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MATHEMATICAL AND NUMERICAL ANALYSIS OF THE TIME-DEPENDENT GINZBURG–LANDAU EQUATIONS IN NONCONVEX POLYGONS BASED ON HODGE DECOMPOSITION

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ABSTRACT. We prove well-posedness of the time-dependent Ginzburg–Landau system in a nonconvex polygonal domain, and decompose the solution as a regular part plus a singular part. We see that the magnetic potential is not in $H^1(\Omega)$ in general, and so the finite element method (FEM) may give incorrect solutions. To overcome this difficulty, we reformulate the equations into an equivalent system of elliptic and parabolic equations based on the Hodge decomposition, which avoids direct calculation of the magnetic potential. The essential unknowns of the reformulated system admit H^1 solutions and can be solved correctly by the FEMs. We then propose a decoupled and linearized FEM to solve the reformulated equations and present error estimates based on the proved regularity of the solution. Numerical examples are provided to support our theoretical analysis and show the efficiency of the method.

1. INTRODUCTION

The Ginzburg–Landau theory, initially introduced by Ginzburg and Landau [25] and subsequently extended to the time-dependent case by Gor'kov and Eliashberg [28], are widely used to describe the phenomena of superconductivity in both low and high temperatures [16,33]. In particular, a superconductor occupying a domain Ω can be described by the time-dependent Ginzburg–Landau equations (TDGL):

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi+i\eta\kappa\psi\phi=0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial\mathbf{A}}{\partial t}+\nabla\times(\nabla\times\mathbf{A})+\nabla\phi+\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=\nabla\times\mathbf{f} \qquad \text{in } \Omega, \tag{1.2}$$

where η and k are given positive physical constants, the order parameter ψ is an unknown complex scalar function and ψ^* denotes the complex conjugate of ψ, the real vector-valued function $\mathbf{A}$ denotes the unknown magnetic potential, the electric potential ϕ is a real scalar unknown, and $\mathbf{f}$ is a given external magnetic field. The

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physically interesting quantities of this model are the superconducting density $|\psi|^2$, the magnetic induction field $\mathbf{B} = \nabla \times \mathbf{A}$, and the electric field $\mathbf{E} = \partial_t \mathbf{A} + \nabla\phi$. The physical boundary conditions are given by

$$\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

$$\mathbf{B}\times\mathbf{n} = \mathbf{f}\times\mathbf{n} \quad \text{on } \partial\Omega, \tag{1.4}$$

$$\mathbf{E}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where $\mathbf{n}$ denotes the unit outward normal vector on the boundary $\partial\Omega$.

It is well known that the TDGL requires an additional condition, called gauge, to determine the solution uniquely. For example, the zero electric potential gauge $\phi = 0$ [22, 27, 32, 35, 37] and the gauge $\phi = -\nabla\cdot\mathbf{A}$ [10, 12, 24] are often used in numerical simulations, and it was proved in [13] that the solutions under the different gauges are theoretically equivalent in producing the physical quantities (such as $|\psi|^2$ and $\mathbf{B}$).

Under the gauge $\phi = 0$, the TDGL can be written as

$$\eta\frac{\partial\psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + (|\psi|^2 - 1)\psi = 0 \quad \text{in } \Omega, \tag{1.6}$$

$$\frac{\partial\mathbf{A}}{\partial t} + \nabla\times(\nabla\times\mathbf{A}) + \mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right] = \nabla\times\mathbf{f} \quad \text{in } \Omega, \tag{1.7}$$

with the boundary conditions

$$\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

$$(\nabla\times\mathbf{A})\times\mathbf{n} = \mathbf{f}\times\mathbf{n} \quad \text{on } \partial\Omega, \tag{1.9}$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.10}$$

provided the initial data of $\mathbf{A}$ satisfies $\mathbf{A}\cdot\mathbf{n} = 0$. Clearly, the equation (1.7) is degenerate parabolic, as $\int_\Omega |\nabla\times\mathbf{A}|^2\,\mathrm{d}x$ is not equivalent to $\int_\Omega |\nabla\mathbf{A}|^2\,\mathrm{d}x$. Due to this degeneracy, both theoretical and numerical analysis of this model are difficult. In smooth domains, existence and uniqueness of solutions of (1.6)-(1.7) have been proved in [17], and finite element approximations have been studied in [18, 20, 36]. In all of these works, convergence of the numerical solution was proved for the sufficiently smooth PDE solution. In a domain with reentrant corners, however, well-posedness of the TDGL and regularity of the solutions remain open. Hence, convergence of the numerical solutions also remains open in this case.

Under the gauge $\phi = -\nabla\cdot\mathbf{A}$, the TDGL reduces to

$$\eta\frac{\partial\psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + (|\psi|^2 - 1)\psi - i\eta\kappa\psi\nabla\cdot\mathbf{A} = 0 \quad \text{in } \Omega, \tag{1.11}$$

$$\frac{\partial\mathbf{A}}{\partial t} + \nabla\times(\nabla\times\mathbf{A}) - \nabla(\nabla\cdot\mathbf{A}) + \mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right] = \nabla\times\mathbf{f} \quad \text{in } \Omega, \tag{1.12}$$

with the boundary conditions

$$\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{1.13}$$

$$(\nabla\times\mathbf{A})\times\mathbf{n} = \mathbf{f}\times\mathbf{n} \qquad \text{on } \partial\Omega, \tag{1.14}$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega. \tag{1.15}$$

Existence, uniqueness and gauge invariance property of the solution were proved in [13] in smooth domains. Error analysis of a Galerkin finite element method (FEM) with an implicit backward Euler time-stepping scheme was presented in [11], where optimal-order convergence rate of the numerical solution was proved for the sufficiently smooth solution. It has been reported in [23, 35] that the numerical solution of the magnetic potential by the FEM often exhibits undesired singularities around a corner. To resolve this problem, a mixed FEM was proposed in [9] to approximate the triple $(\nabla\times\mathbf{A},\, \nabla\cdot\mathbf{A}, \mathbf{A})$ in a finite element subspace of $\mathbf{H}^1(\Omega)\times H^1(\Omega)\times\mathbf{L}^2(\Omega)$, which requires less regularity of $\mathbf{A}$ intuitively, and error estimates of the finite element solution were presented under the assumption that $\mathbf{A}$ is in $\mathbf{H}^1_{\rm n}(\Omega) := \{\mathbf{a}\in\mathbf{H}^1(\Omega) : \mathbf{a}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$. Recently, an optimal-order error estimate of the FEM with a linearized Crank–Nicolson scheme was presented in [23] without restriction on the grid ratio, but the analysis requires stronger regularity of the solution and the domain.

On one hand, existing theoretical and numerical analysis of the model all require the magnetic potential to be in $\mathbf{H}^1_{\rm n}(\Omega)$. In a domain with reentrant corners, however, the magnetic potential may not be in $\mathbf{H}^1_{\rm n}(\Omega)$ and well-posedness of the TDGL remains open. On the other hand, numerical approximations of the TDGL in domains with reentrant corners (e.g., nonconvex polygons) are important for physicists to study the effects of surface defects in superconductivity [3, 37], which are often accomplished by solving (1.16)-(1.17) directly with the finite element or finite difference methods, without being aware of the danger of these numerical methods. In a nonsmooth domain, the magnetic potential $\mathbf{A}$ may be only in $L^\infty((0,T);\mathbf{H}(\text{curl},\text{div}))$, which is not equivalent to $L^\infty((0,T);\mathbf{H}^1(\Omega))$, and so the Galerkin finite element solution may not converge to the solution of TDGL. To solve the TDGL in nonsmooth domains, a mixed FEM was introduced in [24], where convergence of the mixed finite element solutions was illustrated numerically. Some discrete gauge invariant numerical methods introduced in [19, 21] are also promising to approximate the solution correctly in nonsmooth domains. However, theoretical analysis of convergence of these numerical methods in nonsmooth domains still remains open.

In this paper, we prove well-posedness of TDGL and regularity of its solutions in a general two-dimensional polygon, possibly with reentrant corners, and prove convergence of a numerical solution based on the regularity of the PDE's solution proved in this paper. Specifically, with $\mathbf{f} = (0, 0, f)$, the two-dimensional TDGL can be written as

$$\eta\frac{\partial\psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + (|\psi|^2 - 1)\psi - i\eta\kappa\psi\nabla\cdot\mathbf{A} = 0 \qquad \text{in } \Omega, \tag{1.16}$$

$$\frac{\partial\mathbf{A}}{\partial t} + \nabla\times(\nabla\times\mathbf{A}) - \nabla(\nabla\cdot\mathbf{A}) + \mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right] = \nabla\times f \quad \text{in } \Omega, \tag{1.17}$$

with the boundary and initial conditions

$$\nabla\psi\cdot\mathbf{n}=0,\quad \mathbf{A}\cdot\mathbf{n}=0,\quad \nabla\times\mathbf{A}=f \qquad \text{on } \partial\Omega, \tag{1.18}$$

$$\psi(x,0)=\psi_0(x),\quad \mathbf{A}(x,0)=\mathbf{A}_0(x) \qquad \text{in } \Omega, \tag{1.19}$$

where $\mathbf{A}=(A_1,A_2)$ and the following notation is used:

$$\nabla\times\mathbf{A}=\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2},\qquad \nabla\cdot\mathbf{A}=\frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2},$$
$$\nabla\times f=\left(\frac{\partial f}{\partial x_2},-\frac{\partial f}{\partial x_1}\right),\quad \nabla\psi=\left(\frac{\partial\psi}{\partial x_1},\frac{\partial\psi}{\partial x_2}\right).$$

This two-dimensional model has many applications in physics [3, 22, 32, 37]. We prove that the system (1.16)-(1.19) is well-posed, with $\mathbf{A}\in L^\infty((0,T);H^s(\Omega)^2)$ for some $s\in(0,1)$ which depends on the interior angles of the reentrant corners. As shown in the numerical examples (see Section 6 and [31]), with such low-regularity, the FEM may give an incorrect solution for the magnetic potential $\mathbf{A}$, which further pollutes the numerical solution of ψ due to the coupling of equations. We are interested in reformulating (1.16)-(1.19) into an equivalent form which can be solved correctly by the Galerkin FEMs, as they are preferred when using software packages and when other equations are coupled with the Ginzburg–Landau equations. Our idea is to apply the Hodge decomposition

$$\mathbf{A}=\nabla\times u+\nabla v \tag{1.20}$$

and

$$\mathrm{Re}\big[\psi^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\psi\big]=\nabla\times p+\nabla q,$$

and consider the projection of (1.17) onto the divergence-free and curl-free subspaces, respectively. Then (1.16)-(1.19) can be reformulated as

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi-i\eta\kappa\psi\nabla\cdot\mathbf{A}=0, \tag{1.21}$$

$$\Delta p=-\nabla\times\left(\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]\right), \tag{1.22}$$

$$\Delta q=\nabla\cdot\left(\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]\right), \tag{1.23}$$

$$\frac{\partial u}{\partial t}-\Delta u=f-p, \tag{1.24}$$

$$\frac{\partial v}{\partial t}-\Delta v=-q, \tag{1.25}$$

with the boundary and initial conditions

$$\nabla\psi\cdot\mathbf{n}=0,\quad p=0,\quad \nabla q\cdot\mathbf{n}=0,\quad u=0,\quad \nabla v\cdot\mathbf{n}=0,\quad \text{on } \partial\Omega\times(0,T], \tag{1.26}$$

$$\psi(x,0)=\psi_0(x),\quad u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad \text{in } \Omega, \tag{1.27}$$

where u_0 and v_0 are defined by

$$\begin{cases}-\Delta u_0=\nabla\times\mathbf{A}_0 & \text{in } \Omega,\\ u_0=0 & \text{on } \partial\Omega,\end{cases}\qquad\text{and}\qquad \begin{cases}\Delta v_0=\nabla\cdot\mathbf{A}_0 & \text{in } \Omega,\\ \partial_n v_0=0 & \text{on } \partial\Omega,\end{cases}$$

with $\int_\Omega v_0(x)\,\mathrm{d}x = 0$. In this formulation, u and v can be viewed as anti-derivatives of $\mathbf{A}$, which should be in $L^\infty((0,T);H^{1+s}(\Omega))$ and so can be solved by the Galerkin FEMs; p and q are auxiliary unknown variables. Detailed derivation of the new system (1.21)-(1.27) was presented in a separate paper [31], where the superiority of solving this new system was shown via numerical simulations (in comparison with the traditional approaches of solving the TDGL directly under the gauges $\phi = 0$ and $\phi = -\nabla\cdot\mathbf{A}$). In the current paper, we present theoretical and numerical analysis for (1.16)-(1.19) and the new system (1.21)-(1.27). In particular, we prove well-posedness and regularity of solutions for the two systems in a general nonconvex polygon, and also prove the equivalence of their solutions. Then we propose a decoupled and linearized FEM to solve (1.21)-(1.27) and prove the convergence of the finite element solution based on the regularity proved in this paper. The analysis presented in this paper may provide a foundation for future numerical analysis of this model in domains with corners.

Our main results are presented in Section 2, and we prove these results in Sections 3–5. In Section 6, we present numerical examples to support our theoretical analysis.

2. Notation and main results

For any nonnegative integer k, we let $W^{k,p}(\Omega)$ and $\mathcal{W}^{k,p}(\Omega)$ denote the conventional Sobolev spaces of real-valued and complex-valued functions defined in Ω, respectively, with $H^k(\Omega) = W^{k,2}(\Omega)$, $\mathcal{H}^k(\Omega) = \mathcal{W}^{k,2}(\Omega)$, $L^2(\Omega) = H^0(\Omega)$ and $\mathcal{L}^2(\Omega) = \mathcal{H}^0(\Omega)$; see [1]. For a positive real number $s_0 = k + s$, with $s \in (0,1)$, we define $H^{s_0}(\Omega) = (H^k(\Omega), H^{k+1}(\Omega))_{[s]}$ via the complex interpolation; see [5]. We denote $H^{s_0} = H^{s_0}(\Omega)$, $\mathcal{H}^{s_0} = \mathcal{H}^{s_0}(\Omega)$, $L^p = L^p(\Omega)$, $\mathcal{L}^p = \mathcal{L}^p(\Omega)$, and let $\mathring{H}^1$ denote the subspace of H^1 consisting of functions whose traces are zero on $\partial\Omega$. For any two functions $f, g \in \mathcal{L}^2$ we define

$$(f,g) = \int_\Omega f(x)g(x)^*\,\mathrm{d}x,$$

where $g(x)^*$ denotes the complex conjugate of $g(x)$, and define

$$\begin{aligned}
&\mathbf{L}^p = L^p \times L^p, \quad \mathbf{H}^s = H^s \times H^s, \quad \mathbf{H}^1_{\mathrm{n}}(\Omega) := \{\mathbf{a} \in H^1 \times H^1 : \mathbf{a}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}) = \{\mathbf{a} \in \mathbf{L}^2 : \nabla\times\mathbf{a} \in L^2,\ \nabla\cdot\mathbf{a} \in L^2 \text{ and } \mathbf{a}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&H(\mathrm{curl}) = \{g \in L^2 : \nabla\times g \in \mathbf{L}^2\}.
\end{aligned}$$

The definitions of weak solutions for (1.16)-(1.19) and (1.21)-(1.27) are presented below.

