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A Numerical Analysis of the Weak Galerkin Method for the Helmholtz Equation with High Wave Number

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Abstract. We study the error analysis of the weak Galerkin finite element method in [24, 38] (WG-FEM) for the Helmholtz problem with large wave number in two and three dimensions. Using a modified duality argument proposed by Zhu and Wu, we obtain the pre-asymptotic error estimates of the WG-FEM. In particular, the error estimates with explicit dependence on the wave number k are derived. This shows that the pollution error in the broken H^1-norm is bounded by $\mathcal{O}(k(kh)^{2p})$ under mesh condition $k^{7/2}h^2 \leq C_0$ or $(kh)^2+k(kh)^{p+1}\leq C_0$, which coincides with the phase error of the finite element method obtained by existent dispersion analyses. Here h is the mesh size, p is the order of the approximation space and C_0 is a constant independent of k and h. Furthermore, numerical tests are provided to verify the theoretical findings and to illustrate the great capability of the WG-FEM in reducing the pollution effect.

AMS subject classifications: 65N12, 65N15, 65N30, 78A40

Key words: Weak Galerkin finite element method, Helmholtz equation, large wave number, stability, error estimates.

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1 Introduction

Let $\Omega\in\mathbb{R}^d, d=2,3$, be a bounded domain with smooth boundary $\Gamma=\partial\Omega$. We consider the following Helmholtz problem with the Robin boundary condition:

$$-\Delta u-k^2u=f \qquad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}}+\mathbf{i}ku=g \qquad \text{on } \Gamma, \tag{1.2}$$

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where $\mathbf{i}=\sqrt{-1}$ denotes the imaginary unit and $\mathbf{n}$ denotes the unit outward normal to Γ.

The above Helmholtz problem is an approximation of the acoustic scattering problem. The Robin boundary condition (1.2) is known as the first order approximation of the radiation condition [13]. We remark that the Helmholtz problem (1.1)-(1.2) also arises in applications as a consequence of frequency domain treatment of attenuated scalar waves [10].

It is well-known that the finite element method of fixed order for the Helmholtz problem (1.1)-(1.2) at high frequencies ($k\gg 1$) is subject to the effect of pollution: the ratio of the error of the finite element solution to the error of the best approximation from the finite element space cannot be uniformly bounded with respect to k [1,3,4,9,16,18,19]. In other words, the error bound of the finite element solution to the Helmholtz problem (1.1)-(1.2) usually consists of two parts: one is the same order as the error of the best approximation of u from the finite element space, the other dominates the error bound of the finite element solution for large wave number k. The second part is the so-called pollution error (cf. [8,17]). We recall that, the term "asymptotic error estimate" refers to the error estimate without pollution error and the term "preasymptotic error estimate" refers to the estimate with non-negligible pollution effect.

However, the highly indefinite nature of the Helmholtz problem with high wave number makes the error analysis of the FEM (including discontinuous Galerkin methods) very difficult. The reader is referred to [21,22] for the pollution free error estimates of the FEM for the one and higher dimensional Helmholtz problems, and to [11,12,33,34] for the estimates with pollution error of the FEM and CIP-FEM for two and three dimensional Helmholtz problem.

Weak Galerkin finite element methods were first introduced as nonconforming methods in [31] by Wang and Ye in 2013 for second order elliptic equations, which has been used to solve various problems [36–38]. The WG-FEMs admit various finite element meshes, such as a mix of arbitrary shape of polygons and polyhedrons and less number of the degree of freedoms in algebraic system than the general DG methods after parallel computation. The biggest feature of WG-FEMs is their ability to replace the classic derivatives in various variational formulations by the weak derivatives defined in [31]. Wang and Ye have applied the WG formulation for solving the Helmholtz problem in [38] and have shown the error estimates under the mesh condition $k^2h\le C_0$ by using the Schatz argument, where C_0 is a constant independent of the wave number k and the mesh size h of a uniform partition. Since $k^2h\le C_0$ is too strict for large k, later they improved the mesh condition and derived the stability and well-posedness without a mesh size constraint using arguments similar to those provided in [14,15] by Feng and Wu. However, the convergence rate in their estimates lost one order under the general assumption that $kh\le 1$ for the large wave number problem. The goal of this work is to obtain the optimal pre-asymptotic error estimates under the mesh condition $k^{7/2}h^2\le C_0$ or $(kh)^2+k(kh)^{p+1}\le C_0$ by using a modified dual argument like the one recently proposed in [33,34].

Other than the WG-FEMs, various discontinuous Galerkin methods are also nonconforming methods. We refer the reader to [20] by Melenk et al. for the latest asymptotic error analysis of the general DG-method on regular mesh if $k(kh)^p \le C_0$, to [14,15] by Feng and Wu for the stability without any mesh constraint and the broken H^1 error estimate of the interior penalty DG method and to [11,34] by Wu et al. for the pre-asymptotic error estimates under the improved mesh condition $k(kh)^{p+1} \le C_0$ or $k(kh)^{2p} \le C_0$ of the IP-DG methods. For the error analysis of other methods including DG methods, we refer the reader to [6,8,27,35].

The remainder of this paper is organized as follows. The weak Galerkin finite element methods are introduced in Section 2. Some preliminary results, including the stability of the continuous solution, the error estimates of various L^2-projections and elliptic projections defined in the weak Galerkin FEM space are cited or proved in Section 3. The preasymptotic error analysis of WG-FEMs is given in Section 4. Finally, we simulate a model problem in two dimensions on triangulation and verify the theoretical findings in Section 5.

Throughout the paper, C is used to denote a generic positive constant which is independent of h,k,f and g. We also use the shorthand notation $A \lesssim B$ and $A \gtrsim B$ for the inequalities $A \le CB$ and $A \ge B$. $A \eqsim B$ is a shorthand notation for the statement $A \lesssim B$ and $B \lesssim A$. We assume that $k \gg 1$ since we are considering high-frequency problems and that k is constant on Ω for ease of presentation. We also assume that Ω is a strictly star-shaped domain. Here "strictly star-shaped" means that there exist a point $x_\Omega \in \Omega$ and a positive constant c_Ω depending only on Ω such that

$$(x - x_\Omega) \cdot n \ge c_\Omega \quad \forall x \in \Gamma.$$

2 The weak Galerkin FEM methods

We first introduce some notation. The standard Sobolev and Hilbert space, norm and inner product notation are adopted. Their definitions can be found in [5,7]. In particular, $(\cdot,\cdot)_K$ and $\langle \cdot,\cdot \rangle_\Sigma$ for $\Sigma = \partial K$ denotes the L^2-inner product on complex-valued $L^2(K)$ and $L^2(\Sigma)$ spaces, respectively. Denote $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$ and $\langle \cdot,\cdot \rangle := \langle \cdot,\cdot \rangle_\Gamma$. For simplicity, denote $\|\cdot\|_j := \|\cdot\|_{H^j(\Omega)}$ and $|\cdot|_j := |\cdot|_{H^j(\Omega)}$.

