y_k^{(\beta)}) = 0, k = 0,1,2,\dots,N, \tag{3.8}$$

where $w_{N+1}(x)$ *denotes the aforementioned four sets of Chebyshev polynomials, respectively.*

In what follows, we explain how to compute $\{\xi_k^{(\beta)}\}$ in (3.3).

Denote $L_n(x)$, the Legendre polynomial of degree n. In spectral methods, we often use the following four combinations. For convenience, express them as GJFs:

$$\begin{aligned}
L_n(x) &= P_n^{(0,0)}(x) \qquad \text{(Legendre polynomial)},\\
(L_n - L_{n-2})(x) &= P_n^{(-1,-1)}(x) \qquad \text{(Lobatto polynomial)},\\
(L_n + L_{n-1})(x) &= P_n^{(0,-1)}(x) \qquad \text{(left Radau polynomial)},\\
(L_n - L_{n-1})(x) &= P_n^{(-1,0)}(x) \qquad \text{(right Radau polynomial)}.
\end{aligned}$$

From the spectral relationships for the left/right Riemann–Liouville fractional integrals [2, 21], we use the following formulas [13]:

$$P_{n-\beta}^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha-\beta+1)\Gamma(n+1)}{\Gamma(n-\beta+1)\Gamma(n+\alpha+1)} \left(\frac{1+x}{2}\right)^{-\beta} P_n^{(\alpha,-\beta)}(x) \tag{3.9}$$

and

$$ {}_{-1}^{RL}D_x^{\mu} P_{n-\beta}^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+\mu+1)}{2^{\mu}\Gamma(n+\alpha+1)} P_{n-\beta-\mu}^{(\alpha+\mu,\beta+\mu)}(x); \tag{3.10}$$

we can derive

$$ {}_{-1}^{RL}D_x^{\mu} P_n^{(0,0)}(x) = \frac{2^{-\mu}\Gamma(n+1)}{\Gamma(n-\mu+1)} \left(\frac{1+x}{2}\right)^{-\mu} P_n^{(\mu,-\mu)}(x), \tag{3.11}$$

$$ {}_{-1}^{RL}D_x^{\mu} P_n^{(\alpha,-1)}(x) = \frac{(n+\alpha)\Gamma(n)}{2^{\mu}\Gamma(n-\mu+1)} \left(\frac{1+x}{2}\right)^{1-\mu} P_{n-1}^{(\alpha+\mu,1-\mu)}(x), \quad \alpha=-1,0. \tag{3.12}$$

Similarly, we have

$$ {}_{x}^{RL}D_1^{\mu} P_n^{(0,0)}(x) = \frac{2^{-\mu}\Gamma(n+1)}{\Gamma(n-\mu+1)} \left(\frac{1-x}{2}\right)^{-\mu} P_n^{(-\mu,\mu)}(x), \tag{3.13}$$

$$ {}_{x}^{RL}D_1^{\mu} P_n^{(-1,\alpha)}(x) = \frac{(n+\alpha)\Gamma(n)}{2^{\mu}\Gamma(n-\mu+1)} \left(\frac{1-x}{2}\right)^{1-\mu} P_{n-1}^{(1-\mu,\alpha+\mu)}(x), \quad \alpha=-1,0. \tag{3.14}$$

Therefore, we conclude that (3.3) is equivalent to finding zeros of

(1) $P_{N+1}^{(\mu,-\mu)}(x)$ for the Legenrde interpolation, i.e., interpolation at roots of $P_{N+1}^{(0,0)}(x)$;

(2) $(1+x)^{1-\mu}P_N^{(\mu-1,1-\mu)}(x)$ for the Lobatto interpolation, i.e., interpolation at roots of $P_{N+1}^{(-1,-1)}(x)$;

(3) $(1+x)^{1-\mu}P_N^{(\mu,1-\mu)}(x)$ for the left-Radau interpolation, i.e., interpolation at roots of $P_{N+1}^{(0,-1)}(x)$.

Similar results can be obtained for the right fractional derivative ${}_{x}^{RL}D_1^{\mu}$.

We see that a fractional derivative of a polynomial of degree $N+1$ may have $N+1$ roots (compared with N roots for an integer derivative).