Definition 2.1 (Weak solutions of (1.16)-(1.19))**.** Let ω denote the maximal interior angle of the nonconvex polygon Ω. The pair $(\psi, \mathbf{A})$ is called a weak solution of (1.16)-(1.19) if

$$\begin{aligned}
&\psi \in C([0,T];\mathcal{L}^2) \cap L^\infty((0,T);\mathcal{H}^1) \cap L^2((0,T);\mathcal{H}^{1+s}),\\
&\partial_t\psi, \Delta\psi \in L^2((0,T);\mathcal{L}^2), \quad |\psi| \le 1 \text{ a.e. in } \Omega\times(0,T),\\
&\mathbf{A} \in C([0,T];\mathbf{L}^2) \cap L^\infty((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div})),\\
&\partial_t\mathbf{A} \in L^2((0,T);\mathbf{L}^2), \quad \nabla\times\mathbf{A}, \nabla\cdot\mathbf{A} \in L^2((0,T);H^1),
\end{aligned}$$

for any $s \in (1/2, \pi/\omega)$, with $\psi(\cdot,0) = \psi_0$, $\mathbf{A}(\cdot,0) = \mathbf{A}_0$, and the variational equations

$$\begin{aligned}&\int_0^T \left[\left(\eta\frac{\partial\psi}{\partial t},\varphi\right) + \left(\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi, \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\varphi\right)\right] \mathrm{d}t \\ &\qquad + \int_0^T \left[\left((|\psi|^2-1)\psi - i\eta\kappa\psi\nabla\cdot\mathbf{A}, \varphi\right)\right] \mathrm{d}t = 0,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\int_0^T \left[\left(\frac{\partial\mathbf{A}}{\partial t},\mathbf{a}\right) + (\nabla\times\mathbf{A},\nabla\times\mathbf{a}) + (\nabla\cdot\mathbf{A},\nabla\cdot\mathbf{a})\right] \mathrm{d}t \\ &= \int_0^T \left[(f,\nabla\times\mathbf{a}) - \left(\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right], \mathbf{a}\right)\right] \mathrm{d}t,\end{aligned} \tag{2.2}$$

hold for all $\varphi \in L^2((0,T);\mathcal{H}^1)$ and $\mathbf{a} \in L^2((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))$.

Definition 2.2 (Weak solutions of (1.21)-(1.27))**.** Let ω denote the maximal interior angle of the nonconvex polygon Ω. The quintuple (ψ,p,q,u,v) is called a weak solution of (1.21)-(1.27) if

$\psi \in C([0,T];\mathcal{L}^2) \cap L^\infty((0,T);\mathcal{H}^1) \cap L^2((0,T);\mathcal{H}^{1+s})$,
$\partial_t\psi, \Delta\psi \in L^2((0,T);\mathcal{L}^2)$, $\quad |\psi| \le 1$ a.e. in $\Omega\times(0,T)$,
$p \in L^\infty((0,T);\mathring{H}^1)$, $\quad q \in L^\infty((0,T);H^1)$, $\quad u \in C([0,T];\mathring{H}^1)$, $\quad v \in C([0,T];H^1)$,
$\partial_t u, \partial_t v, \Delta u, \Delta v \in L^\infty((0,T);L^2) \cap L^2((0,T);H^1)$,

for any $s \in (1/2,\pi/\omega)$, with $\psi(\cdot,0) = \psi_0$, $u(\cdot,0) = u_0$, $v(\cdot,0) = v_0$, and the variational equations

$$\begin{aligned}&\int_0^T \left[\left(\eta\frac{\partial\psi}{\partial t},\varphi\right) + \left(\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi, \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\varphi\right)\right] \mathrm{d}t \\ &\qquad + \int_0^T \left[\left((|\psi|^2-1)\psi - i\eta\kappa\psi\nabla\cdot\mathbf{A}, \varphi\right)\right] \mathrm{d}t = 0,\end{aligned} \tag{2.3}$$

$$\int_0^T (\nabla p, \nabla\xi)\,\mathrm{d}t = \int_0^T \left(\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right], \nabla\times\xi\right) \mathrm{d}t, \tag{2.4}$$

$$\int_0^T (\nabla q, \nabla\zeta)\,\mathrm{d}t = \int_0^T \left(\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right], \nabla\zeta\right) \mathrm{d}t, \tag{2.5}$$

$$\int_0^T \left[\left(\frac{\partial u}{\partial t},\theta\right) + (\nabla u,\nabla\theta)\right] \mathrm{d}t = \int_0^T (f-p,\theta)\,\mathrm{d}t, \tag{2.6}$$

$$\int_0^T \left[\left(\frac{\partial v}{\partial t},\vartheta\right) + (\nabla v,\nabla\vartheta)\right] \mathrm{d}t = -\int_0^T (q,\vartheta)\,\mathrm{d}t, \tag{2.7}$$

hold for all $\varphi \in L^2((0,T);\mathcal{H}^1)$, $\xi,\theta \in L^2((0,T);\mathring{H}^1)$ and $\zeta,\vartheta \in L^2((0,T);H^1)$.

Our first result is the well-posedness and equivalence of the systems (1.16)-(1.19) and (1.21)-(1.27), which are presented in the following theorem.

Theorem 2.1 (Well-posedness and equivalence of the two systems)**.** *Suppose* $f \in L^\infty((0,T);L^2) \cap L^2((0,T);H(\mathrm{curl}))$, $\psi_0 \in \mathcal{H}^1$, $\mathbf{A}_0 \in \mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div})$ *and* $|\psi_0| \le 1$ *a.e. in* Ω. *Then the system* (1.16)-(1.19) *admits a unique weak solution in the* *sense of Definition* 2.1, *and the system* (1.21)-(1.27) *admits a unique solution which coincides with the solution of* (1.16)-(1.19).

Moreover, if we let x_j, $j = 1, \cdots, m$, *be the reentrant corners of the domain* Ω, *then the solution has the decomposition*

$$\psi(x,t) = \Psi(x,t) + \sum_{j=1}^{m} \alpha_j(t)\Phi(|x-x_j|)|x-x_j|^{\pi/\omega_j}\cos(\pi\Theta_j(x)/\omega_j),$$

$$\mathbf{A} = \nabla \times u + \nabla v$$

with

$$u(x,t) = \widetilde{u}(x,t) + \sum_{j=1}^{m} \beta_j(t)\Phi(|x-x_j|)|x-x_j|^{\pi/\omega_j}\sin(\pi\Theta_j(x)/\omega_j),$$

$$v(x,t) = \widetilde{v}(x,t) + \sum_{j=1}^{m} \gamma_j(t)\Phi(|x-x_j|)|x-x_j|^{\pi/\omega_j}\cos(\pi\Theta_j(x)/\omega_j),$$

where $\Psi \in L^2((0,T);\mathcal{H}^2)$, $\widetilde{u}, \widetilde{v} \in L^\infty((0,T);H^2)$, $\Phi(r)$ *is a given smooth cut-off function which equals* 1 *in a neighborhood of* 0, $\Theta_j(x)$ *is the angle shown in Figure* 2.1, *and* $\alpha_j, \beta_j, \gamma_j \in L^2(0,T)$.

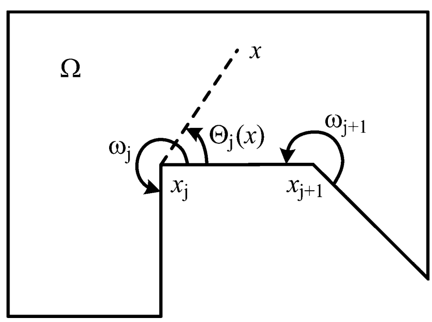

FIGURE 2.1. Illustration of the domain Ω, corner x_j, interior angle ω_j and argument $\Theta_j(x)$.

Further regularity of the solution is presented below, which is needed in the analysis of the convergence of the numerical solution.

Theorem 2.2 (Further regularity)**.** *If* $f \in C([0,T];H(\mathrm{curl}))$, $\nabla \times f \in L^2((0,T);H(\mathrm{curl}))$, $\partial_t f \in L^2((0,T);L^2)$, $\psi_0 \in \mathcal{H}^1$, $\Delta\psi_0 \in \mathcal{L}^2$, $\mathbf{A}_0 \in \mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div})$, $\nabla \cdot \mathbf{A}_0, \nabla \times \mathbf{A}_0 \in \mathbf{H}^1$, $|\psi_0| \le 1$ *a.e. in* Ω, *and the compatibility conditions*

$$\partial_n \psi_0 = 0 \quad \textit{and} \quad \nabla \times \mathbf{A}_0 = f(\cdot,0) \quad \textit{on} \quad \partial\Omega$$

are satisfied, then the solution of (1.21)-(1.27) *possesses the regularity*

$$\psi \in C([0,T];\mathcal{H}^{1+s}), \quad \partial_t\psi \in L^2((0,T);\mathcal{H}^{1+s}), \quad \partial_{tt}\psi \in L^2((0,T);\mathcal{L}^2),$$
$$p, q \in L^\infty((0,T);H^1), \quad u, v \in C([0,T];H^{1+s}),$$
$$\partial_t u, \partial_t v \in L^2((0,T);H^{1+s}), \quad \partial_{tt}u, \partial_{tt}v \in L^2((0,T);L^2)$$

for any $s \in (1/2, \pi/\omega)$.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ and set $\tau = T/N$, where N is a positive integer. Let $w^n(x) = w(x, t_n)$ for any given function w defined on $\Omega \times [0, T]$. For any sequence of functions φ^n we define the backward Euler difference operator $D_\tau \varphi^{n+1} := (\varphi^{n+1} - \varphi^n)/\tau$, and we define a cut-off function $\chi : \mathbb{C} \to \mathbb{C}$ by

$$\chi(z) = z/\max(|z|, 1), \quad \forall\, z \in \mathbb{C},$$

which is Lipschitz continuous and satisfies that $|\chi(z)| \le 1$, $\forall\, z \in \mathbb{C}$. To solve the reformulated system (1.21)-(1.27), we linearize (1.21)-(1.27) in the following way:

$$\eta D_\tau \psi_\tau^{n+1} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}_\tau^n\right)^2 \psi_\tau^{n+1} + (|\psi_\tau^n|^2 - 1)\psi_\tau^{n+1} - i\eta\kappa\psi_\tau^{n+1}\nabla\cdot\mathbf{A}_\tau^n = 0, \tag{2.8}$$

$$\Delta p_\tau^{n+1} = -\nabla \times \left(\mathrm{Re}\left[\chi(\psi_\tau^n)^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}_\tau^n\right)\psi_\tau^{n+1}\right]\right) \tag{2.9}$$

$$\Delta q_\tau^{n+1} = \nabla \cdot \left(\mathrm{Re}\left[\chi(\psi_\tau^n)^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}_\tau^n\right)\psi_\tau^{n+1}\right]\right) \tag{2.10}$$

$$D_\tau u_\tau^{n+1} - \Delta u_\tau^{n+1} = f^{n+1} - p_\tau^{n+1}, \tag{2.11}$$

$$D_\tau v_\tau^{n+1} - \Delta v_\tau^{n+1} = -q_\tau^{n+1}, \tag{2.12}$$

with $\mathbf{A}_\tau^n = \nabla \times u_\tau^n + \nabla v_\tau^n$ and the boundary/initial conditions

$$\partial_n \psi_\tau^{n+1} = \partial_n q_\tau^{n+1} = \partial_n v_\tau^{n+1} = 0 \qquad \text{on} \ \ \partial\Omega, \tag{2.13}$$

$$p_\tau^{n+1} = u_\tau^{n+1} = 0 \qquad \text{on} \ \ \partial\Omega, \tag{2.14}$$

$$\psi_\tau^0 = \psi_0, \quad u_\tau^0 = u_0, \quad v_\tau^0 = v_0, \qquad \text{in} \ \ \Omega. \tag{2.15}$$

Since the exact solution satisfies $|\psi| \le 1$ a.e., the above time discretization scheme is consistent.

We solve the linear equations above by the Galerkin FEM. Specifically, we let π_h be a quasi-uniform triangulation of the domain Ω and denote the mesh size by h, let $\mathcal{V}_h^1$ denote the space of complex-valued C^0 piecewise linear functions subject to the triangulation, let V_h^1 denote the space of real-valued C^0 piecewise linear functions, and set $\mathring{V}_h^1 = \{\varphi \in V_h^1 : \varphi = 0 \text{ on } \partial\Omega\}$. Clearly, $\mathcal{V}_h^1$, $\mathring{V}_h^1$ and V_h^1 are finite dimensional subspaces of $\mathcal{H}^1$, $\mathring{H}^1$ and H^1, respectively. Let I_h be the commonly used Lagrange interpolation operator onto the finite element spaces. We look for $\psi_h^{n+1} \in \mathcal{V}_h^1$, $p_h^{n+1}, u_h^{n+1} \in \mathring{V}_h^1$ and $q_h^{n+1}, v_h^{n+1} \in V_h^1$ satisfying the equations

$$\begin{aligned}&\big(\eta D_\tau \psi_h^{n+1}, \varphi\big) + \big((i\kappa^{-1}\nabla + \mathbf{A}_h^n)\psi_h^{n+1}, (i\kappa^{-1}\nabla + \mathbf{A}_h^n)\varphi\big) \\ &\quad + \big((|\psi_h^n|^2 - 1)\psi_h^{n+1}, \varphi\big) + \big(i\eta\kappa\mathbf{A}_h^n, \nabla((\psi_h^{n+1})^*\varphi)\big) = 0,\end{aligned} \tag{2.16}$$

$$\big(\nabla p_h^{n+1}, \nabla\xi\big) = \big(\mathrm{Re}[\chi(\psi_h^n)^*(i\kappa^{-1}\nabla\psi_h^{n+1} + \mathbf{A}_h^n\psi_h^{n+1})], \nabla\times\xi\big), \tag{2.17}$$

$$\big(\nabla q_h^{n+1}, \nabla\zeta\big) = \big(\mathrm{Re}[\chi(\psi_h^n)^*(i\kappa^{-1}\nabla\psi_h^{n+1} + \mathbf{A}_h^n\psi_h^{n+1})], \nabla\zeta\big), \tag{2.18}$$

$$\big(D_\tau u_h^{n+1}, \theta\big) + \big(\nabla u_h^{n+1}, \nabla\theta\big) = (f^{n+1} - p_h^{n+1}, \theta), \tag{2.19}$$

$$\big(D_\tau v_h^{n+1}, \vartheta\big) + \big(\nabla v_h^{n+1}, \nabla\vartheta\big) = (-q_h^{n+1}, \vartheta), \tag{2.20}$$

for all $\varphi \in \mathcal{V}_h^1$, $\xi, \theta \in \mathring{V}_h^1$ and $\zeta, \vartheta \in V_h^1$, with $\mathbf{A}_h^n = \nabla \times u_h^n + \nabla v_h^n$, where $u_h^0 \in \mathring{V}_h^1$ and $v_h^0 \in V_h^1$ are solved from

$$(\nabla u_h^0, \nabla \xi) = \big(\mathbf{A}_0, \nabla \times \xi\big), \quad \forall\, \xi \in \mathring{V}_h^1, \tag{2.21}$$

$$(\nabla v_h^0, \nabla \zeta) = \big(\mathbf{A}_0, \nabla \cdot \zeta\big), \ \ \forall\, \zeta \in V_h^1, \tag{2.22}$$

and ψ_h^0 is the Lagrange interpolation of ψ^0.