Let $\mathcal{T}_h$ be a triangulation of Ω (cf. [21–23]). Let $\mathcal{E}_h$ be the set of all edges of $\mathcal{T}_h$. For any $K \in \mathcal{T}_h$, h_K denotes its diameter and $|K|$ denotes its area. Similarly, for each edge $e \in \mathcal{E}_h$, define $h_e := \mathrm{diam}(e)$. Let $h = \max_{K \in \mathcal{T}_h} h_K$. Assume that $h_K \eqsim h$. We denote all the boundary edges by $\mathcal{E}_h^B := \{e \in \mathcal{E}_h : e \subset \Gamma\}$ and the interior edges by $\mathcal{E}_h^I := \mathcal{E}_h \backslash \mathcal{E}_h^B$.

Let $\widehat{K}$ denote the reference elements, and let F_K denote the element maps from $\widehat{K}$ to $K \in \mathcal{T}_h$. Let $\mathcal{P}_r(\widehat{K})$ and $\mathcal{P}_r(\hat{e})$ be the set of all polynomials with degrees $\le r$ on $\widehat{K}$ and $\hat{e} \subset \partial \widehat{K}$, respectively. Also, let $\mathcal{P}_r(K)$ and $\mathcal{P}_r(e)$ be the set of functions v_h satisfying $v_h \circ F_K \in \mathcal{P}_p(\widehat{K})$ and $\mathcal{P}_r(\hat{e})$, respectively. Note that the partition $\mathcal{T}_h$ consisting of polygons or polyhedrons introduced in [24,31] is not applied because of the analytic boundary of Ω. The theoretical results of this paper also hold for finite element discretizations on curvilinear Cartesian meshes or isoparametric finite element approximations [5]..

Now we give the definitions of the weak derivative operator and its discrete form [31]. For any $K\in\mathcal{T}_h$, the local weak function space $S(K)$ is defined by

$$S(K)=\big\{v=\{v_0,v_b\}: v_0\in L^2(K), v_b\in L^2(\partial K)\big\}.$$

Then let the weak function space V be the set:

$$V=\{v=\{v_0,v_b\}: v|_K\in S(K)\}.$$

For any $v\in V$ and $K\in\mathcal{T}_h$, the discrete weak gradient $\nabla_{w,r,K}v$ on K is defined by

$$(\nabla_{w,r,K}v,\mathbf{q})_K=-(v_0,\nabla\cdot\mathbf{q})_K+\langle v_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}\quad \forall\mathbf{q}\in[\mathcal{P}_r(K)]^d,$$

where r is a nonnegative integer.

Next, we introduce the discrete forms of the spaces $S(K)$ and V. The definition of the discrete space $S(r,K)$ of $S(K)$ is broken into two cases. In the first case, the local weak finite element space $S_1(r,K)$ is defined as

$$S_1(r,K)=\{v=\{v_0,v_b\}: v_0\in\mathcal{P}_r(K), v_b|_e\in\mathcal{P}_{r-1}(e)\ \forall e\subset\partial K\}.$$

In the second case, $S_2(r,K)$ is defined as

$$S_2(r,K)=\{v=\{v_0,v_b\}: v_0\in\mathcal{P}_r(K), v_b|_e\in\mathcal{P}_r(e)\ \forall e\subset\partial K\}.$$

We also define the approximation space of the weak finite element methods in two cases. In the first case, let V_h^1 be the discrete form of V, that is,

$$V_h^1:=\Big\{v=\{v_0,v_b\}: v|_K\in S_1(p,K),\ v_b|_{K_1\cap e}=v_b|_{K_2\cap e}\quad \forall K,K_1,K_2\in\mathcal{T}_h,\ \forall e\in\mathcal{E}_h^I\Big\}. \tag{2.1}$$

In the second case, the discrete space V_h^2 is defined as

$$V_h^2:=\Big\{v=\{v_0,v_b\}: v|_K\in S_2(p,K),\ v_b|_{K_1\cap e}=v_b|_{K_2\cap e}\quad \forall K,K_1,K_2\in\mathcal{T}_h,\ \forall e\in\mathcal{E}_h^I\Big\}. \tag{2.2}$$

We show some L^2 projections [24] which will be used often in the forthcoming subsection. For any $K\in\mathcal{T}_h$, Q_0 is a L^2 projection from $L^2(K)$ to $\mathcal{P}_p(K)$ and Q_b is a L^2 projection from $L^2(e)$ to $\mathcal{P}_{p-1}(e)$ or $\mathcal{P}_p(e)$ in the two cases for any $e\subset\partial K$, respectively. Then the L^2 projection Q_h of weak FEM in V_h is defined by

$$Q_hv|_K:=\{Q_0v,Q_bv\}\quad \forall v\in V,K\in\mathcal{T}_h. \tag{2.3}$$

Furthermore, $\mathbb{Q}_h:[L^2(\mathcal{T}_h)]^d\rightarrow[\mathcal{P}_{p-1}(\mathcal{T}_h)]^d$ is a L^2 projection defined by:

$$(\mathbb{Q}_h\mathbf{q},\mathbf{p}_h)_K=(\mathbf{q},\mathbf{p}_h)_K \quad \forall\mathbf{q}\in[L^2(K)]^d,\mathbf{p}_h\in[\mathcal{P}_{p-1}(K)]^d,$$

for any $K\in\mathcal{T}_h$.

The following identity holds(cf. [24]):

$$\nabla_{w,p-1,K}(Q_hv)=\mathbb{Q}_h(\nabla v) \quad \forall v\in H^1(K). \tag{2.4}$$

For simplicity of presentation, the sesquilinear form $a(\cdot,\cdot)$ on $V\times V$ is defined as follows:

$$a(u,v):=\sum_{K\in\mathcal{T}_h}(\nabla_{w,p-1,K}u,\nabla_{w,p-1,K}v)_K+s(u,v), \tag{2.5}$$

$$s(u,v):=\rho\sum_{K\in\mathcal{T}_h}h_K^{-1}\langle Q_bu_0-u_b,Q_bv_0-v_b\rangle_{\partial K}, \tag{2.6}$$

where ρ is a positive real number.