3.2. Numerical validation. We report on the numerical results using the aforementioned four sets of Gauss interpolation points in (3.1).

Example 3.1. We choose $u(x) = \frac{1}{100}(x+1)^{10.15}$ in (3.1).

Fractional derivative errors of its interpolant, using Gauss points, Gauss–Lobatto points, and the left Gauss–Radau points with number of points of $N = 12$ are depicted in Figures 1–3, respectively. From Proposition 3.1, we know that for a polynomial interpolant of degree N, the leading error term is a polynomial of degree $N+1$, and there are $N+1$ roots for its fractional derivative. From Figures 1–3, it is clearly seen that there are 13 superconvergence points. Particularly, -1 is a superconvergence point for the Gauss–Lobatto and left Gauss–Radau interpolants. To view those roots near -1, we add zoom-in windows into these figures.

Five different colors of asterisks denoting the superconvergence points, with orders $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$ for Legendre polynomial interpolation, are shown in

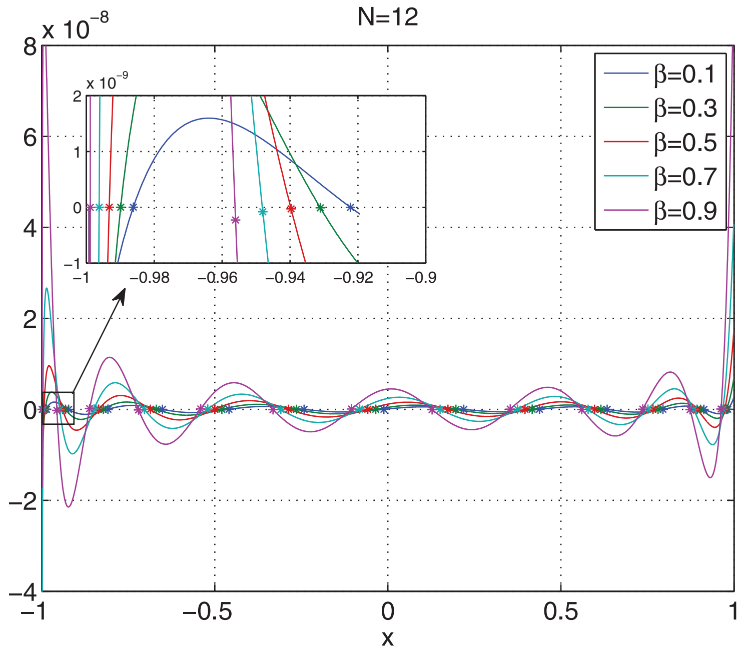

FIG. 1. *Curves of* ${}_{-1}^{RL}\mathcal{D}_x^{\beta}(u-u_N)$ *for Example* 3.1 *using Gauss points, where* $*$ *denotes the corresponding superconvergence points.*

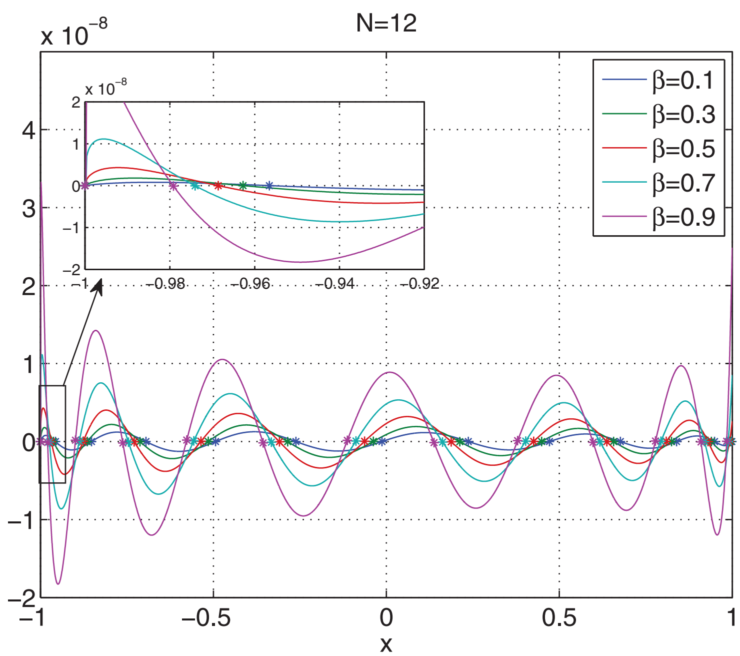

FIG. 2. *Curves of* ${}_{-1}^{RL}\mathcal{D}_x^{\beta}(u-u_N)$ *for Example* 3.1 *using Gauss–Lobatto points, where* $*$ *denotes the corresponding superconvergence points.*