Theorem 2.3 (Convergence of the finite element solution)**.** *The finite element system* (2.16)-(2.20) *admits a unique solution* $(\psi_h^n, p_h^n, q_h^n, u_h^n, v_h^n)$ *when* $\tau < \eta/4$ *and, under the assumptions of Theorem* 2.2*,*

$$\max_{1\le n\le N} \big(\|u^n - u_h^n\|_{H^1} + \|v^n - v_h^n\|_{H^1} + \|\mathbf{A}^n - \mathbf{A}_h^n\|_{L^2} + \|\psi^n - \psi_h^n\|_{\mathcal{L}^2}\big) \\ \le C(\tau + h^s),$$

where C is a positive constant independent of τ and h.

In the remainder of this paper, we prove Theorems 2.1–2.3. To simplify the notation, we denote by C a generic positive constant which may be different at each occurrence but is independent of n, τ and h.

3. Proof of Theorem 2.1

In this section, we prove well-posedness of the Ginzburg–Landau equations in a nonconvex polygon and equivalence of the two formulations (1.16)-(1.19) and (1.21)-(1.27). Compared with smooth domains, in a nonconvex polygon, the space $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$ is not equivalent to $\mathbf{H}_{\mathrm{n}}^1(\Omega)$ and is not embedded into $\mathbf{L}^p$ for large p. Convergence of the nonlinear terms of the approximating solutions needs to be proved based on the weaker embedding $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}) \hookrightarrow\hookrightarrow \mathbf{L}^4$ in the compactness argument, and uniqueness of solution needs to be proved based on weaker regularity of the solution.

3.1. Preliminaries.

First, we cite a lemma concerning the regularity of Poisson's equations in a nonconvex polygon [15, 26].

Lemma 3.1. *The solution of the Poisson equations*

$$\begin{cases} \Delta u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta v = \widetilde{g} & \text{in } \Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies that (*the Neumann problem requires* $\int_\Omega \widetilde{g}(x)\,\mathrm{d}x = \int_\Omega v(x)\,\mathrm{d}x = 0$)

$$\|u\|_{W^{1,p_s}} + \|u\|_{H^{1+s}} \le C_s \|g\|_{L^2}, \quad \forall\, s \in (1/2, \pi/\omega)$$

and

$$\|v\|_{W^{1,p_s}} + \|v\|_{H^{1+s}} \le C_s \|\widetilde{g}\|_{L^2}, \quad \forall\, s \in (1/2, \pi/\omega),$$

where $p_s = 2/(1-s) > 4$ when $s \in (1/2, \pi/\omega)$.

Second, we introduce a lemma concerning the embedding of $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})$ into $\mathbf{H}^s$.

Lemma 3.2. $\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div}) \hookrightarrow \mathbf{H}^s \hookrightarrow \mathbf{L}^{p_s}$ *for any* $s \in (1/2, \pi/\omega)$.

Proof. From [9] we know that $\mathbf{A}$ has the decomposition $\mathbf{A} = \nabla \times u + \nabla v$, where u and v are the solutions of

$$\begin{cases} -\Delta u = \nabla \times \mathbf{A} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta v = \nabla \cdot \mathbf{A} & \text{in } \Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively, with $\int_\Omega v(x)\,\mathrm{d}x = 0$. For the two Poisson equations, Lemma 3.1 implies that

$$\|u\|_{H^{1+s}} \le C_s \|\nabla \times \mathbf{A}\|_{L^2}, \quad \forall\, s \in (1/2, \pi/\omega)$$

and

$$\|v\|_{H^{1+s}} \le C_s \|\nabla \cdot \mathbf{A}\|_{L^2}, \quad \forall\, s \in (1/2, \pi/\omega),$$

which imply

$$\|\mathbf{A}\|_{H^s} \le C_s(\|\nabla \times \mathbf{A}\|_{L^2} + \|\nabla \cdot \mathbf{A}\|_{L^2}) \le C_s \|\mathbf{A}\|_{\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div})}, \quad \forall\, s \in (1/2, \pi/\omega).$$

□

3.2. **Existence of weak solutions for (1.16)-(1.19).** In this subsection, we prove existence of weak solutions for the system (1.16)-(1.19) by constructing approximating solutions in finite dimensional spaces and then applying a compactness argument. First, we need the following lemma to control the order parameter pointwisely.

Lemma 3.3. *For any given* $\mathbf{A} \in L^\infty((0,T); \mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))$*, the equation* (1.16) *has at most one weak solution* $\psi \in L^\infty((0,T); H^1) \cap H^1((0,T); L^2)$ *in the sense of* (2.1). *If the solution exists, then it satisfies that* $|\psi| \le 1$ *a.e. in* $\Omega \times (0,T)$.

Proof. From Lemma 3.2 we see that $\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}) \hookrightarrow \mathbf{H}^s \hookrightarrow \mathbf{L}^4$ and so $\mathbf{A} \in L^\infty((0,T); \mathbf{L}^4)$. Uniqueness of the solution can be proved easily based on the regularity assumption of ψ. To prove $|\psi| \le 1$ a.e. in $\Omega \times (0,T)$, we integrate (1.16) against $\psi^*(|\psi|^2 - 1)_+$ and consider the real part, where $(|\psi|^2 - 1)_+$ denotes the positive part of $|\psi|^2 - 1$. For any $t' \in (0,T)$ we have

$$\begin{aligned}
&\int_\Omega \left(\frac{\eta}{4}(|\psi(x,t')|^2 - 1)_+^2\right) \mathrm{d}x + \int_0^{t'} \int_\Omega (|\psi|^2 - 1)_+^2 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{t'} \mathrm{Re} \int_\Omega \left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\left(-\frac{i}{\kappa}\nabla + \mathbf{A}\right)[\psi^*(|\psi|^2 - 1)_+]\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{t'} \int_\Omega \left|\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right|^2 (|\psi|^2 - 1)_+ \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{t'} \mathrm{Re} \int_{\{|\psi|^2 > 1\}} \left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\psi^* \left(\frac{i}{\kappa}\psi\nabla\psi^* + \frac{i}{\kappa}\psi^*\nabla\psi\right)\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^{t'} \int_\Omega \left|\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right|^2 (|\psi|^2 - 1)_+ \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^{t'} \mathrm{Re} \int_{\{|\psi|^2 > 1\}} \frac{1}{\kappa^2}(|\psi|^2|\nabla\psi|^2 + (\psi^*)^2\nabla\psi\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t \\
&\le 0,
\end{aligned}$$

which implies that $\int_\Omega (|\psi(x,t')|^2 - 1)_+^2\,\mathrm{d}x = 0$. Thus $|\psi| \le 1$ a.e. in $\Omega \times (0,T)$. □

Second, we construct approximating solutions in finite dimensional spaces. For this purpose, we let $\phi_1, \phi_2, \cdots$ be the eigenfunctions of the Neumann Laplacian, which form a basis of $\mathcal{H}^1$. Let $M : \mathbf{H}_{\rm n}({\rm curl},{\rm div}) \to (\mathbf{H}_{\rm n}({\rm curl},{\rm div}))'$ be defined by

$$(M\mathbf{u},\mathbf{v}) = (\nabla\times\mathbf{u},\nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}), \quad \text{for } \mathbf{u},\mathbf{v}\in\mathbf{H}_{\rm n}({\rm curl},{\rm div})\,.$$

Since the bilinear form on the right-hand side is coercive on the space $\mathbf{H}_{\rm n}({\rm curl},{\rm div})$, which is compactly embedded into $\mathbf{L}^2$, the spectrum of M consists of a sequence of eigenvalues which tend to infinity, and the corresponding eigenvectors $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\cdots$ form a basis of $\mathbf{H}_{\rm n}({\rm curl},{\rm div})$ [14, 34].

We define $\mathcal{V}_N = {\rm span}\{\phi_1,\phi_2,\cdots,\phi_N\}$ and $\mathbf{X}_N = {\rm span}\{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_N\}$, which are finite dimensional subspaces of $\mathcal{H}^1$ and $\mathbf{H}_{\rm n}({\rm curl},{\rm div})$, respectively, and we look for $\Psi_N(t)\in\mathcal{V}_N$, $\mathbf{\Lambda}_N(t)\in\mathbf{X}_N$ such that

$$\begin{aligned}&\Big(\eta\frac{\partial\Psi_N}{\partial t},\varphi\Big) + \Big(\Big(\frac{i}{\kappa}\nabla+\mathbf{\Lambda}_N\Big)\Psi_N, \Big(\frac{i}{\kappa}\nabla+\mathbf{\Lambda}_N\Big)\varphi\Big)\\ &\qquad + \Big((|\Psi_N|^2-1)\Psi_N - i\eta\kappa\chi(\Psi_N)\nabla\cdot\mathbf{\Lambda}_N,\varphi\Big) = 0,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\Big(\frac{\partial\mathbf{\Lambda}_N}{\partial t},\mathbf{a}\Big) + (\nabla\times\mathbf{\Lambda}_N,\nabla\times\mathbf{a}) + (\nabla\cdot\mathbf{\Lambda}_N,\nabla\cdot\mathbf{a})\\ &\qquad + \Big({\rm Re}\Big[\Psi_N^*\Big(\frac{i}{\kappa}\nabla+\mathbf{\Lambda}_N\Big)\Psi_N\Big],\mathbf{a}\Big) = (f,\nabla\times\mathbf{a}),\end{aligned} \tag{3.2}$$

for any $\varphi\in\mathcal{V}_N$ and $\mathbf{a}\in\mathbf{X}_N$ at any $t\in(0,T)$, with the initial conditions $\Psi(0)=\Pi_N\psi_0$ and $\mathbf{\Lambda}(0)=\widetilde{\Pi}_N\mathbf{A}_0$, where Π_N and $\widetilde{\Pi}_N$ are the projections of $\mathcal{H}^1$ and $\mathbf{H}_{\rm n}({\rm curl},{\rm div})$ onto the subspaces $\mathcal{V}_N$ and $\mathbf{X}_N$, respectively.

Existence and uniqueness of solutions for the ODE problem (3.1)-(3.2) are obvious. To present estimates of the semi-discrete solution $(\Psi_N,\mathbf{\Lambda}_N)$, we substitute $\varphi=\partial_t\Psi$ and $\mathbf{a}=\partial_t\mathbf{\Lambda}$ into the equations, and sum up the two results. Then we obtain that

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}\int_\Omega\frac{1}{2}\Big(\Big|\frac{i}{\kappa}\nabla\Psi_N+\mathbf{\Lambda}_N\Psi_N\Big|^2+\frac{1}{2}(|\Psi_N|^2-1)^2+|\nabla\times\mathbf{\Lambda}_N-f|^2+|\nabla\cdot\mathbf{\Lambda}_N|^2\Big)\,{\rm d}x\\ &\quad+\int_\Omega\Big(\Big|\frac{\partial\mathbf{\Lambda}_N}{\partial t}\Big|^2+\eta\Big|\frac{\partial\Psi_N}{\partial t}\Big|^2\Big)\,{\rm d}x\\ &=\eta\kappa\int_\Omega{\rm Im}\Big(\chi(\Psi_N)\frac{\partial\Psi_N^*}{\partial t}\Big)\nabla\cdot\mathbf{\Lambda}_N\,{\rm d}x\\ &\le\frac{1}{2}\int_\Omega\eta\Big|\frac{\partial\Psi_N}{\partial t}\Big|^2{\rm d}x+\frac{1}{2}\int_\Omega\eta\kappa^2|\nabla\cdot\mathbf{\Lambda}_N|^2\,{\rm d}x.\end{aligned}$$

By applying Gronwall's inequality, we obtain that

$$\begin{aligned}&\|\Psi_N\|_{L^\infty((0,T);\mathcal{H}^1)}+\|\partial_t\Psi_N\|_{L^2((0,T);\mathcal{L}^2)}\\ &\quad+\|\mathbf{\Lambda}_N\|_{L^\infty((0,T);\mathbf{H}_{\rm n}({\rm curl},{\rm div}))}+\|\partial_t\mathbf{\Lambda}_N\|_{L^2((0,T);L^2)}\le C,\end{aligned}$$

where the constant C does not depend on N.

Third, since $H^1 \hookrightarrow\hookrightarrow L^p$ for any $1 < p < \infty$ and $\mathbf{H}_{\mathbf{n}}(\mathrm{curl}, \mathrm{div}) \hookrightarrow\hookrightarrow L^{4+\varepsilon}$ for some $\varepsilon > 0$, by the Aubin–Lions compactness argument [30], there exist

$$\psi \in L^\infty((0,T);\mathcal{H}^1) \cap H^1((0,T);\mathcal{L}^2), \tag{3.3}$$
$$\mathbf{A} \in L^\infty((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})) \cap H^1((0,T);\mathbf{L}^2), \tag{3.4}$$

and a subsequence of $(\Psi_N, \mathbf{\Lambda}_N)_{N=1}^\infty$, denoted by $(\Psi_{N_m}, \mathbf{\Lambda}_{N_m})_{m=1}^\infty$, such that

$$\begin{aligned}
&\Psi_{N_m} \rightharpoonup \psi \quad \text{weakly}^* \text{ in } L^\infty((0,T);\mathcal{H}^1),\\
&\Psi_{N_m} \rightharpoonup \psi \quad \text{weakly in } L^p((0,T);\mathcal{H}^1) \text{ for any } 1 < p < \infty,\\
&\partial_t\Psi_{N_m} \rightharpoonup \partial_t\psi \quad \text{weakly in } L^2((0,T);\mathcal{L}^2),\\
&\Psi_{N_m} \to \psi \quad \text{strongly in } L^p((0,T);\mathcal{L}^p) \text{ for any } 1 < p < \infty,\\
&\mathbf{\Lambda}_{N_m} \rightharpoonup \mathbf{A} \quad \text{weakly}^* \text{ in } L^\infty((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})),\\
&\mathbf{\Lambda}_{N_m} \rightharpoonup \mathbf{A} \quad \text{weakly in } L^p((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})) \text{ for any } 1 < p < \infty,\\
&\partial_t\mathbf{\Lambda}_{N_m} \rightharpoonup \partial_t\mathbf{A} \quad \text{weakly in } L^2((0,T);\mathbf{L}^2),\\
&\mathbf{\Lambda}_{N_m} \to \mathbf{A} \quad \text{strongly in } L^p((0,T);\mathbf{L}^{4+\varepsilon}) \text{ for any } 1 < p < \infty,
\end{aligned}$$

which further imply that

$$\begin{aligned}
&\Psi_{N_m}\mathbf{\Lambda}_{N_m} \to \psi\mathbf{A} \quad \text{strongly in } L^2((0,T);\mathcal{L}^2 \times \mathcal{L}^2),\\
&\nabla\Psi_{N_m} \cdot \mathbf{\Lambda}_{N_m} \rightharpoonup \nabla\psi \cdot \mathbf{A} \quad \text{weakly in } L^2((0,T);\mathcal{L}^{4/3}),\\
&\Psi_{N_m}|\mathbf{\Lambda}_{N_m}|^2 \to \psi|\mathbf{A}|^2 \quad \text{strongly in } L^2((0,T);\mathcal{L}^{4/3}),\\
&\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N\right)\Psi_N \rightharpoonup \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi \quad \text{weakly in } L^2((0,T);\mathcal{L}^2),\\
&\Psi_N^*\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N\right)\Psi_N \rightharpoonup \psi^*\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi \quad \text{weakly in } L^2((0,T);\mathcal{L}^{4/3} \times \mathcal{L}^{4/3}),\\
&\left(\frac{i}{\kappa}\nabla + \mathbf{\Lambda}_N\right)\Psi_N \cdot \mathbf{\Lambda}_N \rightharpoonup \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi \cdot \mathbf{A} \quad \text{weakly in } L^2((0,T);\mathcal{L}^{4/3}).
\end{aligned}$$

For any given $\varphi \in L^2((0,T);\mathcal{V}_N) \hookrightarrow L^2((0,T);\mathcal{L}^4)$ and $\mathbf{a} \in L^2((0,T);\mathbf{X}_N) \hookrightarrow L^2((0,T);\mathbf{L}^4)$, integrating (3.1)-(3.2) with respect to time and letting $N = N_m \to \infty$, we derive (2.1)-(2.2). In other words, $\psi \in L^\infty((0,T);\mathcal{H}^1) \cap H^1((0,T);\mathcal{L}^2)$ is a weak solution of (1.16) in the sense of (2.1), and $\mathbf{A} \in L^\infty((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})) \cap H^1((0,T);\mathbf{L}^2)$ is a weak solution of (1.17) in the sense of (2.2). The conditions of Lemma 3.3 are satisfied, which implies that $|\psi| \le 1$ a.e. in $\Omega \times (0,T)$.