Then the weak Galerkin finite element methods are defined as such: find $u_h=\{u_0,u_b\}\in V_h^1$ or V_h^2 such that

$$a(u_h,v_h)-k^2(u_0,v_0)+\mathbf{i}k\langle u_b,v_b\rangle=(f,v_0)+\langle g,v_b\rangle, \tag{2.7}$$

holds for any $v_h=\{v_0,v_b\}\in V_h^1$ or V_h^2, respectively.

The following norms are useful later:

$$\|v_h\|_{1,h}:=\left(\sum_{K\in\mathcal{T}_h}\|v_0\|_{L^2(K)}^2+h_K^{-1}\|Q_bv_0-v_b\|_{L^(\partial K)}^2\right)^{1/2},$$

$$|||v_h|||:=\left(|a(v_h,v_h)|+k\|v_b\|_{L^2(\Gamma)}^2\right)^{1/2} \quad \forall v_h=\{v_0,v_b\}\in V_h.$$

3 Preliminary lemmas

In this section, we recall some preliminary lemmas about stability estimates of the continuous problem and the approximation estimates of the discrete space V_h.

The following lemma (cf. [22, Theorem 4.10]) states that the solution u to the continuous problem (1.1)-(1.2) can be decomposed into the sum of an elliptic part and an analytic part $u=u_{\mathcal{E}}+u_{\mathcal{A}}$, where $u_{\mathcal{E}}$ is usually nonsmooth but the H^2-bound of $u_{\mathcal{E}}$ is independent of k and $u_{\mathcal{A}}$ is oscillatory but the H^j-bound of $u_{\mathcal{A}}$ is available for any integer $j\geq 0$.

Lemma 3.1. *Assume that Ω is a strictly star-shaped domain with an analytic boundary. Suppose $f\in L^2(\Omega)$ and $g\in H^{1/2}(\Gamma)$. Then the solution u to the problem* (1.1)-(1.2) *can be written as $u=u_{\mathcal{E}}+u_{\mathcal{A}}$ and satisfies*

$$|u_{\mathcal{E}}|_j\lesssim k^{j-2}C_{f,g}, \quad j=0,1,2, \tag{3.1}$$

$$|u_{\mathcal{A}}|_j\lesssim k^{j-1}C_{f,g} \quad \forall j\in\mathbb{N}_0. \tag{3.2}$$

Here $C_{f,g}:=\|f\|_0+\|g\|_{H^{1/2}(\Gamma)}$.

The following lemma shows the regularity of u in norms of high order [11].

Lemma 3.2. *Suppose* $s\geq 2$ *and* $f\in H^{s-2}(\Omega)$ *and* $g\in H^{s-3/2}(\Gamma)$*. Then the solution* u *to the problem* (1.1)-(1.2) *satisfies the following stability estimate:*

$$\|u\|_s \lesssim k^{s-1} C_{s-2,f,g}, \tag{3.3}$$

where $C_{s-2,f,g}:=\|f\|_0+\|g\|_{L^2(\Gamma)}+\sum_{j=0}^{s-2}k^{-(j+1)}\big(\|f\|_j+\|g\|_{H^{j+1/2}(\Gamma)}\big)$.

Since the partition $\mathcal{T}_h$ is a curvilinear triangulation, the following estimates can be easily obtained for the L^2-projections. We omit the details of proof and refer the reader to [5].

Lemma 3.3. *Let* $1\leq s\leq p+1$*. Then we have:*

$$\|\nabla u-\mathbb{Q}_h(\nabla u)\|_0+h\Big(\sum_{K\in\mathcal{T}_h}|\nabla u-\mathbb{Q}_h(\nabla u)|^2_{H^1(K)}\Big)^{1/2}\lesssim h^{s-1}|u|_s,$$

$$\|u-Q_0u\|_0+h\Big(\sum_{K\in\mathcal{T}_h}|u-Q_0u|^2_{H^1(K)}\Big)^{1/2}\lesssim h^{s}|u|_s,$$

for $u\in H^s(\Omega)$, *and*

$$\|u-Q_bu\|_{L^2(\Gamma)}\lesssim h^{s-1}\|u\|_{H^{s-1}(\Gamma)},$$

for $u\in H^{s-1}(\Gamma)$ *and* $Q_bu|_e\in\mathcal{P}_{p-1}(e)$ $\forall e\in\mathcal{E}_h$.

If u is the exact solution satisfying the decomposition $u=u_{\mathcal{E}}+u_{\mathcal{A}}$ as in Lemma 3.1, then we may approximate u and ∇u by $Q_hu=Q_hu_{\mathcal{E}}+Q_hu_{\mathcal{A}}$ and $\mathbb{Q}_h(\nabla u)=\mathbb{Q}_h(\nabla u_{\mathcal{E}})+\mathbb{Q}_h(\nabla u_{\mathcal{A}})$ to show the following estimate (cf. [21,22]).

Lemma 3.4. *Let* u *be the solution to the problem* (1.1)-(1.2)*. Suppose* $f\in L^2(\Omega)$ *and* $g\in H^{1/2}(\Gamma)$*. We have*

$$\Big(\sum_{K\in\mathcal{T}_h}h_K\|\nabla u-\mathbb{Q}_h(\nabla u)\|^2_{\partial K}\Big)^{1/2}\lesssim (h+(kh)^p)C_{f,g}, \tag{3.4}$$

$$\Big(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0u-u\|^2_{\partial K}\Big)^{1/2}\lesssim (h+(kh)^p)C_{f,g}. \tag{3.5}$$

Lemma 3.5 can be easily obtained by some arguments similar to those in Lemma 3.1 in [24].

Lemma 3.5. *Assume that* u *is the solution to the Helmholtz problem* (1.1)-(1.2) *and* u_h *is the WG-FEM solution in* V_h^1 *or* V_h^2*. Denote by* $e_h=\{e_0,e_b\}:=Q_hu-u_h=\{Q_0u-u_0,Q_bu-u_b\}$*, We have the following error equality:*

$$a(e_h,v_h)-k^2(e_0,v_0)+\mathbf{i}k\langle e_b,v_b\rangle=l_u(v_h)+s(Q_hu,v_h)\quad \forall v_h\in V_h, \tag{3.6}$$

where $l_w(v_h):=\sum_{K\in\mathcal{T}_h}\langle(\nabla w-\mathbb{Q}_h\nabla w)\cdot\mathbf{n},v_0-v_b\rangle_{\partial K}$.