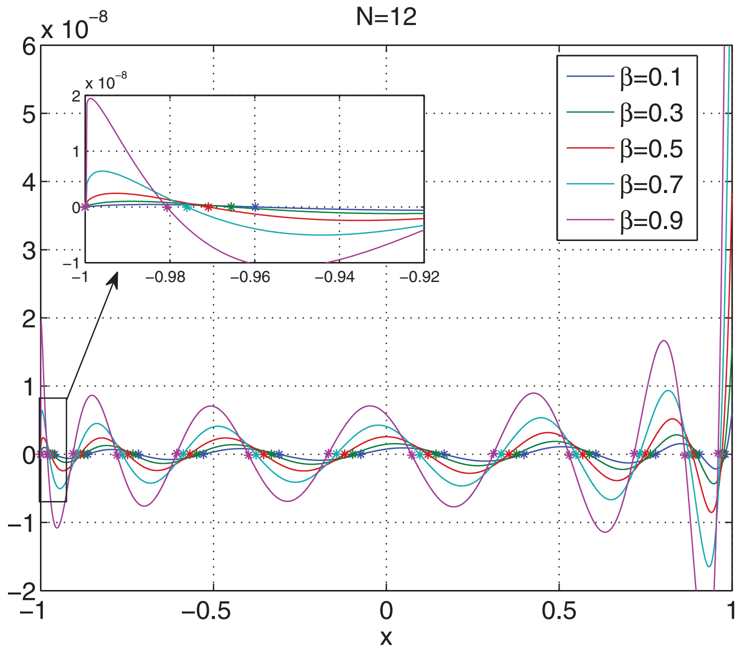

FIG. 3. *Curves of* ${}_{-1}^{RL}\mathcal{D}_x^{\beta}(u-u_N)$ *for Example* 3.1 *using left Gauss–Radau points, where* $*$ *denotes the corresponding superconvergence points.*

Figure 1. It is easily shown that for different orders of fractional derivatives, the errors at the superconvergence points are significantly smaller than the maximum error. In addition, it is also indicated that when the fractional order becomes smaller, the error curve goes lower.

Similar situations are easily drawn from Figures 2–3 for the Legendre–Lobatto polynomial and the left Legendre–Radau polynomial, respectively. For approximations using Gauss points, the errors on the boundaries are larger than that using Gauss–Lobatto points and left Gauss–Radau points.

4. Fractional derivative interpolation. In this section, we present the interpolation for the fractional Riemann–Liouville derivatives of a smooth function using GJFs as basis functions and investigate superconvergence points for the function value approximation and fractional derivatives. In particular, we construct u_N such that

$$\begin{aligned} &{}_{x}^{RL}\mathcal{D}_1^s u_N(x_k) = {}_{x}^{RL}\mathcal{D}_1^s u(x_k) = f(x_k), \quad u_N(1) = u(1) = 0, \\ &\qquad k = 1, 2, \ldots, N, \quad -1 \le x_1 < \cdots < x_N \le 1, \end{aligned} \tag{4.1}$$

which is related to finding the solution for the following factional initial value problem of order $s \in (0,1)$:

$$ {}_{x}^{RL}\mathcal{D}_1^s u(x) = f(x), \quad u(1) = 0, \quad x \in [-1,1]. \tag{4.2}$$

4.1. Petrov–Galerkin spectral method for (4.2). We follow the same methodology used in [5] for solving (4.2), to find $u_N \in {}^{+}\mathcal{F}_N^{(-s,-s)}$ such that $u_N(1) = 0$ and

$$ ({}_{x}^{RL}\mathcal{D}_1^s u_N, v) = (f, v) \ \forall v \in \mathcal{P}_N. \tag{4.3}$$