Finally, we prove the additional regularity of the solution specified in Definition 2.1. From Lemma 3.2 we see that

$$\mathbf{A} \in L^\infty((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl}, \mathrm{div})) \hookrightarrow L^\infty((0,T);\mathbf{H}^s) \hookrightarrow L^\infty((0,T);\mathbf{L}^4)$$

for any $s \in (1/2, \pi/\omega)$. From (1.16) we see that

$$\frac{1}{\kappa^2}\Delta\psi = \eta\partial_t\psi + \frac{i}{\kappa}\nabla\cdot(\mathbf{A}\psi) + \frac{i}{\kappa}\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi + (|\psi|^2-1)\psi - i\eta\kappa\psi\nabla\cdot\mathbf{A}$$

which implies that

$$\begin{aligned}
\|\Delta\psi\|_{\mathcal{L}^2} &\le C\|\partial_t\psi\|_{\mathcal{L}^2} + C\|\nabla\cdot\mathbf{A}\|_{L^2}\|\psi\|_{\mathcal{L}^\infty} \\
&\qquad + C\|\mathbf{A}\|_{L^4}\|\nabla\psi\|_{\mathcal{L}^4} + C\|\mathbf{A}\|_{L^4}^2 + C\|(|\psi|^2-1)\psi\|_{\mathcal{L}^2} \\
&\le C + C\|\partial_t\psi\|_{L^2} + C\|\nabla\psi\|_{\mathcal{L}^4} \\
&\le C + C\|\partial_t\psi\|_{L^2} + C\|\nabla\psi\|_{\mathcal{L}^2}^{(1-4/p_s)/(2-4/p_s)}\|\nabla\psi\|_{\mathcal{L}^{p_s}}^{1/(2-4/p_s)} \\
&\le C + \eta\|\partial_t\psi\|_{\mathcal{L}^2} + C\|\nabla\psi\|_{\mathcal{L}^{p_s}}^{1/(2-4/p_s)} \\
&\le C + C\|\partial_t\psi\|_{\mathcal{L}^2} + C\|\Delta\psi\|_{\mathcal{L}^2}^{1/(2-4/p_s)},
\end{aligned}$$

where we have used (3.3) and Lemma 3.1 in the last inequality. Since $1/(2-4/p_s) < 1$, the last inequality implies $\|\Delta\psi\|_{\mathcal{L}^2} \le C + C\|\partial_t\psi\|_{\mathcal{L}^2}$, and so

$$\|\Delta\psi\|_{L^2((0,T);\mathcal{L}^2)} \le C + C\|\partial_t\psi\|_{L^2((0,T);\mathcal{L}^2)} \le C,$$

which further implies $\psi \in L^2((0,T);\mathcal{H}^{1+s})$ by Lemma 3.1. From (1.17) we see that

$$\begin{aligned}
&\|\nabla\times(\nabla\times\mathbf{A}) - \nabla(\nabla\cdot\mathbf{A})\|_{L^2((0,T);\mathbf{L}^2)} \\
&\le C\|\partial_t\mathbf{A}\|_{L^2((0,T);\mathbf{L}^2)} + C\|\psi^*(i\kappa^{-1}\nabla\psi + \mathbf{A}\psi)\|_{L^2((0,T);\mathbf{L}^2)} + C\|\nabla\times f\|_{L^2((0,T);\mathbf{L}^2)} \\
&\le C\|\partial_t\mathbf{A}\|_{L^2((0,T);\mathbf{L}^2)} + C\|\nabla\psi\|_{L^2((0,T);\mathcal{L}^2)} \\
&\quad + C\|\mathbf{A}\|_{L^2((0,T);\mathbf{L}^2)} + C\|\nabla\times f\|_{L^2((0,T);\mathbf{L}^2)} \\
&\le C.
\end{aligned}$$

Note that $w = \nabla\times\mathbf{A} - f$ satisfies the equation

$$-\Delta w = \nabla\times(\nabla\times w) = \nabla\times\mathbf{f},$$

with $w = 0$ on $\partial\Omega$ and $\mathbf{f} = \nabla\times(\nabla\times\mathbf{A}) - \nabla(\nabla\cdot\mathbf{A}) - \nabla\times f \in L^2((0,T);\mathbf{L}^2)$. The energy estimate of w gives

$$\|w\|_{L^2((0,T);H^1)} \le C\|\mathbf{f}\|_{L^2((0,T);\mathbf{L}^2)} \le C.$$

Thus $\nabla(\nabla\cdot\mathbf{A}) = \nabla\times w - \mathbf{f} \in L^2((0,T);\mathbf{L}^2)$, which indicates that $\nabla\cdot\mathbf{A} \in L^2((0,T);H^1)$.

Existence of a weak solution of (1.16)-(1.19) in the sense of Definition 2.1 has been proved.

3.3. **Uniqueness of the weak solution.** Suppose that there are two solutions $(\psi, \mathbf{A})$ and $(\Psi, \mathbf{\Lambda})$ for the system (1.16)-(1.19) in the sense of Definition 2.1. Let

$e = \psi - \Psi$ and $\mathbf{E} = \mathbf{A} - \mathbf{\Lambda}$. Then we have

$$
\begin{aligned}
&\int_0^T \Big[\big(\eta\partial_t e, \varphi\big) + \frac{1}{\kappa^2}(\nabla e, \nabla\varphi) + \big(|\mathbf{A}|^2 e, \varphi\big)\Big]\,\mathrm{d}t \\
&= \int_0^T \Big[-\frac{i}{\kappa}\big(\mathbf{A}\cdot\nabla e, \varphi\big) - \frac{i}{\kappa}\big(\mathbf{E}\cdot\nabla\Psi, \varphi\big) + \frac{i}{\kappa}\big(e\mathbf{\Lambda}, \nabla\varphi\big) + \frac{i}{\kappa}\big(\Psi\mathbf{E}, \nabla\varphi\big) \\
&\quad - \big((|\mathbf{A}|^2 - |\mathbf{\Lambda}|^2)\Psi, \varphi\big) - \big(|\mathbf{\Lambda}|^2 e, \varphi\big) - \big((|\psi|^2-1)\psi - (|\Psi|^2-1)\Psi, \varphi\big)\Big]\,\mathrm{d}t \\
&\quad - \int_0^T \big(i\eta\kappa\psi\nabla\cdot\mathbf{E} + i\eta\kappa e\nabla\cdot\mathbf{\Lambda}, \varphi\big)\,\mathrm{d}t,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\int_0^T \Big[\big(\partial_t\mathbf{E}, \mathbf{a}\big) + \big(\nabla\times\mathbf{E}, \nabla\times\mathbf{a}\big) + \big(\nabla\cdot\mathbf{E}, \nabla\cdot\mathbf{a}\big)\Big]\,\mathrm{d}t \\
&= -\int_0^T \mathrm{Re}\Big(\frac{i}{\kappa}(\psi^*\nabla\psi - \Psi^*\nabla\Psi) + \mathbf{A}(|\psi|^2 - |\Psi|^2) + |\Psi|^2\mathbf{E}\,,\,\mathbf{a}\Big)\,\mathrm{d}t,
\end{aligned}
\tag{3.6}
$$

for any $\varphi \in L^2((0,T);\mathcal{H}^1)$ and $\mathbf{a} \in L^2((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))$. Let $1_{(0,t')}(t)$ denote the function of time which equals one when $t \in (0,t')$ while it equals zero when $t \notin (0,t')$. By choosing $\varphi(x,t) = e(x,t)1_{(0,t')}(t)$ and $\mathbf{a}(x,t) = \mathbf{E}(x,t)1_{(0,t')}(t)$, and using the regularity estimate

$$
\operatorname*{ess\,sup}_{t\in(0,T)} \big(\|\nabla\psi\|_{L^2} + \|\nabla\Psi\|_{L^2} + \|\mathbf{A}\|_{L^4} + \|\mathbf{\Lambda}\|_{L^4}\big) \le C,
$$

we obtain that

$$
\begin{aligned}
&\frac{\eta}{2}\|e(\cdot,t')\|_{L^2}^2 + \int_0^{t'} \Big(\frac{1}{\kappa^2}\|\nabla e\|_{L^2}^2 + \|\mathbf{A}e\|_{L^2}^2\Big)\,\mathrm{d}t \\
&\le \int_0^{t'} \Big(C\|\mathbf{A}\|_{L^4}\|\nabla e\|_{L^2}\|e\|_{L^4} + C\|\mathbf{E}\|_{L^4}\|\nabla\Psi\|_{L^2}\|e\|_{L^4} \\
&\qquad + C\|e\|_{L^4}\|\mathbf{\Lambda}\|_{L^4}\|\nabla e\|_{L^2} + C\|\mathbf{E}\|_{L^2}\|\nabla e\|_{L^2} \\
&\qquad + C(\|\mathbf{A}\|_{L^4} + \|\mathbf{\Lambda}\|_{L^4})\|\mathbf{E}\|_{L^2}\|e\|_{L^4} + C\|e\|_{L^2}^2 + C\|\nabla\cdot\mathbf{E}\|_{L^2}\|e\|_{L^2}\Big)\,\mathrm{d}t \\
&\le \int_0^{t'} \Big(C\|\nabla e\|_{L^2}(\epsilon^{-1}\|e\|_{L^2} + \epsilon\|\nabla e\|_{L^2}) + C\|\mathbf{E}\|_{\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div})}(\epsilon^{-1}\|e\|_{L^2} + \epsilon\|\nabla e\|_{L^2}) \\
&\qquad + C\|\nabla e\|_{L^2}(\epsilon^{-1}\|e\|_{L^2} + \epsilon\|\nabla e\|_{L^2}) + C\|\mathbf{E}\|_{L^2}\|\nabla e\|_{L^2} \\
&\qquad + C\|\mathbf{E}\|_{L^2}(\epsilon^{-1}\|e\|_{L^2} + \epsilon\|\nabla e\|_{L^2}) + C\|e\|_{L^2}^2 + C\|\nabla\cdot\mathbf{E}\|_{L^2}\|e\|_{L^2}\Big)\,\mathrm{d}t \\
&\le \int_0^{t'} \Big(\epsilon\|\nabla e\|_{L^2}^2 + \epsilon\|\nabla\times\mathbf{E}\|_{L^2}^2 + \epsilon\|\nabla\cdot\mathbf{E}\|_{L^2}^2 \\
&\qquad + (C + C\epsilon^{-3})\|e\|_{L^2}^2 + (C + C\epsilon^{-1})\|\mathbf{E}\|_{L^2}^2\Big)\,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{E}(\cdot,t')\|_{L^2}^2+\int_0^{t'}\Big(\|\nabla\times\mathbf{E}\|_{L^2}^2+\|\nabla\cdot\mathbf{E}\|_{L^2}^2\Big)\,\mathrm{d}t\\
&\leq\int_0^{t'}\Big(C\|e\|_{L^4}\|\nabla\psi\|_{L^2}\|\mathbf{E}\|_{L^4}+\|\nabla e\|_{L^2}\|\mathbf{E}\|_{L^2}+(\|e\|_{L^4}\|\mathbf{A}\|_{L^4}+\|\mathbf{E}\|_{L^2})\|\mathbf{E}\|_{L^2}\Big)\,\mathrm{d}t\\
&\leq\int_0^{t'}\Big(C(\epsilon^{-1}\|e\|_{L^2}+\epsilon\|\nabla e\|_{L^2})\|\mathbf{E}\|_{\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div})}+\|\nabla e\|_{L^2}\|\mathbf{E}\|_{L^2}\\
&\quad+(\|e\|_{L^2}+\|\nabla e\|_{L^2}+\|\mathbf{E}\|_{L^2})\|\mathbf{E}\|_{L^2}\Big)\,\mathrm{d}t\\
&\leq\int_0^{t'}\Big(\epsilon\|\nabla e\|_{L^2}^2+\epsilon\|\nabla\times\mathbf{E}\|_{L^2}+\epsilon\|\nabla\cdot\mathbf{E}\|_{L^2}\\
&\qquad\qquad+(C+C\epsilon^{-3})\|e\|_{L^2}^2+(C+C\epsilon^{-1})\|\mathbf{E}\|_{L^2}^2\Big)\,\mathrm{d}t,
\end{aligned}
$$

where ϵ is an arbitrary positive number. By choosing $\epsilon<\frac{1}{4}\min(1,\kappa^{-2})$ and summing up the last two inequalities, we obtain that

$$
\frac{\eta}{2}\|e(\cdot,t')\|_{L^2}^2+\frac{1}{2}\|\mathbf{E}(\cdot,t')\|_{L^2}^2\leq\int_0^{t'}\Big(C\|e\|_{L^2}^2+C\|\mathbf{E}\|_{L^2}^2\Big)\,\mathrm{d}t,
$$

which implies

$$
\max_{t\in(0,T)}\left(\frac{\eta}{2}\|e\|_{L^2}^2+\frac{1}{2}\|\mathbf{E}\|_{L^2}^2\right)=0
$$

via Gronwall's inequality. Uniqueness of the weak solution is proved.