Before showing the error analysis for the Helmholtz problem (1.1)-(1.2), we introduce two kinds of elliptic projections similar to those in [33, 34]. For any $u\in V$, we define its elliptic projections $u_h^+=\{u_0^+,u_b^+\}$, $u_h^-=\{u_0^-,u_b^-\}\in V_h^1$ or V_h^2 as the WG finite element approximations to the following Poisson problem:

$$\begin{aligned}-\Delta u&=F \quad \text{in } \Omega,\\ \frac{\partial u}{\partial n}\pm\mathbf{i}ku&=G \quad \text{on } \Gamma,\end{aligned}$$

for some functions F and G which are determined by u, i.e. the following equalities hold for any $v_h=\{v_0,v_b\}\in V_h^1$ or V_h^2

$$a(u_h^+,v_h)+\mathbf{i}k\langle u_b^+,v_b\rangle=(F,v_0)+\langle G,v_b\rangle, \tag{3.7}$$

$$a(v_h,u_h^-)+\mathbf{i}k\langle v_b,u_b^-\rangle=(v_0,F)+\langle v_b,G\rangle, \tag{3.8}$$

respectively.

We establish approximation properties of these elliptic projections in the following lemma.

Lemma 3.6. *Let u be any function in $H^2(\Omega)$ and $u_h^\pm$ be its elliptic projections defined by* (3.7) *and* (3.8)*, respectively. Then we have the following estimates:*

(i) for $u_h^\pm\in V_h^1$ with $p=1$,

$$\begin{aligned}&|||Q_hu-u_h^\pm|||\lesssim E(u),\\ &\|Q_0u-u_0^\pm\|_0\lesssim h\left(E(u)+\left\|\frac{\partial u}{\partial \mathbf{n}}-Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)}\right);\end{aligned}$$

(ii) for $u_h^\pm\in V_h^1$ with $p\geq 2$ or $u_h^\pm\in V_h^2$,

$$|||Q_hu-u_h^\pm|||+h^{-1}\|Q_0u-u_0^\pm\|_0\lesssim E(u).$$

Here

$$E(u)=\left(\sum_{K\in\mathcal{T}_h}h_K\|\nabla u-\mathbb{Q}_h\nabla u\|_{L^2(\partial K)}^2\right)^{1/2}+\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0u-u\|_{\partial K}^2\right)^{1/2}.$$

Proof. We only give the details of proof for the elliptic projection u_h^+ of u. The proof of u_h^- can be completed by similar arguments.

Define $\eta=\{\eta_0,\eta_b\}:=Q_hu-u_h^+$. By some arguments used in Lemma 3.5, we have

$$a(\eta,v_h)+\mathbf{i}k\langle\eta_b,v_b\rangle=l_u(v_h)+s(Q_hu,v_h)\quad\forall v_h=\{v_0,v_b\}\in V_h, \tag{3.9}$$

where $l_u(v_h)$ is defined in Lemma 3.5. Take $v_h=\eta$, we have

$$
\begin{aligned}
|||\eta|||^2 &= \Re\big(a(\eta,\eta)+\mathbf{i}k\langle\eta_b,\eta_b\rangle\big)+\Im\big(a(\eta,\eta)+\mathbf{i}k\langle\eta_b,\eta_b\rangle\big)\\
&\lesssim |l_u(\eta)|+|s(Q_hu,\eta)|\\
&\lesssim |||\eta|||\Bigg[\Big(\sum_{K\in\mathcal{T}_h} h_K\|\nabla u-\mathbb{Q}_h\nabla u\|^2_{L^2(\partial K)}\Big)^{1/2}\\
&\qquad +\Big(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|Q_bQ_0u-Q_bu\|^2_{L^2(\partial K)}\Big)^{1/2}\Bigg],
\end{aligned}
\tag{3.10}
$$

where the inequality given in [30]

$$
|l_u(v_h)|\lesssim\Big(\sum_{K\in\mathcal{T}_h} h_K\|\nabla u-\mathbb{Q}_h\nabla u\|^2_{L^2(\partial K)}\Big)^{1/2}\cdot|||v_h|||\ \ \forall v_h\in V_h^1\cup V_h^2 \tag{3.11}
$$

is used. By the fact that

$$
\begin{aligned}
\|Q_bQ_0u-Q_bu\|^2_{L^2(\partial K)} &= \sum_{e\in\mathcal{E}_h,e\subset\partial K}\langle Q_b(Q_0u-u),Q_bQ_0u-Q_bu\rangle_e\\
&= \sum_{e\in\mathcal{E}_h,e\subset\partial K}\langle Q_0u-u,Q_bQ_0u-Q_bu\rangle_e\\
&\le \|Q_0u-u\|_{L^2(\partial K)}\|Q_bQ_0u-Q_bu\|_{L^2(\partial K)},
\end{aligned}
\tag{3.12}
$$

we get

$$
|||\eta|||\lesssim E(u). \tag{3.13}
$$

Next, we estimate the L^2-norm of η_0 in Ω. Consider the following auxiliary problem:

$$
-\Delta\Psi=\eta_0 \quad \text{in }\ \Omega, \tag{3.14}
$$

$$
\frac{\partial\Psi}{\partial\mathbf{n}}-\mathbf{i}k\Psi=0 \quad \text{on }\ \Gamma. \tag{3.15}
$$

It is known that $\|\Psi\|_2\lesssim\|\eta_0\|_0$ (cf. [34]).

Testing the conjugated (3.14) by η_0, by (2.4) and (3.9) we have

$$
\begin{aligned}
\|\eta\|_0^2 &= -(\eta_0,\Delta\Psi)=\sum_{K\in\mathcal{T}_h}(\nabla\eta_0,\nabla\Psi)_K-\sum_{K\in\mathcal{T}_h}\Big\langle\eta_0,\frac{\partial\Psi}{\partial\mathbf{n}_e}\Big\rangle_{\partial K}\\
&= \sum_{K\in\mathcal{T}_h}(\nabla\eta_0,\mathbb{Q}_h\nabla\Psi)_K-\sum_{K\in\mathcal{T}_h}\Big\langle\eta_0-\eta_b,\frac{\partial\Psi}{\partial\mathbf{n}_e}\Big\rangle_{\partial K}-\sum_{e\in\mathcal{E}_h^B}\Big\langle\eta_b,\frac{\partial\Psi}{\partial\mathbf{n}_e}\Big\rangle_e\\
&= \sum_{K\in\mathcal{T}_h}(\nabla_{w,p-1,K}\eta,\mathbb{Q}_h\nabla\Psi)-\overline{l_\Psi(\eta)}+\mathbf{i}k\langle\eta_b,Q_b\Psi\rangle\\
&= a(\eta,Q_h\Psi)+\mathbf{i}k\langle\eta,Q_b\Psi\rangle-\overline{l_\Psi(\eta)}-s(\eta,Q_h\Psi)\\
&= l_u(Q_h\Psi)+s(Q_hu,Q_h\Psi)-\overline{l_\Psi(\eta)}-s(\eta,Q_h\Psi).
\end{aligned}
\tag{3.16}
$$