We write the numerical solution as an expansion of the GJF basis

$$u_N(x) = \sum_{n=0}^{N} \tilde{u}_n^{(s)}\, {}^{+}J_n^{(-s,-s)}(x) \in {}^{+}\mathcal{F}_N^{(-s,-s)}(\Lambda). \tag{4.4}$$

Taking $v_N = P_k^{(0,0)}$ in (4.3) and using the orthogonality of Legendre polynomials, we derive from (2.7) that

$$\tilde{u}_n^{(s)} = \frac{n!}{\Gamma(n+s+1)} \tilde{f}_n, \quad 0 \le n \le N, \tag{4.5}$$

where $\{\tilde{f}_n\}$ is the sequence of coefficients for the Legendre expansion of function $f(x)$. Substituting (4.5) into (4.4), we obtain the numerical solution

$$u_N(x) = \sum_{n=0}^{N} \frac{n!}{\Gamma(n+s+1)} \tilde{f}_n\, {}^{+}J_n^{(-s,-s)}(x). \tag{4.6}$$

We would like to indicate the relation of the spectral collocation method (4.1) and the Petrov–Galerkin method (4.3). We select collocation points x_k in (4.1) as the Gauss points, i.e., the roots of the Legendre polynomial of degree N, and construct an interpolant $I_N f \in \mathcal{P}_N$ such that $(I_N u)(1) = 0$ and $(I_N f)(x_k) = f(x_k)$, for $k = 1, 2, \ldots, N$. Denote w_k as weights of the N-point Gauss quadrature. Next, we multiply both sides of (4.1) by $L_j(x_k) w_k$, sum over k, and obtain

$$({}^{RL}_{\ x}\mathcal{D}_1^s u_N, L_j) = (I_N f, L_j) \ \forall j = 1, 2, \ldots, N-1, \tag{4.7}$$

$$({}^{RL}_{\ x}\mathcal{D}_1^s u_N, L_N)^* = (I_N f, L_N)^*. \tag{4.8}$$

Here $*$ indicates that the integration is carried out by the numerical quadrature. Note that the N-point Gauss quadrature is exact for polynomials of degree $2N-1$. We see that (4.1) is "almost" equivalent to (4.3) in that only one term (4.8) is done by the N-point Gauss quadrature.

Now, we recall the error estimate of fractional derivative in the L^2-norm as follows.

THEOREM 4.1 (see [5]). *Let u and u_N be the solutions of* (4.2) *and* (4.3), *respectively. If $f \in C(\bar{\Lambda})$ and $f^{(l)} \in L^2_{w^{(l-1,l-1)}}(\Lambda)$ for all $1 \le l \le m$, then we have that for $1 \le m \le N+1$,*

$$\| {}^{RL}_{\ x}\mathcal{D}_1^s (u - u_N)\| + \|u - u_N\| \le c N^{-m} \|f^{(m)}\|_{\omega^{(m-1,m-1)}},$$

where c is a positive constant independent of u, N, and m.

The above error estimate leads to both the approximation errors of u using the GJF basis and ${}^{RL}_{\ x}\mathcal{D}_1^s u$ using Legendre polynomials. In order to investigate the superconvergence property for the fractional derivative of u, one observes from the right-hand side of (4.3) that the approximation of ${}^{RL}_{\ x}\mathcal{D}_1^s u$ is actually the same as the approximation error of function f using Legendre polynomials. Thus, it naturally yields to the following superconvergence points for fractional derivatives.

PROPOSITION 4.2. *For the Petrov–Galerkin spectral method for* (4.2), *the left fractional Riemann–Liouville derivative of numerical solution ${}^{RL}_{\ x}\mathcal{D}_1^s u_N$ superconverges* *to* ${}^{RL}_{\ x}\mathcal{D}^s_1 u$ *at the zeros of Legendre polynomial* $P^{(0,0)}_{N+1}(x)$*, namely, the points* $\{\zeta^{(0)}_k\}$ *satisfying*

$$P^{(0,0)}_{N+1}(\zeta^{(0)}_k) = 0, \quad k = 1, 2, \ldots, N+1,$$

which are usually called Gauss points.