3.4. **Equivalence of (1.16)-(1.19) and (1.21)-(1.27).** Let $(\psi,\mathbf{A})$ be the unique solution of (1.16)-(1.19) and, for the given ψ and $\mathbf{A}$, we let (p,q,u,v) be the solution of (1.22)-(1.25). Since $\mathrm{Re}\big[\psi^*\big(\frac{i}{\kappa}\nabla+\mathbf{A}\big)\psi\big]\in L^\infty((0,T);L^2)$, the standard regularity estimates of Poisson's equations yield that

$$
\begin{aligned}
&p,q\in L^\infty((0,T);H^1),\\
&u,v\in L^\infty((0,T);H^1)\cap L^2((0,T);H^{1+s}),\quad \partial_t u,\partial_t v,\Delta u,\Delta v\in L^2((0,T);L^2).
\end{aligned}
$$

By setting $\widetilde{\mathbf{A}}=\nabla\times u+\nabla v$, we have $\widetilde{\mathbf{A}}\in L^\infty((0,T);\mathbf{L}^2)\cap L^2((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))$ and $\partial_t\widetilde{\mathbf{A}}\in L^2((0,T);(\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))')$. Since $\mathrm{Re}\big[\psi^*\big(\frac{i}{\kappa}\nabla+\mathbf{A}\big)\psi\big]=\nabla\times p+\nabla q$, the integration of (1.24) against $\nabla\times\mathbf{a}$ minus the integration of (1.25) against $\nabla\cdot\mathbf{a}$ gives

$$
\begin{aligned}
&\int_0^T\left[\left(\frac{\partial\widetilde{\mathbf{A}}}{\partial t},\mathbf{a}\right)+\big(\nabla\times\widetilde{\mathbf{A}},\nabla\times\mathbf{a}\big)+\big(\nabla\cdot\widetilde{\mathbf{A}},\nabla\cdot\mathbf{a}\big)\right]\mathrm{d}t\\
&=\int_0^T(f,\nabla\times\mathbf{a})\,\mathrm{d}t-\int_0^T\left(\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right],\mathbf{a}\right)\mathrm{d}t
\end{aligned}
$$

for any $\mathbf{a}\in L^2((0,T);\mathbf{H}_{\mathrm{n}}(\mathrm{curl},\mathrm{div}))$, with $\widetilde{\mathbf{A}}_0=\mathbf{A}_0$. Comparing the above equation with (2.2), we derive that $\widetilde{\mathbf{A}}=\mathbf{A}$. Thus $\Delta u=\nabla\times\mathbf{A}\in L^\infty((0,T);L^2)\cap L^2((0,T);H^1)$ and $\Delta v=\nabla\cdot\mathbf{A}\in L^\infty((0,T);L^2)\cap L^2((0,T);H^1)$, and from (1.24)-(1.25) we further derive that $\partial_t u,\partial_t v\in L^\infty((0,T);L^2)\cap L^2((0,T);H^1)$.

Overall, (1.21)-(1.27) has a solution (ψ,p,q,u,v) which possesses the regularity specified in Definition 2.2, satisfying (2.3)-(2.7) with $\mathbf{A}=\nabla\times u+\nabla v$, where

$(\psi, \mathbf{A})$ coincides with the unique solution of (1.16)-(1.19). Based on the regularity of ψ, p, q, u and v, uniqueness of the solution for (1.21)-(1.27) can be proved in a similar way as Section 3.3. We omit the proof due to the limitation on pages.

3.5. **Singularity of the solution.** From the analysis in the last two subsections we see that

$$\begin{cases} -\Delta u = \nabla \times \mathbf{A} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta v = \nabla \cdot \mathbf{A} & \text{in } \Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nabla \times \mathbf{A}, \nabla \cdot \mathbf{A} \in L^\infty((0,T); L^2)$. For each fixed t, the solutions of the two Poisson equations have the decomposition [29]

$$u(x,t) = \sum_{j=1}^m \beta_j(t)\Phi(|x-x_j|)|x-x_j|^{\pi/\omega_j}\sin(\pi\Theta_j(x)/\omega_j) + \widetilde{u}(x,t),$$

$$v(x,t) = \sum_{j=1}^m \gamma_j(t)\Phi(|x-x_j|)|x-x_j|^{\pi/\omega_j}\cos(\pi\Theta_j(x)/\omega_j) + \widetilde{v}(x,t),$$

where

$$\sum_{j=1}^m |\beta_j(t)| + \sum_{j=1}^m |\gamma_j(t)| + \|\widetilde{u}(\cdot,t)\|_{H^2} + \|\widetilde{v}(\cdot,t)\|_{H^2} \le C\|\nabla\times\mathbf{A}\|_{L^2} + C\|\nabla\cdot\mathbf{A}\|_{L^2}.$$

Thus

$$\begin{aligned} &\|\beta_j\|_{L^\infty(0,T)} + \|\gamma_j\|_{L^\infty(0,T)} + \|\widetilde{u}\|_{L^\infty((0,T);H^2)} + \|\widetilde{v}\|_{L^\infty((0,T);H^2)} \\ &\le C(\|\nabla\times\mathbf{A}\|_{L^\infty((0,T);L^2)} + \|\nabla\cdot\mathbf{A}\|_{L^\infty((0,T);L^2)}) \le C. \end{aligned}$$

The singular part of ψ can be derived in a similar way.

The proof of Theorem 2.1 is completed.

4. Proof of Theorem 2.2

In this section, we prove further regularity of the solution under some compatibility conditions. We need the following lemma concerning the maximal L^p regularity of parabolic equations in a Lipscthiz domain [38].

Lemma 4.1. *The solution of the equation*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x,0) = 0 & \text{for } x \in \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t v - \Delta v = g & \text{in } \Omega, \\ \partial_n v = 0 & \text{on } \partial\Omega, \\ v(x,0) = 0 & \text{for } x \in \Omega, \end{cases}$$

satisfy that

$$\begin{aligned} &\|\partial_t u\|_{L^p((0,T);L^2)} + \|\Delta u\|_{L^p((0,T);L^2)} \le C_p\|f\|_{L^p((0,T);L^2)}, \\ &\|\partial_t v\|_{L^p((0,T);L^2)} + \|\Delta v\|_{L^p((0,T);L^2)} \le C_p\|g\|_{L^p((0,T);L^2)}, \end{aligned}$$

for any $1 < p < \infty$.

Rewriting (1.16) as

$$\eta\frac{\partial(\psi-\psi_0)}{\partial t} - \frac{1}{\kappa^2}\Delta(\psi-\psi_0) = -g,$$

with

$$g = \frac{i}{\kappa}\nabla\cdot(\mathbf{A}\psi) + \frac{i}{\kappa}\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi + (|\psi|^2-1)\psi - i\eta\kappa\psi\nabla\cdot\mathbf{A} - \frac{1}{\kappa^2}\Delta\psi_0,$$

and applying Lemma 4.1 (here we need the compatibility condition $\partial_n \psi_0 = 0$ on $\partial\Omega$), we derive that, for any given $1 < p < \infty$,

$$
\begin{aligned}
&\|\partial_t(\psi-\psi_0)\|_{L^p((0,T);L^2)} + \|\Delta(\psi-\psi_0)\|_{L^p((0,T);L^2)} \\
&\le C\|g\|_{L^p((0,T);L^2)} \\
&\le C\|\nabla\cdot\mathbf{A}\|_{L^p((0,T);L^2)} + C\|\mathbf{A}\|_{L^\infty((0,T);L^4)}\|\nabla\psi\|_{L^p((0,T);L^4)} + C\|\mathbf{A}\|^2_{L^{2p}((0,T);L^4)} + C \\
&\le C + C\|\nabla\psi\|^{(1-4/p_s)/(2-4/p_s)}_{L^p((0,T);L^2)}\|\nabla\psi\|^{1/(2-4/p_s)}_{L^p((0,T);L^{p_s})} + C \\
&\le C\|\nabla\psi\|^{1/(2-4/p_s)}_{L^p((0,T);L^{p_s})} + C \\
&\le C\|\Delta\psi\|^{1/(2-4/p_s)}_{L^p((0,T);L^2)} + C,
\end{aligned}
$$

which implies that $\|\partial_t\psi\|_{L^p((0,T);L^2)} + \|\Delta\psi\|_{L^p((0,T);L^2)} \le C$. In other words, we have

$$
\psi \in \bigcap_{p>1} W^{1,p}((0,T);L^2) \cap L^p((0,T);H^{1+s}) \hookrightarrow L^\infty((0,T);W^{1,4}). \tag{4.1}
$$

Let $\overline{w} = \nabla\cdot\mathbf{A}$ and consider the divergence of (1.17), i.e.,

$$
\frac{\partial\overline{w}}{\partial t} - \Delta\overline{w} = -\mathrm{Re}\left[\nabla\psi^*\cdot\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right) + \psi^*\left(\frac{i}{\kappa}\Delta\psi + \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi\right)\right],
$$

with the boundary condition $\partial_n w = 0$ on $\partial\Omega$. The standard energy estimates of the above equation give

$$
\begin{aligned}
&\|\partial_t\overline{w}\|_{L^2((0,T);L^2)} + \|\Delta\overline{w}\|_{L^2((0,T);L^2)} \\
&\le C\|\overline{w}_0\|_{H^1} + C\big\|\nabla\psi^*\cdot\big(i\kappa^{-1}\nabla\psi + \mathbf{A}\psi\big) + \psi^*\big(i\kappa^{-1}\Delta\psi + \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi\big)\big\|_{L^2((0,T);L^2)} \\
&\le C\|\nabla\cdot\mathbf{A}_0\|_{H^1} + C\|\nabla\psi^*\|_{L^4((0,T);L^4)}(\|\nabla\psi\|_{L^4((0,T);L^4)} + \|\mathbf{A}\|_{L^4((0,T);L^4)}) \\
&\quad + C(\|\Delta\psi\|_{L^2((0,T);L^2)} + \|\nabla\cdot\mathbf{A}\|_{L^2((0,T);L^2)} + \|\mathbf{A}\|_{L^4((0,T);L^4)}\|\nabla\psi\|_{L^4((0,T);L^4)}) \\
&\le C.
\end{aligned}
$$

If we let $w = \nabla\times\mathbf{A} - f$ and consider the curl of (1.17), in a similar way one can prove

$$
\|\partial_t w\|_{L^2((0,T);L^2)} + \|\Delta w\|_{L^2((0,T);L^2)} \le C.
$$

The last two inequalities imply that

$$
\partial_t\mathbf{A} \in L^2((0,T);\mathbf{H}_\mathrm{n}(\mathrm{curl},\mathrm{div})) \hookrightarrow L^2((0,T);\mathbf{L}^4). \tag{4.2}
$$

Consider the time derivative of (1.16) and denote $\dot\psi = \partial_t\psi$. We have

$$
\eta\frac{\partial\dot\psi}{\partial t} - \frac{1}{\kappa^2}\Delta\dot\psi = -\dot g,
$$

with the boundary condition $\partial_n\dot\psi = 0$ on $\partial\Omega$, where

$$
\begin{aligned}
\dot g &= \big(i\kappa^{-1} - i\eta\kappa\big)\dot\psi\nabla\cdot\mathbf{A} + \big(i\kappa^{-1} - i\eta\kappa\big)\psi\nabla\cdot\dot{\mathbf{A}} + 2i\kappa^{-1}\dot{\mathbf{A}}\cdot\nabla\psi + 2i\kappa^{-1}\mathbf{A}\cdot\nabla\dot\psi \\
&\quad + 2\mathbf{A}\cdot\dot{\mathbf{A}}\psi + |\mathbf{A}|^2\dot\psi + (\dot\psi\psi^* + \psi\dot\psi^*)\psi + (|\psi|^2 - 1)\dot\psi.
\end{aligned}
$$

The energy estimates of the equation give that

$$
\begin{aligned}
&\|\partial_t\dot{\psi}\|_{L^2((0,T);L^2)}+\|\Delta\dot{\psi}\|_{L^2((0,T);L^2)}+\|\nabla\dot{\psi}\|_{L^\infty((0,T);L^2)}\\
&\le C\|g\|_{L^2((0,T);L^2)}\\
&\le C\|\dot{\psi}\|_{L^\infty((0,T);L^2)}\|\nabla\cdot\mathbf{A}\|_{L^2((0,T);L^\infty)}+C\|\nabla\cdot\dot{\mathbf{A}}\|_{L^2((0,T);L^2)}\\
&\quad+C\|\dot{\mathbf{A}}\|_{L^2((0,T);L^4)}\|\nabla\psi\|_{L^\infty((0,T);L^4)}+C\|\mathbf{A}\|_{L^\infty((0,T);L^4)}\|\nabla\dot{\psi}\|_{L^2((0,T);L^4)}\\
&\quad+C\|\mathbf{A}\|_{L^\infty((0,T);L^4)}\|\dot{\mathbf{A}}\|_{L^2((0,T);L^4)}+C\|\mathbf{A}\|^2_{L^\infty((0,T);L^4)}\|\dot{\psi}\|_{L^2((0,T);L^\infty)}\\
&\quad+C\|\dot{\psi}\|_{L^2((0,T);L^2)}\\
&\le C\|\dot{\psi}\|^{1/2}_{L^2((0,T);L^2)}\|\partial_t\dot{\psi}\|^{1/2}_{L^2((0,T);L^2)}+C+C+C\|\nabla\dot{\psi}\|_{L^2((0,T);L^4)}\\
&\quad+C+C\|\nabla\dot{\psi}\|_{L^2((0,T);L^4)}\\
&\le C\|\partial_t\dot{\psi}\|^{1/2}_{L^2((0,T);L^2)}+C\|\nabla\dot{\psi}\|^{\frac{1-4/p_s}{2-4/p_s}}_{L^2((0,T);L^2)}\|\nabla\dot{\psi}\|^{\frac{1}{2-4/p_s}}_{L^2((0,T);L^{p_s})}+C\\
&\le C\|\partial_t\dot{\psi}\|^{1/2}_{L^2((0,T);L^2)}+C\|\dot{\psi}\|^{\frac{(1-4/p_s)s}{(2-4/p_s)(1+s)}}_{L^2((0,T);L^2)}\|\dot{\psi}\|^{\frac{1-4/p_s}{(2-4/p_s)(1+s)}}_{L^2((0,T);H^{1+s})}\|\nabla\dot{\psi}\|^{\frac{1}{2-4/p_s}}_{L^2((0,T);L^{p_s})}+C\\
&\le C\|\partial_t\dot{\psi}\|^{1/2}_{L^2((0,T);L^2)}+C\|\Delta\dot{\psi}\|^{1-\frac{(1-4/p_s)s}{(2-4/p_s)(1+s)}}_{L^2((0,T);L^2)}+C,
\end{aligned}
$$

which reduces to

$$\|\partial_t\dot{\psi}\|_{L^2((0,T);L^2)}+\|\Delta\dot{\psi}\|_{L^2((0,T);L^2)}+\|\nabla\dot{\psi}\|_{L^\infty((0,T);L^2)}\le C.$$

In other words, we have

$$\|\partial_{tt}\psi\|_{L^2((0,T);L^2)}+\|\partial_t\psi\|_{L^2((0,T);H^{1+s})}+\|\partial_t\psi\|_{L^\infty((0,T);H^1)}\le C. \tag{4.3}$$