We estimate each term in the right hand side of (3.16). We first estimate $l_u(Q_h\Psi)$ in two cases, respectively. In the first case where $u_h\in V_h^1$, the fact that $Q_b v\in\mathcal{P}_{p-1}(e)$, $\forall v\in L^2(e)$, $e\in\mathcal{E}_h$ implies

$$\begin{aligned}
|l_u(Q_h\Psi)|\le&\left|\sum_{K\in\mathcal{T}_h}\langle(u-\mathbb{Q}_h u)\cdot\mathbf{n}_e,Q_0\Psi-\Psi\rangle_{\partial K}\right|\\
&+\left|\sum_{K\in\mathcal{T}_h}\langle(u-\mathbb{Q}_h u)\cdot\mathbf{n}_e,\Psi-Q_b\Psi\rangle_{\partial K}\right|\\
\le&\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|\Psi-Q_0\Psi\|_{L^2(\partial K)}^2\right)^{1/2}\\
&\cdot\left(\sum_{K\in\mathcal{T}_h}h_K\|\nabla u-\mathbb{Q}_h\nabla u\|_{L^2(\partial K)}^2\right)^{1/2}+\left|\sum_{e\in\mathcal{E}_h^B}\left\langle\frac{\partial u}{\partial\mathbf{n}},\Psi-Q_b\Psi\right\rangle_e\right|\\
\lesssim& h\|\Psi\|_2E(u)+\left|\sum_{e\in\mathcal{E}_h^B}\left\langle\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}},\Psi-Q_b\Psi\right\rangle_e\right|.
\end{aligned}\tag{3.17}$$

If $p=1$, $Q_b\Psi\in\mathcal{P}_0(e)$ implies that

$$\left|\sum_{e\in\mathcal{E}_h^B}\left\langle\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}},\Psi-Q_b\Psi\right\rangle_e\right|\lesssim h\|\Psi\|_{H^1(\Gamma)}\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}.$$

If $p\ge2$, $Q_b\Psi\in\mathcal{P}_1(e)$ implies that

$$\begin{aligned}
\left|\sum_{e\in\mathcal{E}_h^B}\left\langle\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}},\Psi-Q_b\Psi\right\rangle_e\right|=&\left|\sum_{e\in\mathcal{E}_h^B}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n},\Psi-Q_b\Psi\rangle_e\right|\\
\lesssim& h\|\Psi\|_2\left(\sum_{e\in\mathcal{E}_h^B}h_e\|\nabla u-\mathbb{Q}_h\nabla u\|_{L^2(e)}^2\right)^{1/2}.
\end{aligned}$$

Therefore, in the first case we have

$$|l_u(Q_h\Psi)|\lesssim h\|\eta_0\|_0\left(E(u)+\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}\right)$$

for $p=1$, and

$$|l_u(Q_h\Psi)|\lesssim h\|\eta_0\|_0E(u)$$

for $p\ge2$.

In the second case where $u_h^+\in V_h^2$,

$$
\begin{aligned}
|l_u(Q_h\Psi)| &= \Big|\sum_{K\in\mathcal{T}_h}\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n},Q_b(Q_0\Psi-\Psi)\rangle_{\partial K}\Big|\\
&\le\Big(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0\Psi-\Psi\|^2_{L^2(\partial K)}\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}h_K\|\nabla u-\mathbb{Q}_h\nabla u\|^2_{L^2(\partial K)}\Big)^{1/2}\\
&\lesssim h\|\eta_0\|_0E(u).
\end{aligned}
$$

For $|s(Q_hu,Q_h\Psi)|$, the following estimate holds for both $u_h^+\in V_h^1$ and V_h^2

$$
\begin{aligned}
|s(Q_hu,Q_h\Psi)| &\le\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_b(Q_0u-u)\|_{\partial K}\|Q_b(Q_0\Psi-\Psi)\|_{\partial K}\\
&\lesssim\Big(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0u-u\|^2_{\partial K}\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0\Psi-\Psi\|^2_{\partial K}\Big)^{1/2}\\
&\lesssim h\|\eta_0\|_0E(u).
\end{aligned}
\tag{3.18}
$$

From (3.11) and (3.13) it is easy to see that for both $u_h^+\in V_h^1$ and V_h^2

$$
|l_\Psi(\eta)|+|s(\eta,Q_h\Psi)|\lesssim h\|\Psi\|_2|||\eta|||\lesssim h\|\eta_0\|_0E(u). \tag{3.19}
$$

Combining (3.16)-(3.19), we have

(i) for $u_h^+\in V_h^1$ with $p=1$,

$$
\|\eta\|_0\lesssim h\bigg(\Big\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\Big\|_{L^2(\Gamma)}+\big(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0u-u\|^2_{\partial K}\big)^{1/2}+|||\eta|||\bigg). \tag{3.20}
$$

(ii) for $u_h^+\in V_h^1$ with $p\ge 2$ or $u_h^+\in V_h^2$,

$$
\begin{aligned}
\|\eta\|_0\lesssim h\bigg(&\big(\sum_{K\in\mathcal{T}_h}h_K\|\nabla u-\mathbb{Q}_h\nabla u\|^2_{L^2(\partial K)}\big)^{1/2}\\
&+\big(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|Q_0u-u\|^2_{\partial K}\big)^{1/2}+|||\eta|||\bigg).
\end{aligned}
\tag{3.21}
$$

Finally, by combining (3.13) we complete the proof. □

4 Pre-asymptotic error estimates

The duality argument (or Aubin-Nitsche trick) (cf. [2,10,19,21,22,26]), that is estimating the L^2-error of the finite element solution by its H^1-error, is one crucial step in asymptotic error analyses of FEM for scattering problems. Based on the standard duality argument, the stability of Melenk and Sauter [21, 22] (cf. Lemma 3.1) leads to pollution-free estimates under the condition that $k^{p+1}h^p$ is sufficiently small instead. In [34], Zhu and Wu develop a modified duality argument which uses some specially designed elliptic projections in the duality-argument step so that we can bound the L^2-error of the discrete solution by using the errors of the elliptic projections of the exact solution u and obtain the first preasymptotic error estimates for the FEM in higher dimensions under the condition that $k^{p+2}h^{p+1}$ is sufficiently small. We use this kind of modified duality argument to obtain the error estimate for the WG-FEM.