Due to the special design of basis functions based on different fractional derivatives, the resulting superconvergence points are the same Gauss points for different orders of fractional derivatives.

From (4.5) we see that the resulting system is diagonal, which indicates the direct expansion of the numerical solution using the GJF basis $\{{}^+J^{(-s,-s)}_n(x)\}$. Therefore, the expansion of numerical solution u_N in (4.4) shows that the leading term of the approximation error is ${}^+J^{(-s,-s)}_{N+1}(x)$, if one finds the solution in ${}^+\mathcal{F}^{(-s,-s)}_N(\Lambda)$. Therefore, the superconvergence phenomenon for the numerical solution is observed as follows.

PROPOSITION 4.3. *For the Petrov–Galerkin spectral method for* (4.2)*, the numerical solution* u_N *superconverges to the exact solution at the zeros of* ${}^+J^{(-s,-s)}_{N+1}(x)$*, namely, the points* $\{\eta^{(s)}_k\}$ *satisfying*

$${}^+J^{(-s,-s)}_{N+1}(\eta^{(s)}_k) = 0, \quad k = 1, 2, \ldots, N+1.$$

In the above proposition, superconvergence points $\{\eta^{(s)}_k\}$, for the function value, depend on the order of fractional derivative. On the contrary, Proposition 4.2 tells us that the fractional derivative of the approximation superconverges at Gauss points, which are s-independent, for any $0 < s < 1$. The properties of the two kinds of superconvergence points manifest the essential differences of the estimates in Theorem 4.1.

The following remark gives us the opportunity to see further how the GJFs of (2.5) and (2.6) becoming excellent candidates for solving the fractional equation with underlying singularities at the boundaries.

Remark 4.4. For solving problem (4.2), the GJF Petrov–Galerkin spectral method with (2.5) as basis functions are applied. While, for the following initial value problem involving the left Riemann–Liouville fractional derivative,

$$ {}^{RL}_{-1}\mathcal{D}^s_x u(x) = f(x), \quad u(-1) = 0, \quad x \in [-1, 1], \tag{4.9}$$

one use (2.6) as a basis to construct the Petrov–Galerkin spectral method, and the main results of error estimates and superconvergence points could also be proved.

In the last part of this section, we present numerical examples to confirm the theoretical results and show the superconvergence points for the function value and fractional derivatives.

4.2. Numerical examples. In this subsection, we report on numerical results for problem (4.2) with three different cases of f.

Example 4.1. Take $f(x) = 1 + x + \cos(x) + \sin(x)$ in (4.2).

Example 4.2. Take $f(x) = e^{\sin(x)+2}$ in (4.2).

Example 4.3. Take $f(x) = (1 + x)^{7.89}$ in (4.2).

From Theorem 4.1, we know that the errors of the function value and its fractional derivatives decay exponentially regardless of whether the unknown solution is singular at the boundaries or not. Here, we focus on displaying the errors and the superconvergence points for these two cases.

We compute a reference exact solution with $N = 41$ in (4.6). Maximum errors between the exact solution and the numerical solution with $N = 9$ for the above