Now we consider the time derivative of (1.22)-(1.25), i.e.,

$$\Delta\dot{p}=-\nabla\times\mathrm{Re}\big[\psi^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\psi\big]^{\boldsymbol{\cdot}}, \tag{4.4}$$

$$\Delta\dot{q}=\nabla\cdot\mathrm{Re}\big[\psi^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\psi\big]^{\boldsymbol{\cdot}}, \tag{4.5}$$

$$\frac{\partial\dot{u}}{\partial t}-\Delta\dot{u}=\dot{f}-\dot{p}, \tag{4.6}$$

$$\frac{\partial\dot{v}}{\partial t}-\Delta\dot{v}=-\dot{q}, \tag{4.7}$$

with the boundary conditions $\dot{p}=0$, $\partial_n\dot{q}=0$, $\dot{u}=0$ and $\partial_n\dot{v}=0$ on $\partial\Omega$. In particular, the boundary condition $\dot{u}=0$ on $\partial\Omega$ at the time $t=0$ requires the compatibility condition $\nabla\times\mathbf{A}_0=f_0$ on $\partial\Omega$. Since

$$
\begin{aligned}
&\big\|\big[\psi^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\psi\big]^{\boldsymbol{\cdot}}\big\|_{L^2((0,T);L^2)}\\
&=\big\|\dot{\psi}^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\psi+\psi^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\dot{\psi}+|\psi|^2\dot{\mathbf{A}}\big\|_{L^2((0,T);L^2)}\\
&\le\|\dot{\psi}^*\|_{L^2((0,T);L^\infty)}\|i\kappa^{-1}\nabla\psi+\mathbf{A}\psi\|_{L^\infty((0,T);L^2)}\\
&\quad+\|i\kappa^{-1}\nabla\dot{\psi}\|_{L^2((0,T);L^2)}+\|\mathbf{A}\|_{L^\infty((0,T);L^2)}\|\dot{\psi}\|_{L^2((0,T);L^\infty)}+\|\dot{\mathbf{A}}\|_{L^2((0,T);L^2)}\\
&\le C,
\end{aligned}
$$

the energy estimates of (4.4)-(4.5) give

$$\|\nabla\dot{p}\|_{L^2((0,T);L^2)}+\|\nabla\dot{q}\|_{L^2((0,T);L^2)}\le\big\|\big[\psi^*\big(i\kappa^{-1}\nabla+\mathbf{A}\big)\psi\big]^{\boldsymbol{\cdot}}\big\|_{L^2((0,T);L^2)}\le C,$$

and then the energy estimates of (4.6)-(4.7) give

$$\|\partial_t \dot u\|_{L^2((0,T);L^2)}+\|\Delta \dot u\|_{L^2((0,T);L^2)}+\|\nabla \dot u\|_{L^\infty((0,T);L^2)}\le C\|\dot f-\dot p\|_{L^2((0,T);L^2)}\le C,$$
$$\|\partial_t \dot v\|_{L^2((0,T);L^2)}+\|\Delta \dot v\|_{L^2((0,T);L^2)}+\|\nabla \dot v\|_{L^\infty((0,T);L^2)}\le C\|\dot q\|_{L^2((0,T);L^2)}\le C,$$

which further imply that $\partial_t u, \partial_t v \in L^2((0,T);H^{1+s})$.

The proof of Theorem 2.2 is completed.

5. Proof of Theorem 2.3

The proof consists of two parts. In the first part, we prove the boundedness of the finite element solution and the invertibility of the linear systems, which are independent of the regularity of the exact solution. In the second part, we present error estimates of the finite element solution based on a mathematical induction on the L^4 norm of $\mathbf{A}_h^n = \nabla \times u_h^n + \nabla v_h^n$, which is needed to control the nonlinear terms in the equations.

The following lemma of discrete Sobolev embedding will be used in this section.

Lemma 5.1. *Let $\theta_h \in \mathring{V}_h^1$, $\vartheta_h \in V_h^1$ with $\int_\Omega \vartheta_h(x)\,\mathrm{d}x = 0$. If we define $\Delta_h\theta_h \in \mathring{V}_h^1$ and $\Delta_h\vartheta_h \in V_h^1$ by*

$$(\Delta_h\theta_h,\varphi) = -(\nabla\theta_h,\nabla\varphi), \quad \forall \varphi\in \mathring{V}_h^1,$$
$$(\Delta_h\vartheta_h,\varphi) = -(\nabla\vartheta_h,\nabla\varphi), \quad \forall \varphi\in V_h^1,$$

then

$$\|\nabla\theta_h\|_{L^4}\le C\|\Delta_h\theta_h\|_{L^2} \qquad \text{and} \qquad \|\nabla\vartheta_h\|_{L^4}\le C\|\Delta_h\vartheta_h\|_{L^2}.$$

Proof. Let θ be the solution of the Poisson equation

$$\Delta\theta = \Delta_h\theta_h$$

with the Dirichlet boundary condition $\theta = 0$ on $\partial\Omega$. Then $(\nabla(\theta-\theta_h),\nabla\xi_h) = 0$ for any $\xi_h\in\mathring{V}_h^1$, which implies that, via the standard H^1-norm error estimate and Lemma 3.1,

$$\|\nabla(\theta-\theta_h)\|_{L^2}\le C\|\theta\|_{H^{1+s}}h^s\le C\|\Delta_h\theta_h\|_{L^2}h^s.$$

Since $s>1/2$, by applying the inverse inequality we obtain that

$$\|\nabla(I_h\theta-\theta_h)\|_{L^4}\le Ch^{-1/2}\|\nabla(I_h\theta-\theta_h)\|_{L^2}\le C\|\Delta_h\theta_h\|_{L^2}h^{s-1/2}\le C\|\Delta_h\theta_h\|_{L^2}.$$

Thus $\|\nabla\theta_h\|_{L^4}\le\|\nabla(I_h\theta-\theta_h)\|_{L^4}+\|\nabla I_h\theta\|_{L^4}\le C\|\Delta_h\theta_h\|_{L^2}$. The proof for ϑ_h is similar. ☐

5.1. **Stability of the finite element solution.** Substituting $\varphi=\psi_h^{n+1}$ into (2.16) and considering the real part, we derive that

$$D_\tau\left(\frac{\eta}{2}\|\psi_h^{n+1}\|_{L^2}^2\right)+\|(i\kappa^{-1}\nabla+\mathbf{A}_h^n)\psi_h^{n+1}\|_{L^2}^2+\int_\Omega|\psi_h^n|^2|\psi_h^{n+1}|^2\,\mathrm{d}x=\|\psi_h^{n+1}\|_{L^2}^2,$$

which together with the discrete Gronwall's inequality implies that, when $\tau < \eta/4$,

$$\max_{0\le n\le N-1} \|\psi_h^{n+1}\|_{L^2}^2 + \sum_{n=0}^{N-1} \tau\|(i\kappa^{-1}\nabla + \mathbf{A}_h^n)\psi_h^{n+1}\|_{L^2}^2 \le C. \tag{5.1}$$

Since $|\chi(\psi_h^n)| \le 1$, by substituting $\xi = p_h^{n+1}$ into (2.17) and substituting $\zeta = q_h^{n+1}$ into (2.18), we obtain

$$\|\nabla p_h^{n+1}\|_{L^2} + \|\nabla q_h^{n+1}\|_{L^2} \le C\|(i\kappa^{-1}\nabla + \mathbf{A}_h^n)\psi_h^{n+1}\|_{L^2},$$

which together with (5.1) gives

$$\sum_{n=0}^{N-1} \tau\|\nabla p_h^{n+1}\|_{L^2} + \sum_{n=0}^{N-1} \tau\|\nabla q_h^{n+1}\|_{L^2} \le C. \tag{5.2}$$

Then, substituting $\theta = D_\tau u_h^{n+1}$ into (2.19) and $\vartheta = D_\tau v_h^{n+1}$ into (2.20), we derive that

$$\begin{aligned}&\sum_{n=0}^{N-1} \tau\big(\|D_\tau u_h^{n+1}\|_{L^2}^2 + \|D_\tau v_h^{n+1}\|_{L^2}^2\big) + \max_{0\le n\le N-1} \big(\|\nabla u_h^{n+1}\|_{L^2}^2 + \|\nabla v_h^{n+1}\|_{L^2}^2\big)\\ &\le C\sum_{n=0}^{N-1} \tau\|f^{n+1}\|_{L^2} + C\sum_{n=0}^{N-1} \tau\|p_h^{n+1}\|_{L^2} + C\sum_{n=0}^{N-1} \tau\|q_h^{n+1}\|_{L^2} \le C.\end{aligned} \tag{5.3}$$

From the above derivations it is not difficult to see that the linear systems defined by (2.16)-(2.20) are invertible when $\tau < \eta/4$, and the discrete solution $(\psi_h^n, p_h^n, q_h^n, u_h^n, v_h^n)$ solved from (2.16)-(2.20) is uniformly bounded in $L_\tau^\infty(\mathcal{L}^2) \times L_\tau^2(H^1) \times L_\tau^2(H^1) \times L_\tau^\infty(H^1) \times L_\tau^\infty(H^1)$ with respect to the time-step size τ and spatial mesh size h.

5.2. **Error estimates.** Note that the exact solution (ψ, p, q, u, v) satisfies the equations

$$\begin{aligned}&\big(\eta D_\tau\psi^{n+1}, \varphi\big) + \big((i\kappa^{-1}\nabla + \mathbf{A}^n)\psi^{n+1}, (i\kappa^{-1}\nabla + \mathbf{A}^n)\varphi\big)\\ &\quad + \big((|\psi^n|^2 - 1)\psi^{n+1}, \varphi\big) + \big(i\eta\kappa\mathbf{A}^n, \nabla((\psi^{n+1})^*\varphi)\big) = \big(E_\psi^{n+1}, \varphi\big),\end{aligned} \tag{5.4}$$

$$\begin{aligned}&(\nabla p^{n+1}, \nabla\xi)\\ &= \big(\mathrm{Re}[\chi(\psi^n)^*(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})], \nabla\times\xi\big) + \big(E_p^{n+1}, \nabla\times\xi\big),\end{aligned} \tag{5.5}$$

$$(\nabla q^{n+1}, \nabla\zeta) = \big(\mathrm{Re}[\chi(\psi^n)^*(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})], \nabla\zeta\big) + \big(E_q^{n+1}, \nabla\zeta\big), \tag{5.6}$$

$$\big(D_\tau u^{n+1}, \theta\big) + \big(\nabla u^{n+1}, \nabla\theta\big) = (f^{n+1} - p^{n+1}, \theta) + \big(E_u^{n+1}, \theta\big), \tag{5.7}$$

$$\big(D_\tau v^{n+1}, \vartheta\big) + \big(\nabla v^{n+1}, \nabla\vartheta\big) = (-q^{n+1}, \vartheta) + \big(E_v^{n+1}, \vartheta\big), \tag{5.8}$$

for all $\varphi \in \mathcal{V}_h^1$, $\xi, \theta \in \mathring{V}_h^1$ and $\zeta, \vartheta \in V_h^1$, with

$$(\nabla u^0, \nabla \xi) = \big(\mathbf{A}_0, \nabla \times \xi\big), \quad \forall\, \xi \in \mathring{V}_h^1, \tag{5.9}$$

$$(\nabla v^0, \nabla \zeta) = \big(\mathbf{A}_0, \nabla \cdot \zeta\big), \quad \forall\, \zeta \in V_h^1, \tag{5.10}$$

where

$$\begin{aligned}
E_\psi^{n+1} &= \eta(D_\tau \psi^{n+1} - \partial_t \psi^{n+1}) + \frac{i}{\kappa}\nabla \cdot ((\mathbf{A}^n - \mathbf{A}^{n+1})\psi^{n+1}) \\
&\quad + \frac{i}{\kappa}(\mathbf{A}^n - \mathbf{A}^{n+1}) \cdot \nabla \psi^{n+1} \\
&\quad + (|\mathbf{A}^n|^2 - |\mathbf{A}^{n+1}|^2)\psi^{n+1} + (|\psi^n|^2 - |\psi^{n+1}|^2)\psi^{n+1} \\
&\quad + i\eta\kappa\psi^{n+1}\nabla \cdot (\mathbf{A}^{n+1} - \mathbf{A}^n), \\
E_p^{n+1} &= \mathrm{Re}[i\kappa^{-1}(\psi^{n+1} - \psi^n)^* \nabla \psi^{n+1} + ((\psi^{n+1})^* \mathbf{A}^{n+1} - (\psi^n)^* \mathbf{A}^n)\psi^{n+1})], \\
E_q^{n+1} &= \mathrm{Re}[i\kappa^{-1}(\psi^{n+1} - \psi^n)^* \nabla \psi^{n+1} + ((\psi^{n+1})^* \mathbf{A}^{n+1} - (\psi^n)^* \mathbf{A}^n)\psi^{n+1})], \\
E_u^{n+1} &= D_\tau u^{n+1} - \partial_t u^{n+1}, \\
E_v^{n+1} &= D_\tau v^{n+1} - \partial_t v^{n+1},
\end{aligned}$$

are truncation errors due to the time discretization, which satisfy that

$$\sum_{n=0}^{N-1} \tau\big(\|E_\psi^{n+1}\|_{L^2}^2 + \|E_p^{n+1}\|_{L^2}^2 + \|E_q^{n+1}\|_{L^2}^2 + \|E_u^{n+1}\|_{L^2}^2 + \|E_v^{n+1}\|_{L^2}^2\big) \le C\tau^2.$$

Let $R_h : \mathcal{H}^1 \to \mathcal{V}_h^1$ and $\mathring{R}_h : \mathring{H}^1 \to \mathring{V}_h^1$ denote the Ritz projection operator onto the finite element spaces, i.e.,

$$\begin{aligned}
(\nabla(\phi - R_h\phi), \nabla\varphi) &= 0 \quad \forall\, \phi \in \mathcal{H}^1 \text{ and } \varphi \in \mathcal{V}_h^1, \\
(\nabla(\phi - \mathring{R}_h\phi), \nabla\varphi) &= 0 \quad \forall\, \phi \in \mathring{H}^1 \text{ and } \varphi \in \mathring{V}_h^1.
\end{aligned}$$

Then R_h, restricted to H^1, is just the Ritz projection from H^1 onto V_h^1, and we have [7, 8]

$$\begin{aligned}
&\|\phi - R_h\phi\|_{\mathcal{L}^2} + h^s\|\nabla(\phi - R_h\phi)\|_{\mathcal{L}^2} \le Ch^{2s}\|\phi\|_{\mathcal{H}^{1+s}}, && \forall\, \phi \in \mathcal{H}^{1+s}, \\
&\|\phi - \mathring{R}_h\phi\|_{L^2} + h^s\|\nabla(\phi - \mathring{R}_h\phi)\|_{L^2} \le Ch^{2s}\|\phi\|_{H^{1+s}}, && \forall\, \phi \in \mathring{H}^1 \cap H^{1+s}.
\end{aligned}$$