Theorem 4.1. *Assume that u is the solution to (1.1)-(1.2) with $f\in L^2(\Omega)$, $g\in H^{1/2}(\Gamma)$ and $u_h=\{u_0,u_b\}$ be the WG-FEM solution. There exists a constant C_0 independent of k and h, such that*

(i) for $u_h\in V_h^1$ with $p=1$, if $k^{7/2}h^2\le C_0$,

$$\|Q_0u-u_0\|_0\lesssim (h+kh)\left(E(u)+k^{-1/2}\left\|\frac{\partial u}{\partial \mathbf{n}}-Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)}\right),$$
$$|||Q_hu-u_h|||\lesssim (1+k^2h)E(u)+k^{3/2}h\left\|\frac{\partial u}{\partial \mathbf{n}}-Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)};$$

(ii) for $u_h\in V_h^1$ with $p\ge 2$, if $(kh)^2+k(kh)^{p+1}\le C_0$,

$$\|Q_0u-u_0\|_0\lesssim \big(h+(kh)^p\big)E(u)+\big(h^{3/2}+k^{-1/2}(kh)^p\big)\left\|\frac{\partial u}{\partial \mathbf{n}}-Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)},$$
$$|||Q_hu-u_h|||\lesssim \big(1+k(kh)^p\big)E(u)+\big(h^{1/2}(kh)+k^{1/2}(kh)^p\big)\left\|\frac{\partial u}{\partial \mathbf{n}}-Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)};$$

(iii) for $u_h\in V_h^2$, if $(kh)^2+k(kh)^{p+1}\le C_0$,

$$\|Q_0u-u_0\|_0\lesssim \big(h+(kh)^p\big)E(u),$$
$$|||Q_hu-u_h|||\lesssim \big(1+k(kh)^p\big)E(u),$$

where $E(u)$ is defined in Lemma 3.6.

Proof. Let $e_h=\{e_0,e_b\}:=Q_hu-u_h$ and u_h^+ be the elliptic projections of u defined by (3.7).
Consider the dual problem:

$$-\Delta\phi-k^2\phi=e_0 \quad \text{in } \Omega,$$
$$\frac{\partial \phi}{\partial \mathbf{n}}-\mathbf{i}k\phi=0 \quad \text{on } \Gamma.$$

Let ϕ_h^- be the elliptic projection of ϕ defined by (3.8). By Lemma 3.1, Lemma 3.3 and Lemma 3.6, we show the error estimates between ϕ and ϕ_h^-:

(i) for $\phi_h^- \in V_h^1$ with $p=1$,

$$\left|\!\left|\!\left| Q_h\phi-\phi_h^- \right|\!\right|\!\right| \lesssim (h+kh)\|e_0\|_0, \tag{4.1}$$

$$\left\| Q_h\phi-\phi_h^- \right\|_0 \lesssim h^2k^{3/2}\|e_0\|_0; \tag{4.2}$$

(ii) for $\phi_h^- \in V_h^1$ with $p\geq 2$ or $\phi_h^- \in V_h^2$,

$$\left|\!\left|\!\left| Q_h\phi-\phi_h^- \right|\!\right|\!\right| \lesssim (h+(kh)^p)\|e_0\|_0, \tag{4.3}$$

$$\left\| Q_h\phi-\phi_h^- \right\|_0 \lesssim h(h+(kh)^p)\|e_0\|_0. \tag{4.4}$$

Note that we have used the boundary condition $\frac{\partial\phi}{\partial\mathbf{n}}=\mathbf{i}k\phi$ to estimate (4.2).

Testing the conjugated dual problem by e_0 and by arguments similar to (3.16), we get

$$\|e_0\|_0^2 = a(e_h,Q_h\phi)-k^2(e_0,Q_0\phi)+\mathbf{i}k\langle e_b,Q_b\phi\rangle-\overline{l_\phi(e_h)}-s(e_h,Q_h\phi). \tag{4.5}$$

From Lemma 3.5, (3.9) and (4.5), we have

$$\begin{aligned}\|e_0\|_0^2 &= a(e_h,Q_h\phi-\phi_h^-)-k^2(e_0,Q_0\phi-\phi_0^-)+\mathbf{i}k\langle e_b,Q_b\phi-\phi_b^-\rangle\\ &\quad +l_u(\phi_h^-)+s(Q_hu,\phi_h^-)-\overline{l_\phi(e_h)}-s(e_h,Q_h\phi)\\ &= a(Q_hu-u_h^+,Q_h\phi-\phi_h^-)+\mathbf{i}k\langle Q_bu-u_b^+,Q_b\phi-\phi_b^-\rangle-k^2(e_0,Q_0\phi-\phi_0^-)\\ &\quad +\overline{l_\phi(u_h^+-u_h)}+\overline{s(Q_h\phi,u_h^+-u_h)}+l_u(\phi_h^-)+s(Q_hu,\phi_h^-)\\ &\quad -\overline{l_\phi(e_h)}-s(e_h,Q_h\phi).\end{aligned} \tag{4.6}$$

Combining the equation above and the fact that

$$\begin{aligned}&\overline{l_\phi(u_h^+-u_h)}-\overline{l_\phi(e_h)}=\overline{l_\phi(u_h^+-u_h-e_h)}=\overline{l_\phi(u_h^+-Q_hu)},\\ &\overline{s(Q_h\phi,u_h^+-u_h)}-s(e_h,Q_h\phi)=s(u_h^+-Q_hu,Q_h\phi),\end{aligned}$$

we have

$$\begin{aligned}\|e_0\|_0^2 &\leq \left|\!\left|\!\left| Q_hu-u_h^+ \right|\!\right|\!\right| \left|\!\left|\!\left| Q_h\phi-\phi_h^- \right|\!\right|\!\right| + k^2\|e_0\|_0\left\|Q_0\phi-\phi_0^-\right\|_0\\ &\quad +\left|l_\phi(u_h^+-Q_hu)\right|+\left|s(u_h^+-Q_hu,Q_h\phi)\right|+\left|l_u(\phi_h^-)\right|+\left|s(Q_hu,\phi_h^-)\right|.\end{aligned} \tag{4.7}$$