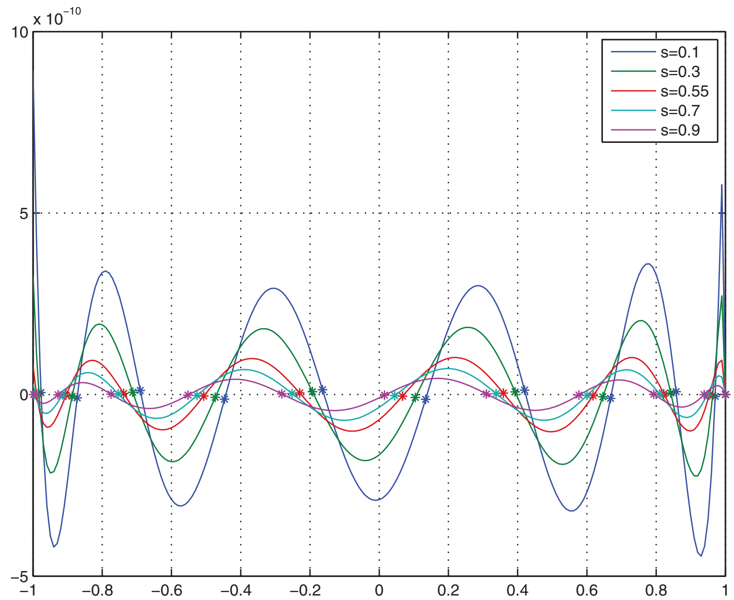

FIG. 4. *Curves of* $u - u_N$ *with* $N = 9$ *for Example* 4.1.

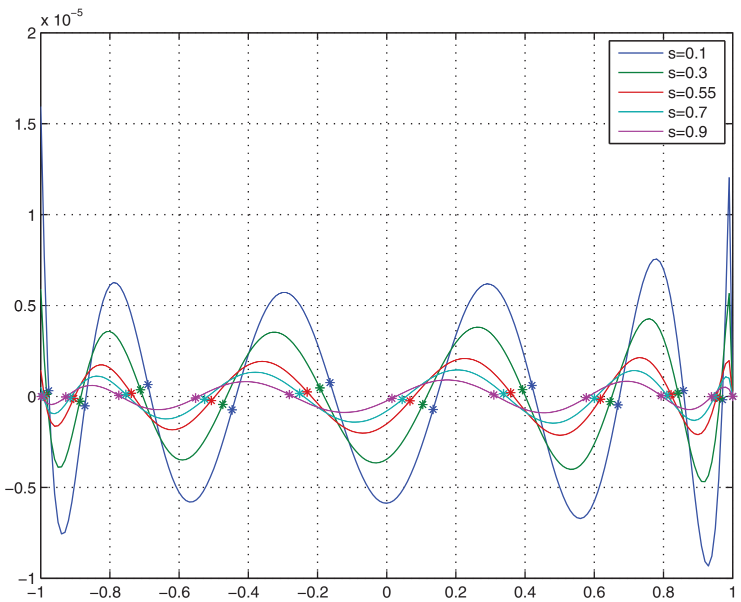

FIG. 5. *Curves of* $u - u_N$ *with* $N = 9$ *for Example* 4.2.

examples are demonstrated in Figures 4–6. For different orders of fractional derivatives as $s = 0.1, 0.3, 0.55, 0.7, 0.9$, the superconvergence points are marked in different colors of asterisks, respectively. It is easily seen that the errors at these points are much smaller than the global maximum errors. An observation worth noting is that when the order $s \to 1$, the amplitudes of the error curves get smaller, each series of the superconvergence points seems to converge to the standard case of $s = 1$.

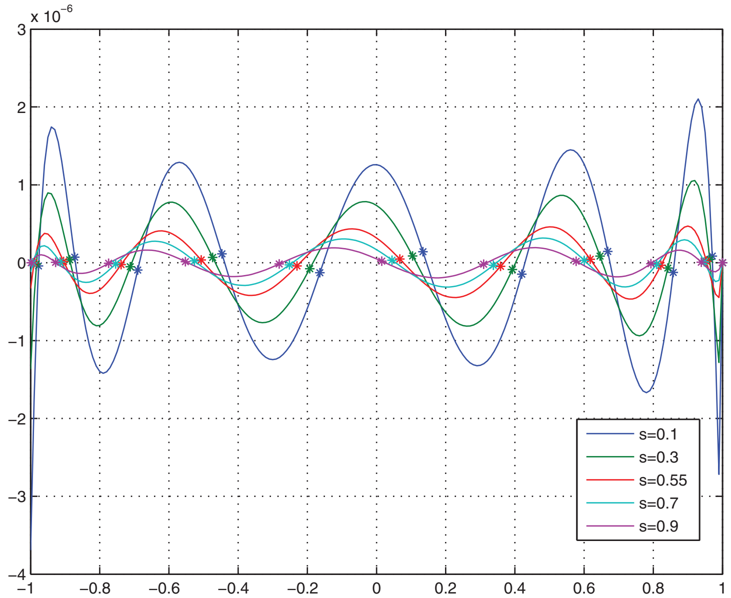

FIG. 6. *Curves of* $u - u_N$ *with* $N = 9$ *for Example* 4.3.