Let $e_{\psi,h}^{n+1} = \psi_h^{n+1} - R_h\psi^{n+1}$, $e_{p,h}^{n+1} = p_h^{n+1} - \mathring{R}_h p^{n+1}$, $e_{q,h}^{n+1} = q_h^{n+1} - R_h q^{n+1}$, $e_{u,h}^{n+1} = u_h^{n+1} - \mathring{R}_h u^{n+1}$, $e_{v,h}^{n+1} = v_h^{n+1} - R_h v^{n+1}$. The difference between (2.16)-(2.22) and (5.4)-(5.10) gives that $u_h^0 = \mathring{R}_h u^0$, $v_h^0 = R_h v^0$ and

$$
\begin{aligned}
&\big(\eta D_\tau e_{\psi,h}^{n+1},\varphi\big) + \kappa^{-2}\big(\nabla e_{\psi,h}^{n+1},\nabla\varphi\big)\\
&= \big(\eta D_\tau(\psi^{n+1} - R_h\psi^{n+1}),\varphi\big) - \big(E_\psi^{n+1},\varphi\big) - \frac{i}{\kappa}\big(\mathbf{A}_h^n\cdot\nabla e_{\psi,h}^{n+1},\varphi\big)\\
&\quad + \frac{i}{\kappa}\big(\mathbf{A}_h^n\cdot\nabla(\psi^{n+1} - R_h\psi^{n+1}),\varphi\big) - \frac{i}{\kappa}\big((\mathbf{A}_h^n - \mathbf{A}^n)\cdot\nabla\psi^{n+1},\varphi\big)\\
&\quad + \frac{i}{\kappa}\big(e_{\psi,h}^{n+1}\mathbf{A}_h^n,\nabla\varphi\big) - \frac{i}{\kappa}\big((\psi^{n+1} - R_h\psi^{n+1})\mathbf{A}_h^n,\nabla\varphi\big)\\
&\quad + \frac{i}{\kappa}\big(\psi^{n+1}(\mathbf{A}_h^n - \mathbf{A}^n),\nabla\varphi\big)\\
&\quad - \big((|\mathbf{A}_h^n|^2 - |\mathbf{A}^n|^2)\psi^{n+1},\varphi\big) - \big(|\mathbf{A}_h^n|^2 e_{\psi,h}^{n+1},\varphi\big)\\
&\quad - \big(|\mathbf{A}_h^n|^2(\psi^{n+1} - R_h\psi^{n+1}),\varphi\big)\\
&\quad - \big((|\psi_h^n|^2 - 1)\psi_h^{n+1} - (|\psi^n|^2 - 1)\psi^{n+1},\varphi\big) - \big(i\eta\kappa\mathbf{A}_h^n,\nabla((e_{\psi,h}^{n+1})^*\varphi)\big)\\
&\quad + \big(i\eta\kappa\mathbf{A}_h^n,\nabla((\psi^{n+1} - R_h\psi^{n+1})^*\varphi)\big) - \big(i\eta\kappa(\mathbf{A}_h^n - \mathbf{A}^n),\nabla((\psi^{n+1})^*\varphi)\big),
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
&(\nabla e_{p,h}^{n+1},\nabla\xi)\\
&= -\big(E_p^{n+1},\nabla\times\xi\big)\\
&\quad + \big(\mathrm{Re}[((\chi(\psi_h^n)^* - \chi(R_h\psi^n)^*)(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})],\nabla\times\xi\big)\\
&\quad - \big(\mathrm{Re}[(\chi(\psi^n)^* - \chi(R_h\psi^n)^*)(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})],\nabla\times\xi\big)\\
&\quad + \big(\mathrm{Re}[\chi(\psi_h^n)^*(i\kappa^{-1}\nabla e_{\psi,h}^{n+1} + (\mathbf{A}_h^n - \mathbf{A}^n)\psi^{n+1} + \mathbf{A}_h^n e_{\psi,h}^{n+1})],\nabla\times\xi\big)\\
&\quad - \big(\mathrm{Re}[\chi(\psi_h^n)^*(i\kappa^{-1}\nabla + \mathbf{A}_h^n)(\psi^{n+1} - R_h\psi^{n+1})],\nabla\times\xi\big),
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
&(\nabla e_{q,h}^{n+1},\nabla\zeta)\\
&= -\big(E_q^{n+1},\nabla\zeta\big)\\
&\quad + \big(\mathrm{Re}[((\chi(\psi_h^n)^* - \chi(R_h\psi^n)^*)(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})],\nabla\zeta\big)\\
&\quad - \big(\mathrm{Re}[(\chi(\psi^n)^* - \chi(R_h\psi^n)^*)(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})],\nabla\zeta\big)\\
&\quad + \big(\mathrm{Re}[\chi(\psi_h^n)^*(i\kappa^{-1}\nabla e_{\psi,h}^{n+1} + (\mathbf{A}_h^n - \mathbf{A}^n)\psi^{n+1} + \mathbf{A}_h^n e_{\psi,h}^{n+1}],\nabla\zeta\big)\\
&\quad - \big(\mathrm{Re}[\chi(\psi_h^n)^*(i\kappa^{-1}\nabla(\psi^{n+1} - R_h\psi^{n+1}) + \mathbf{A}_h^n(\psi^{n+1} - R_h\psi^{n+1})],\nabla\zeta\big),
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
&\big(D_\tau e_{u,h}^{n+1},\theta\big) + \big(\nabla e_{u,h}^{n+1},\nabla\theta\big)\\
&= \big(D_\tau(u^{n+1} - \mathring{R}_h u^{n+1}),\theta\big) + \big(p^{n+1} - p_h^{n+1},\theta\big) - \big(E_u^{n+1},\theta\big),
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&\big(D_\tau e_{v,h}^{n+1},\vartheta\big) + \big(\nabla e_{v,h}^{n+1},\nabla\vartheta\big)\\
&= \big(D_\tau(v^{n+1} - R_h v^{n+1}),\vartheta\big) + (q^{n+1} - q_h^{n+1},\vartheta) - \big(E_v^{n+1},\vartheta\big),
\end{aligned}
\tag{5.15}
$$

for all $\varphi \in \mathcal{V}_h^1$, $\xi,\theta \in \mathring{V}_h^1$ and $\zeta,\vartheta \in V_h^1$, with $\|e_{\psi,h}^0\|_{\mathcal{L}^2} \le Ch^{2s}$, $\|e_{\psi,h}^0\|_{\mathcal{H}^1} \le Ch^s$ and $e_{u,h}^0 = e_{v,h}^0 = 0$.

Substituting $\theta = D_\tau e_{u,h}^{n+1}$ and $\vartheta = D_\tau e_{v,h}^{n+1}$ into (5.14)-(5.15), we get

$$\begin{aligned}
&\|D_\tau e_{u,h}^{n+1}\|_{L^2}^2 + \|\Delta_h e_{u,h}^{n+1}\|_{L^2}^2 + D_\tau\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 \\
&\leq C\|D_\tau(u^{n+1} - \mathring{R}_h u^{n+1})\|_{L^2}^2 + C\|p^{n+1} - p_h^{n+1}\|_{L^2}^2 + C\|E_u^{n+1}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
&\|D_\tau e_{v,h}^{n+1}\|_{L^2}^2 + \|\Delta_h e_{v,h}^{n+1}\|_{L^2}^2 + D_\tau\|\nabla e_{v,h}^{n+1}\|_{L^2}^2 \\
&\leq C\|D_\tau(v^{n+1} - R_h v^{n+1})\|_{L^2}^2 + C\|q^{n+1} - q_h^{n+1}\|_{L^2}^2 + C\|E_v^{n+1}\|_{L^2}^2,
\end{aligned}$$

where $\Delta_h e_{u,h}^{n+1}$ and $\Delta_h e_{v,h}^{n+1}$ are defined in Lemma 5.1. By Lemma 5.1, the last two inequalities imply that

$$\begin{aligned}
&C^{-1}\|e_{u,h}^{n+1}\|_{W^{1,4}}^2 + D_\tau\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 \\
&\leq C\|p^{n+1} - p_h^{n+1}\|_{L^2}^2 + C\|D_\tau(u^{n+1} - \mathring{R}_h u^{n+1})\|_{L^2}^2 + C\|E_u^{n+1}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
&C^{-1}\|e_{v,h}^{n+1}\|_{W^{1,4}}^2 + D_\tau\|\nabla e_{v,h}^{n+1}\|_{L^2}^2 \\
&\leq C\|q^{n+1} - q_h^{n+1}\|_{L^2}^2 + C\|D_\tau(v^{n+1} - R_h v^{n+1})\|_{L^2}^2 + C\|E_v^{n+1}\|_{L^2}^2.
\end{aligned}$$

The sum of the last two inequalities gives

$$\begin{aligned}
&C^{-1}\|e_{\mathbf{A},h}^{n+1}\|_{L^4}^2 + D_\tau\big(\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \|\nabla e_{v,h}^{n+1}\|_{L^2}^2\big) \\
&\leq C\|p^{n+1} - p_h^{n+1}\|_{L^2}^2 + C\|q^{n+1} - q_h^{n+1}\|_{L^2}^2 + C\|E_u^{n+1}\|_{L^2}^2 + C\|E_v^{n+1}\|_{L^2}^2 \\
&\quad + C\|D_\tau(u^{n+1} - \mathring{R}_h u^{n+1})\|_{L^2}^2 + C\|D_\tau(v^{n+1} - R_h v^{n+1})\|_{L^2}^2.
\end{aligned} \tag{5.16}$$

At this moment, we invoke a mathematical induction on

$$\|\mathbf{A}_h^n\|_{L^4} \leq \max_{0\leq n\leq N}\|\mathbf{A}^n\|_{L^4} + 1. \tag{5.17}$$

Since

$$\begin{aligned}
\|\mathbf{A}_h^0 - \mathbf{A}^0\|_{L^4} &\leq \|\nabla\times(\mathring{R}_h u^0 - u^0)\|_{L^4} + \|\nabla(R_h v^0 - v^0)\|_{L^4} \\
&\leq Ch^{s-1/2}(\|u^0\|_{H^{1+s}} + \|v^0\|_{H^{1+s}}),
\end{aligned}$$

there exists a positive constant h_1 such that (5.17) holds for $n = 0$ when $h < h_1$. In the following, we present estimates of the finite element solution by assuming that (5.17) holds for $0 \leq n \leq m$, for some nonnegative integer m. We shall see that if (5.17) holds for $0 \leq n \leq m$, then it also holds for $n = m + 1$.

Substituting $\xi = e_{p,h}^{n+1}$ in (5.12), it is not difficult to derive that

$$\begin{aligned}
\|\nabla e_{p,h}^{n+1}\|_{L^2} &\leq C\|E_p^{n+1}\|_{L^2} + C\|e_{\psi,h}^n\|_{L^4}\|i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1}\|_{L^4} \\
&\quad + C\|\psi^n - R_h\psi^n\|_{L^4}\|(i\kappa^{-1}\nabla\psi^{n+1} + \mathbf{A}^n\psi^{n+1})\|_{L^4} \\
&\quad + C\big(\|\nabla e_{\psi,h}^{n+1}\|_{L^2} + \|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2} + \|\mathbf{A}_h^n\|_{L^4}\|e_{\psi,h}^{n+1}\|_{L^4} \\
&\quad + C\big(\|\nabla(\psi^{n+1} - R_h\psi^{n+1})\|_{L^2} + \|\mathbf{A}_h^n\|_{L^4}\|\psi^n - R_h\psi^n\|_{L^4}\big) \\
&\leq C\|E_p^{n+1}\|_{L^2} + C\|e_{\psi,h}^n\|_{H^1} + C\|\psi^{n+1} - R_h\psi^{n+1}\|_{H^1} + C\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}.
\end{aligned} \tag{5.18}$$

Similarly, by substituting $\zeta = e_{q,h}^{n+1}$ in (5.12), one can derive that

$$\|\nabla e_{q,h}^{n+1}\|_{L^2} \leq C\|E_q^{n+1}\|_{L^2} + C\|e_{\psi,h}^n\|_{H^1} + C\|\psi^n - R_h\psi^n\|_{H^1} + C\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}. \tag{5.19}$$

Substituting $\varphi = e_{\psi,h}^{n+1}$ in (5.11), we obtain that

$$
\begin{aligned}
& D_\tau\left(\frac{\eta}{2}\|e_{\psi,h}^{n+1}\|_{L^2}^2\right) + \kappa^{-2}\|\nabla e_{\psi,h}^{n+1}\|_{L^2}^2 \\
& \leq C\|D_\tau(\psi^{n+1} - R_h\psi^{n+1})\|_{L^2}^2 + C\|e_{\psi,h}^{n+1}\|_{L^2}^2 + C\|E_\psi^{n+1}\|_{L^2}^2 \\
& \quad + C\|\mathbf{A}_h^n\|_{L^4}\|\nabla e_{\psi,h}^{n+1}\|_{L^2}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + C\|\mathbf{A}_h^n\|_{L^4}\|\nabla(\psi_h^{n+1} - R_h\psi^{n+1})\|_{L^2}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + C\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}\|\nabla\psi^{n+1}\|_{L^4}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + C\|e_{\psi,h}^{n+1}\|_{L^4}\|\mathbf{A}_h^n\|_{L^4}\|\nabla e_{\psi,h}^{n+1}\|_{L^2} \\
& \quad + C\|\psi^{n+1} - R_h\psi^{n+1}\|_{L^4}\|\mathbf{A}_h^n\|_{L^4}\|\nabla e_{\psi,h}^{n+1}\|_{L^2} \\
& \quad + C\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}\|\nabla e_{\psi,h}^{n+1}\|_{L^2} \\
& \quad + C(\|\mathbf{A}_h^n\|_{L^4} + \|\mathbf{A}^n\|_{L^4})\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + C\|\mathbf{A}_h^n\|_{L^4}^2\|e_{\psi,h}^{n+1}\|_{L^4}^2 + C\|\mathbf{A}_h^n\|_{L^4}^2\|\psi^{n+1} - R_h\psi^{n+1}\|_{L^4}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + (C\|\psi_h^n\|_{L^4}^2 + C)\|e_{\psi,h}^{n+1}\|_{L^4}^2 + (C\|\psi_h^n\|_{L^4} + C)\|e_{\psi,h}^n\|_{L^2}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + C\|\mathbf{A}_h^n\|_{L^4}\|\nabla e_{\psi,h}^{n+1}\|_{L^2}\|e_{\psi,h}^{n+1}\|_{L^4} \\
& \quad + C\|\mathbf{A}_h^n\|_{L^4}\|\psi^{n+1} - R_h\psi^{n+1}\|_{H^1}\|e_{\psi,h}^{n+1}\|_{H^1} \\
& \quad + C\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}(\|e_{\psi,h}^{n+1}\|_{L^2} + \|\nabla e_{\psi,h}^{n+1}\|_{L^2}) \\
& \leq \epsilon\|\nabla e_{\psi,h}^{n+1}\|_{L^2}^2 + C_\epsilon\|e_{\psi,h}^{n+1}\|_{L^2}^2 + C_\epsilon\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}^2 \\
& \quad + C\big(\|E_\psi^{n+1}\|_{L^2}^2 + \|D_\tau(\psi^{n+1} - R_h\psi^{n+1})\|_{L^2}^2 + \|\psi_h^{n+1} - R_h\psi^{n+1}\|_{H^1}^2\big),
\end{aligned} \tag{5.20}
$$

for any small positive number $\epsilon \in (0,1)$. Substituting (5.18)-(5.19) into (5.16), then (5.16) times ε_1 plus (5.20) gives

$$
\begin{aligned}
& \varepsilon_1 C^{-1}\|e_{\mathbf{A},h}^{n+1}\|_{L^4}^2 + \kappa^{-2}\|\nabla e_{\psi,h}^{n+1}\|_{L^2}^2 \\
& + D_\tau\big(\varepsilon_1\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \varepsilon_1\|\nabla e_{v,h}^{n+1}\|_{L^2}^2 + \frac{\eta}{2}\|e_{\psi,h}^{n+1}\|_{L^2}^2\big) \\
& \leq C\|E_p^{n+1}\|_{L^2}^2 + C\|E_q^{n+1}\|_{L^2}^2 + C\|E_u^{n+1}\|_{L^2}^2 + C\|E_v^{n+1}\|_{L^2}^2 + C\|E_\psi^{n+1}\|_{L^2}^2 \\
& \quad + C\|\psi^{n+1} - R_h\psi^{n+1}\|_{H^1}^2 + C\|D_\tau(\psi^{n+1} - R_h\psi^{n+1})\|_{L^2}^2 \\
& \quad + C\|D_\tau(u^{n+1} - \mathring{R}_h u^{n+1})\|_{L^2}^2 + C\|D_\tau(v^{n+1} - R_h v^{n+1})\|_{L^2}^2 \\
& \quad + (C\varepsilon_1 + \epsilon)\|\nabla e_{\psi,h}^{n+1}\|_{L^2}^2 + (C\varepsilon_1 + C_\epsilon)\|e_{\psi,h}^{n+1}\|_{L^2}^2 + C_\epsilon\|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^2}^2.
\end{aligned} \tag{5.21}
$$