Now we estimate the terms in the right hand side of the inequality above. From (3.11) and Lemma 3.6, it is easy to obtain the following inequalities in both cases $u_h\in V_h^1$ and $u_h\in V_h^2$:

$$\begin{aligned}\left|l_\phi(u_h^+-Q_hu)\right| &\lesssim E(\phi)\left|\!\left|\!\left| u_h^+-Q_hu \right|\!\right|\!\right|\\ &\lesssim \left(h+(kh)^p\right)\|e_0\|_0 E(u),\end{aligned} \tag{4.8}$$

$$\begin{aligned}\left|s(u_h^+ - Q_h u, Q_h\phi)\right| &\lesssim E(\phi)\,|||u_h^+ - Q_h u||| \\ &\lesssim \left(h+(kh)^p\right)\|e_0\|_0 E(u),\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\left|s(Q_h u, \phi_h^-)\right| &\le \left|s(Q_h u, Q_h\phi - \phi_h^-)\right| + |s(Q_h u, Q_h\phi)| \\ &\lesssim \Big(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|Q_0 u - u\|_{\partial K}^2\Big)^{1/2}\Big(|||Q_h\phi - \phi_h^-||| + E(\phi)\Big) \\ &\lesssim \left(h+(kh)^p\right)\|e_0\|_0 E(u).\end{aligned} \tag{4.10}$$

Then we estimate $l_u(\phi_h^-)$ in two cases $u_h \in V_h^1$ and $u_h \in V_h^2$, respectively. First, it follows from (3.11), (4.1) and (4.3) that

$$\begin{aligned}\left|l_u(\phi_h^-)\right| &\le \left|l_u(\phi_h^- - Q_h\phi)\right| + |l_u(Q_h\phi)| \\ &\lesssim \Big(\sum_{K\in\mathcal{T}_h} h_K\|\nabla u - \mathbb{Q}_h\nabla u\|_{L^2(\partial K)}\Big)^{1/2}\cdot|||\phi_h^- - Q_h\phi||| + |l_u(Q_h\phi)| \\ &\lesssim \left(h+(kh)^p\right)\|e_0\|_0 E(u) + |l_u(Q_h\phi)|.\end{aligned} \tag{4.11}$$

So we only need to estimate $l_u(Q_h\phi)$ to complete the inequality above. By a argument similar to (3.17) we get

$$\begin{aligned}|l_u(Q_h\phi)| &\lesssim E(u)E(\phi) + \left|\sum_{e\in\mathcal{E}_h^B}\left\langle \frac{\partial u}{\partial \mathbf{n}} - Q_b\frac{\partial u}{\partial \mathbf{n}}, \phi - Q_b\phi\right\rangle_e\right| \\ &\lesssim \left(h+(kh)^p\right)\|e_0\|_0 E(u) + \left\|\frac{\partial u}{\partial \mathbf{n}} - Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)}\|\phi - Q_b\phi\|_{L^2(\Gamma)}.\end{aligned} \tag{4.12}$$

In the first case that $Q_b\phi \in \mathcal{P}_{p-1}(e)$, that is $u_b \in V_h^1$. From Lemma 3.1, we know that the solution ϕ to the dual problem also can be written as $\phi = \phi_\mathcal{A} + \phi_\mathcal{E}$ satisfying

$$\begin{aligned}|\phi_\mathcal{E}| &\lesssim k^{j-2}\|e_0\|_0, \quad j=0,1,2, \\ |\phi_\mathcal{A}| &\lesssim k^{j-1}\|e_0\|_0 \quad \forall j \in \mathbb{N}_0.\end{aligned}$$

If $p=1$, we have

$$\begin{aligned}\|\phi - Q_b\phi\|_{L^2(\Gamma)} &\lesssim h\|\phi_\mathcal{E}\|_{H^1(\Gamma)} + h\|\phi_\mathcal{E}\|_{H^1(\Gamma)} \\ &\lesssim k^{-1/2}(h+kh)\|e_0\|_0,\end{aligned}$$

which implies

$$|l_u(Q_h\phi)| \lesssim (h+kh)\left(E(u) + k^{-1/2}\left\|\frac{\partial u}{\partial \mathbf{n}} - Q_b\frac{\partial u}{\partial \mathbf{n}}\right\|_{L^2(\Gamma)}\right)\|e_0\|_0. \tag{4.13}$$

If $p\geq 2$, we know that $Q_b\phi_{\mathcal{E}}$ is at least in $\mathcal{P}_1(e)$ for any $e\in\mathcal{E}_h^B$ and get

$$\begin{aligned}\|\phi-Q_b\phi\|_{L^2(\Gamma)}&\lesssim h^{3/2}\|\phi_{\mathcal{E}}\|_2+h^p\|\phi_{\mathcal{A}}\|_{H^p(\Gamma)}\\&\lesssim(h^{3/2}+h^pk^{p-1/2})\|e_0\|_0,\end{aligned}$$

which implies

$$|l_u(Q_h\phi)|\lesssim\left(\left(h+(kh)^p\right)E(u)+\left(h^{3/2}+h^pk^{p-1/2}\right)\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}\right)\|e_0\|_0. \tag{4.14}$$

In the second case that $Q_b\phi\in\mathcal{P}_p(e)$, that is $u_h\in V_h^2$. The fact that

$$\begin{aligned}\|\phi-Q_b\phi\|_{L^2(\Gamma)}&\lesssim\|\phi-Q_0\phi\|_{L^2(\Gamma)},\\\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}&\lesssim\|\nabla u-\mathbb{Q}_h\nabla u\|_{L^2(\Gamma)},\end{aligned}$$

implies that

$$|l_u(Q_h\phi)|\lesssim\left(h+(kh)^p\right)E(u)+k^2\|e_0\|_0. \tag{4.15}$$

By combining (4.7)-(4.15) and Lemma 3.6, we obtain:

(i) for $u_h\in V_h^1$ with $p=1$,

$$\begin{aligned}\|e_0\|_0^2\lesssim&(h+kh)\left(E(u)+k^{-1/2}\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}\right)\|e_0\|_0\\&+k^{2+3/2}h^2\|e_0\|_0^2,\end{aligned}$$

which implies that there exists a constant C_0 independent of k and h, such that if $k^{7/2}h^2\leq C_0$,

$$\|e_0\|_0\lesssim(h+kh)\left(E(u)+k^{-1/2}\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}\right); \tag{4.16}$$