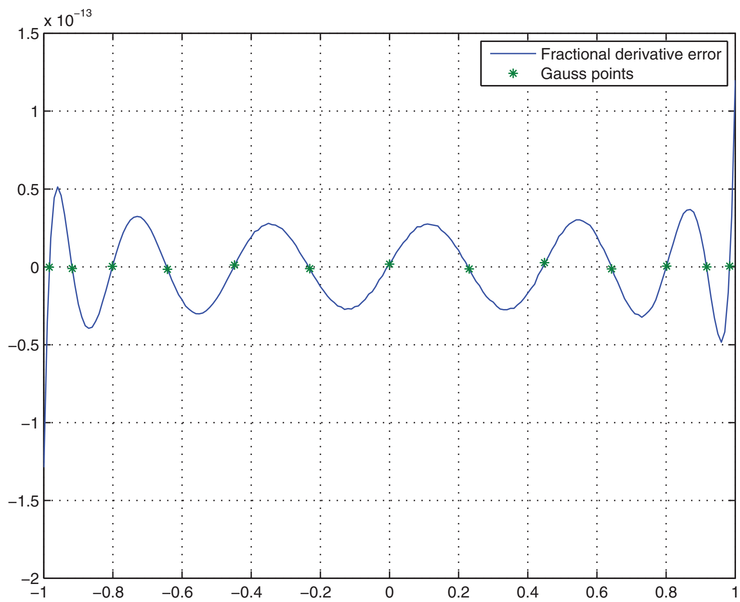

FIG. 7. *Curves of* ${}^{RL}_{x}\mathcal{D}^{s}_{1}(u - u_N)$ *with* $N = 12$ *for Example* 4.1.

For the purpose of comparison, we present fractional derivative errors in Figures 7–9 for three examples. In contrast to the function value approximation, superconvergence points are the same for different orders of fractional derivatives, which confirms the conclusion drawn in Proposition 4.2.

Remark 4.5. We would like to indicate that although our analysis is for a very special case (4.2), our findings of superconvergence points are actually valid for more

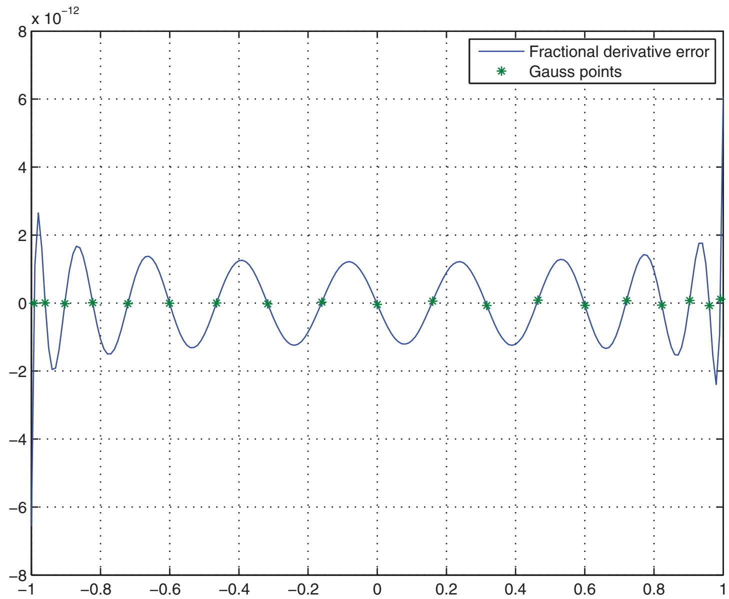

FIG. 8. *Curves of* ${}^{RL}_{\ x}\mathcal{D}^s_1(u-u_N)$ *with* $N=18$ *for Example* 4.2.