By choosing ε_1 and ϵ small enough, the term $(C\varepsilon_1+\epsilon)\|\nabla e_{\psi,h}^{n+1}\|_{L^2}^2$ on the right-hand side of the last inequality can be eliminated by the left-hand side. Since

$$
\begin{aligned}
& \|\mathbf{A}_h^n - \mathbf{A}^n\|_{L^4} \\
& \leq C\|\nabla e_{u,h}^n\|_{L^2} + C\|\nabla e_{v,h}^n\|_{L^2} + C\|\nabla\times(u^n - \mathring{R}_h u^n)\|_{L^4} + C\|\nabla(v^n - R_h v^n)\|_{L^4} \\
& \leq C\|\nabla e_{u,h}^n\|_{L^2} + C\|\nabla e_{v,h}^n\|_{L^2} + C(\|u^{n+1}\|_{H^{1+s}} + \|v^{n+1}\|_{H^{1+s}})h^s,
\end{aligned}
$$

the inequality (5.21) reduces to

$$
\begin{aligned}
&\frac{\varepsilon_1}{C}\|e_{\mathbf{A},h}^{n+1}\|_{L^4}^2 + \frac{1}{2\kappa^2}\|\nabla e_{\psi,h}^{n+1}\|_{L^2}^2 + D_\tau\big(\varepsilon_1\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \varepsilon_1\|\nabla e_{v,h}^{n+1}\|_{L^2}^2 + \frac{\eta}{2}\|e_{\psi,h}^{n+1}\|_{L^2}^2\big)\\
&\le C\|\nabla e_{u,h}^{n}\|_{L^2}^2 + C\|\nabla e_{v,h}^{n}\|_{L^2}^2 + C\|e_{\psi,h}^{n+1}\|_{L^2}^2\\
&\quad + C\|E_p^{n+1}\|_{L^2}^2 + C\|E_q^{n+1}\|_{L^2}^2 + C\|E_u^{n+1}\|_{L^2}^2 + C\|E_v^{n+1}\|_{L^2}^2 + C\|E_\psi^{n+1}\|_{L^2}^2\\
&\quad + C(\|\psi^{n+1}\|_{H^{1+s}}^2 + \|u^{n+1}\|_{H^{1+s}}^2 + \|v^{n+1}\|_{H^{1+s}}^2)h^{2s}\\
&\quad + C(\|D_\tau\psi^{n+1}\|_{H^{1+s}}^2 + \|D_\tau u^{n+1}\|_{H^{1+s}}^2 + \|D_\tau v^{n+1}\|_{H^{1+s}}^2)h^{4s}.
\end{aligned}
$$

By applying Gronwall's inequality, there exists a positive constant τ_1 such that when $\tau < \tau_1$ we have

$$
\begin{aligned}
&\max_{0\le n\le m}\big(\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \|\nabla e_{v,h}^{n+1}\|_{L^2}^2 + \|e_{\psi,h}^{n+1}\|_{L^2}^2\big) + \sum_{n=0}^{m}\tau\|e_{\mathbf{A},h}^{n+1}\|_{L^4}^2\\
&\le C_1(\tau^2 + h^{2s})
\end{aligned}
\tag{5.22}
$$

for some positive constant C_1. In particular, the last inequality implies that

$$
\max_{0\le n\le m}\|e_{\mathbf{A},h}^{n+1}\|_{L^2}^2 + \sum_{n=0}^{m}\tau\|e_{\mathbf{A},h}^{n+1}\|_{L^4}^2 \le C(\tau^2 + h^{2s}).
$$

If $\tau \ge h$, then we have

$$
\|e_{\mathbf{A},h}^{m+1}\|_{L^4}^2 \le \frac{1}{\tau}\sum_{n=0}^{m}\tau\|e_{\mathbf{A},h}^{n+1}\|_{L^4}^2 \le C(\tau + h^{2s}/\tau) \le C(\tau + h^{2s-1})\,;
$$

if $\tau \le h$, then we have

$$
\|e_{\mathbf{A},h}^{m+1}\|_{L^4}^2 \le Ch^{-1}\|e_{\mathbf{A},h}^{m+1}\|_{L^2}^2 \le C(\tau^2/h + h^{2s-1}) \le C(h + h^{2s-1})\,.
$$

Overall, we have $\|e_{\mathbf{A},h}^{m+1}\|_{L^4}^2 \le C(\tau + h + h^{2s-1})$ and so

$$
\begin{aligned}
&\|\mathbf{A}_h^{m+1} - \mathbf{A}^{m+1}\|_{L^4}\\
&\le \|e_{\mathbf{A},h}^{m+1}\|_{L^4} + \|\nabla\times(u^{m+1} - \mathring{R}_h u^{m+1})\|_{L^4} + \|\nabla(v^{m+1} - R_h v^{m+1})\|_{L^4}\\
&\le C(\tau^{1/2} + h^{1/2} + h^{s-1/2}).
\end{aligned}
$$

There exist positive constants τ_2 and h_2 such that when $\tau < \tau_2$ and $h < h_2$ we have

$$
\|\mathbf{A}_h^{m+1} - \mathbf{A}^{m+1}\|_{L^4} \le 1\,,
$$

and this completes the mathematical induction on (5.17) in the case that $\tau < \tau_2$ and $h < h_2$. Thus (5.22) holds for $m = N-1$ with the same constant C_1, provided $\tau < \tau_2$ and $h < h_2$.

If $\tau \ge \tau_2$ or $h \ge h_2$, from (5.1)-(5.3) we see that

$$
\begin{aligned}
&\max_{0\le n\le N-1}\big(\|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \|\nabla e_{v,h}^{n+1}\|_{L^2}^2 + \|e_{\psi,h}^{n+1}\|_{L^2}^2\big)\\
&\le C_2\\
&\le C_2\big(\tau_2^{-2} + h_2^{-2s}\big)(\tau^2 + h^{2s})
\end{aligned}
\tag{5.23}
$$

for some positive constant C_2. From (5.22) and (5.23) we see that for any τ and h we have

$$\begin{aligned}&\max_{0\le n\le N-1}\left(\|\nabla e_{u,h}^{n+1}\|_{L^2}^2+\|\nabla e_{v,h}^{n+1}\|_{L^2}^2+\|e_{\psi,h}^{n+1}\|_{L^2}^2\right)\\&\le\left[C_1+C_2\left(\tau_2^{-2}+h_2^{-2s}\right)\right](\tau^2+h^{2s}).\end{aligned}$$

The proof of Theorem 2.3 is completed.

6. Numerical example

We consider an artificial example, the equations

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi-i\eta\kappa\psi\nabla\cdot\mathbf{A}=g, \tag{6.1}$$

$$\frac{\partial\mathbf{A}}{\partial t}+\nabla\times(\nabla\times\mathbf{A})-\nabla(\nabla\cdot\mathbf{A})+\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=\mathbf{g}+\nabla\times f, \tag{6.2}$$

in an L-shape domain Ω whose longest side has unit length, centered at the origin, with $\eta=1$ and $k=10$. The functions $f=\nabla\times\mathbf{A}\in C^1([0,T];\mathbf{H}^2)$, $g\in C([0,T];L^2)$ and $\mathbf{g}\in C([0,T];\mathbf{L}^2)$ are chosen corresponding to the exact solution

$$\begin{aligned}&\psi=t^2\Phi(r)r^{2/3}\cos(2\theta/3),\\&\mathbf{A}=\begin{pmatrix}\left(4t^2\Phi(r)r^{-1/3}/3+t^2\Phi'(r)r^{2/3}\right)\cos(\theta/3)\\\left(4t^2\Phi(r)r^{-1/3}/3+t^2\Phi'(r)r^{2/3}\right)\sin(\theta/3)\end{pmatrix},\end{aligned}$$

where (r,θ) denotes the polar coordinates, the cut-off function $\Phi(r)$ is defined by

$$\Phi(r)=\begin{cases}0.1 & \text{if } r<0.1,\\ \Upsilon(r) & \text{if } 0.1\le r\le 0.4,\\ 0 & \text{if } r>0.4,\end{cases}$$

and $\Upsilon(r)$ is the unique seventh order polynomial satisfying the conditions $\Upsilon'(0.1)=\Upsilon''(0.1)=\Upsilon'''(0.1)=\Upsilon(0.4)=\Upsilon'(0.4)=\Upsilon''(0.4)=\Upsilon'''(0.4)=0$ and $\Upsilon(0.1)=0.1$. It is easy to check that the exact solution $(\psi,\mathbf{A})$ satisfies the boundary and initial conditions (1.18)-(1.19) with $\psi_0=0$ and $\mathbf{A}_0=(0,0)$.

The L-shape domain is triangulated quasi-uniformly, as shown in Figure 6.2, with M nodes per unit length on each side, and we denote by $h=1/M$ for simplicity.

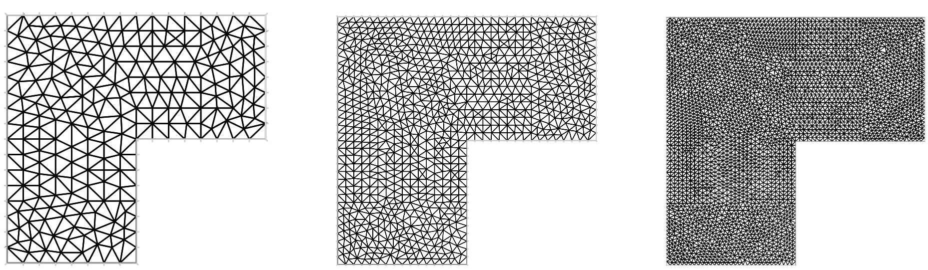

FIGURE 6.2. Quasi-uniform triangulations with $M=16,32,64$.

First, we solve (6.1)-(6.2) directly by the FEM with piecewise linear finite elements and a linearized backward Euler scheme, and we denote the numerical solution by $(\widetilde{\psi}_h^N, \widetilde{\mathbf{A}}_h^N)$. In a convex or smooth domain, convergence of the numerical solution $(\widetilde{\psi}_h^N, \widetilde{\mathbf{A}}_h^N)$ can be proved based on the method of [11, 18]. Here we are interested in the question of whether the numerical solution converges to the correct solution in a nonconvex polygonal domain. To answer this question, we present the errors of the numerical solution in Table 6.1 with $\tau = h$ for several different h. One can see that the errors do not decrease as the mesh is refined. In other words, the numerical solution $(\widetilde{\psi}_h^N, \widetilde{\mathbf{A}}_h^N)$ does not converge to the correct solution, nor does the physical quantity $|\widetilde{\psi}_h^N|$ converge to $|\psi^N|$.

Second, we solve the projected TDGL corresponding to (6.1)-(6.2) by the proposed method and denote the numerical solution by $(\psi_h^N, \mathbf{A}_h^N)$. We present the errors of the numerical solution in Table 6.2, where the convergence rate of ψ_h^N is calculated by the formula

$$\text{convergence rate of } \psi_h^N = \log(\|\psi_h^N - \psi^N\|_{L^2} / \|\psi_{h/2}^N - \psi^N\|_{L^2}) / \log 2$$

based on the finest mesh size h (the same formula is used for $|\psi_h^N|$ and $\mathbf{A}_h^N$). We see that the convergence rates of ψ_h^N, $|\psi_h^N|$ and $\mathbf{A}_h^N$ are better than $O(h^{2/3})$, which is the worst convergence rate proved in Theorem 2.3. The numerical results are consistent with our theoretical analysis and indicate that our method is efficient for solving the Ginzburg–Landau equations in a domain with reentrant corners.

TABLE 6.1. Errors of the finite element solution $(\widetilde{\psi}_h^N, \widetilde{\mathbf{A}}_h^N)$ with $\tau = h$.

h	$\|\widetilde{\psi}_h^N - \psi^N\|_{L^2}$	$\||\widetilde{\psi}_h^N| - |\psi^N|\|_{L^2}$	$\|\widetilde{\mathbf{A}}_h^N - \mathbf{A}^N\|_{L^2}$
1/16	4.2113E-03	3.7007E-03	8.3961E-02
1/32	3.1847E-03	2.0651E-03	8.1396E-02
1/64	2.9884E-03	1.6286E-03	7.9709E-02
1/128	2.9170E-03	1.4624E-03	7.8779E-02
1/256	2.8734E-03	1.3875E-03	7.8210E-02
convergence rate	$O(h^{0.02})$	$O(h^{0.07})$	$O(h^{0.01})$

TABLE 6.2. Errors of the finite element solution $(\psi_h^N, \mathbf{A}_h^N)$ with $\tau = h$.

h	$\|\psi_h^N - \psi^N\|_{L^2}$	$\||\psi_h^N| - |\psi^N|\|_{L^2}$	$\|\mathbf{A}_h^N - \mathbf{A}^N\|_{L^2}$
1/16	2.7608E-03	2.4889E-03	2.9448E-02
1/32	8.0517E-04	7.0163E-04	1.4861E-02
1/64	3.1147E-04	2.8685E-04	8.0870E-03
1/128	1.3066E-04	1.2664E-04	4.3397E-03
1/256	6.1047E-05	6.0252E-05	2.3748E-03
convergence rate	$O(h^{1.09})$	$O(h^{1.07})$	$O(h^{0.87})$

7. Conclusions and remarks

Based on the Hodge decomposition, we reformulated the TDGL into an equivalent system of equations, which avoids direct calculation of the singular magnetic potential. Then we proposed a decoupled and linearized time-stepping scheme, with Galerkin FEM for the space discretization, to solve the reformulated system of equations. Instead of solving the vector equation of $\mathbf{A}$ directly, our new approach solves several scalar heat and Poisson equations at each iteration step, which allows one to use software packages, especially fast solvers, reducing manpower and computational time dramatically. Convergence of the numerical solution was proved based on the regularity of the solution proved in this paper. We have used a cut-off function to guarantee the boundedness of the numerical solution which helps to prove convergence of the numerical solution. In practical computations, the numerical solutions with and without the cut-off function are almost the same.

For simplicity, we have focused on simply connected nonconvex polygons in this paper. In multi-connected domains, well-posedness of (1.16)-(1.19) can be proved in the same way, but the system (1.21)-(1.27) is no longer equivalent to (1.16)-(1.19). In the three-dimensional space, the Hodge decomposition (1.20) needs to be replaced by

$$\mathbf{A} = \nabla \times \mathbf{u} + \nabla v$$

where $\mathbf{u}$ is a vector field rather than a scalar variable. In this case, the Hodge decomposition approach does not reduce the problem to scalar equations, and the Galerkin FEM still may not converge for the reformulated vector equation of $\mathbf{u}$ in nonsmooth and nonconvex domains.

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