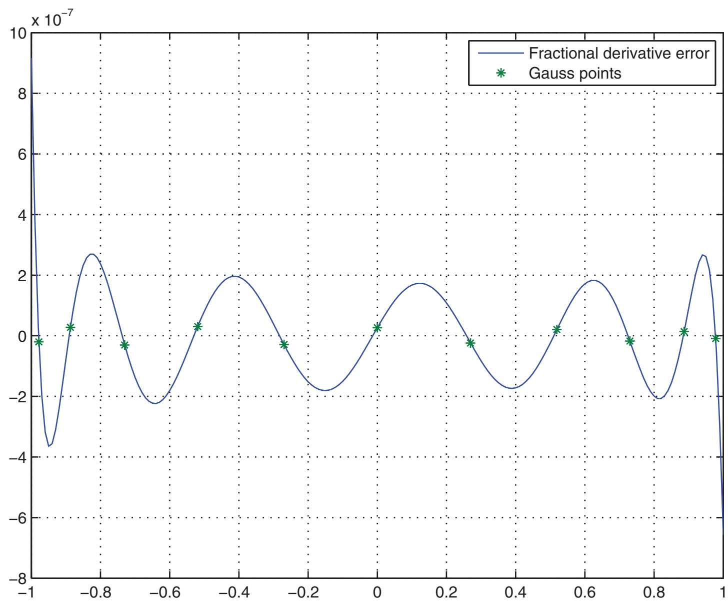

FIG. 9. *Curves of* ${}^{RL}_{\ x}\mathcal{D}^s_1(u-u_N)$ *with* $N=10$ *for Example* 4.3.

general situations as evidence by the following example. Consider

$$ {}^{RL}_{\ x}\mathcal{D}^s_1 u(x) + u(x) = f(x), \quad u(1)=0, \quad x\in[-1,1]. \tag{4.10}$$

We apply the Petrov–Galerkin spectral method (as described above) with $N=9$ for the exact solution $u(x)=(1-x)^{12+\alpha}$. The error curves of function values for different orders of fractional derivative are displayed in Figure 10, where stars mark the same

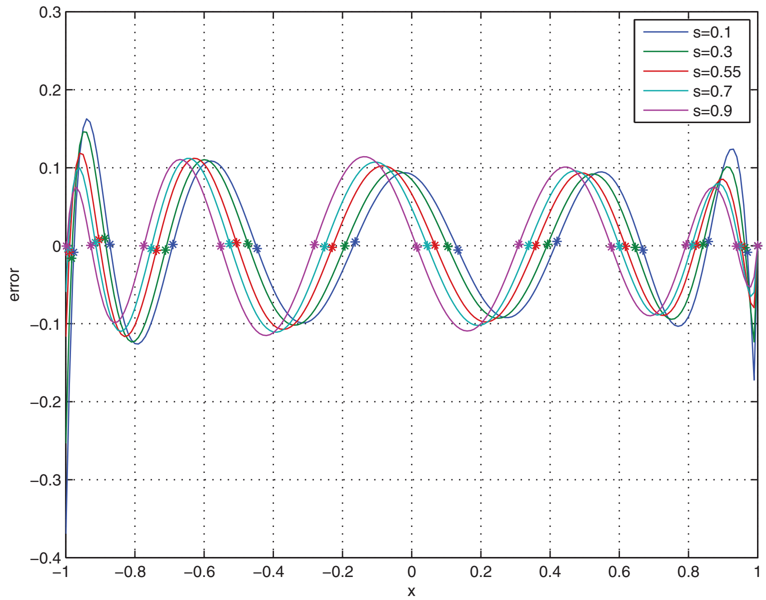

Fig. 10. *Curves of* $u - u_N$ *with* $N = 9$ *for* (4.10).

points as in Proposition 4.3. We notice that superconvergence points are the same as for problem (4.10).

5. Concluding remarks. In this paper, we discuss superconvergence phenomena for two kinds of spectral interpolations involving fractional derivatives. The intended application is the development of high-order methods for fractional problems using spectral methods.

When interpolating function values, superconvergence points for the Legendre basis are located by error analysis. This analysis unifies the identification of superconvergence points for fractional and first-order derivatives. When interpolating the fractional derivative values, we found significant differences between superconvergence points for function values and fractional derivatives. Numerical comparisons between the two cases are provided.

In the future, we plan to study more general fractional differential equations such as (4.10).

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