(ii) for $u_h\in V_h^1$ with ≥ 2,

$$\begin{aligned}\|e_0\|_0^2\lesssim&\left(\left(h+(kh)^p\right)E(u)+\left(h^{3/2}+h^pk^{p-1/2}\right)\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}\right)\|e_0\|_0\\&+((kh)^2+k(kh)^{p+1})\|e_0\|_0,\end{aligned}$$

which implies that there exists a constant C_0 independent of k and h, such that if $(kh)^2+k(kh)^{p+1}\leq C_0$,

$$\|e_0\|_0\lesssim\left(\left(h+(kh)^p\right)E(u)+\left(h^{3/2}+k^{p-1/2}h^p\right)\left\|\frac{\partial u}{\partial\mathbf{n}}-Q_b\frac{\partial u}{\partial\mathbf{n}}\right\|_{L^2(\Gamma)}\right); \tag{4.17}$$

(iii) for $u_h \in V_h^2$,

$$\|e_0\|_0^2 \lesssim (h+(kh)^p)E(u)\|e_0\|_0 + ((kh)^2+k(kh)^{p+1})\|e_0\|_0^2,$$

which implies that there exists a constant C_0 independent of k and h, such that if $(kh)^2+k(kh)^{p+1} \leq C_0$,

$$\|e_0\|_0 \lesssim (h+(kh)^p)E(u). \tag{4.18}$$

From (4.16)-(4.18), we complete the proof for $\|e_0\|_0$.

Finally, the estimate of $|||e_h|||$ can be easily obtained by its definition and Lemma 3.5,

$$\begin{aligned}
|||e_h|||^2 &= \Re\big(a(e_h,e_h)+\mathbf{i}k\langle e_b,e_b\rangle\big)+\Im\big(a(e_h,e_h)+\mathbf{i}k\langle e_b,e_b\rangle\big)\\
&= \Re\big(a(e_h,e_h)-k^2(e_0,e_0)+\mathbf{i}k\langle e_b,e_b\rangle\big)+k^2(e_0,e_0)\\
&\quad +\Im\big(a(e_h,e_h)-k^2(e_0,e_0)+\mathbf{i}k\langle e_b,e_b\rangle\big)\\
&= \Re\big(l_u(e_h)+s(Q_hu,e_h)\big)+\Im\big(l_u(e_h)+s(Q_hu,e_h)\big)+k^2\|e_0\|_0^2\\
&\lesssim E(u)|||e_h|||+k^2\|e_0\|_0^2.
\end{aligned} \tag{4.19}$$

By combining (4.16)-(4.19), we complete the proof for $|||e_h|||$. □

Remark 4.1. (a) The mesh condition $k^{7/2}h^2 \leq C_0$ and $(kh)^2+k(kh)^{p+1} \leq C_0$ in Theorem 4.1 and Corollary 4.1-4.2 may not be "optimal". For the CIP-FEM (including the classic FEM) and the high order IP-DG, the pre-asymptotic error estimates under a mesh condition $k(kh)^{2p} \leq C_0$ have been established [11,12].

(b) From the details of proof, it is easy to see that the same error estimates also can be obtained for $u_h \in V_h^2$ by using fewer degree of freedoms if V_h^2 (2.2) is redefined as

$$V_h^2 := \{v=\{v_0,v_b\} : v|_K \in S_2(p,K) \text{ and } v_b|_e \in \mathcal{P}_{p-1}(e)\ \forall K \in \mathcal{T}_h, e \in \mathcal{E}_h^I\}.$$

By combining Lemma 3.2, with the approximating properties of the L^2 projections $\mathbb{Q}_h$ and Q_h and Theorem 4.1, we have the following corollary, which gives pre-asymptotic estimates for H^{p+2}-regular solutions.

Corollary 4.1. *Let u be the solution to* (1.1)-(1.2) *and let u_h be the weak FEM solution. Assume that u is in $H^{p+2}(\Omega)$ for $u_h \in V_h^1$ and in $H^{p+1}(\Omega)$ for $u_h \in V_h^2$. There exist constants C_0,C_1 and C_2 independent of k and h such that the following estimates hold:*

(i) for $u_h \in V_h^1$ with $p=1$, if $k^{7/2}h^2 \leq C_0$,

$$\begin{aligned}
\|Q_0u-u_0\|_0 &\lesssim (kh)^2 C_{1,f,g},\\
|||Q_hu-u_h||| &\lesssim \big(kh+k(kh)^2\big)C_{1,f,g};
\end{aligned}$$

(ii) for $u_h \in V_h^1$ with $p \geq 2$, if $(kh)^2 + k(kh)^{p+1} \leq C_0$,

$$\|Q_0 u - u_0\|_0 \lesssim \big(h(kh)^p + (kh)^{2p}\big) C_{p,f,g},$$
$$|||Q_h u - u_h||| \lesssim \big((kh)^p + k(kh)^{2p}\big) C_{p,f,g};$$

(iii) for $u_h \in V_h^2$, if $(kh)^2 + k(kh)^{p+1} \leq C_0$,

$$\|Q_0 u - u_0\|_0 \lesssim \big(h(kh)^p + (kh)^{2p}\big) C_{p-1,f,g},$$
$$|||Q_h u - u_h||| \lesssim \big((kh)^p + k(kh)^{2p}\big) C_{p-1,f,g}.$$

Remark 4.2. (a) By dispersion analysis, an important tool to understand numerical behaviors in short wave computations, it has been found that error between the wave number k of the continuous problem and some discrete number ω [1,9,16,18,19,28,29] for the FEM is

$$k - \omega = O\big(k^{2p+1} h^{2p}\big) \quad \text{if } kh \ll 1,$$

which coincides with our estimates.

(b) Corollary 4.1 shows that $u \in H^{p+2}(\Omega)$ is needed for $u_h \in V_h^1$ rather than $u \in H^{p+1}(\Omega)$ for $u_h \in V_h^2$. Whether the estimates are sharp with respect to the regularity of u needs further verification in the future work.

By combining Lemmas 3.1 and Theorem 4.1 we have the following stability estimates for the WG-FEM.

Corollary 4.2. *Suppose the solution $u \in H^2(\Omega)$. Under the mesh conditions of Theorem 4.1, the following estimate holds:*

$$|||u_h||| + k\|u_h\|_0 \lesssim C_{1,f,g} \quad \forall u_h \in V_h^1,$$
$$|||u_h||| + k\|u_h\|_0 \lesssim C_{f,g} \quad \forall u_h \in V_h^2.$$

This Corollary shows that the WG-FEM is well-posed under the mesh condition in Theorem 4.1.

5 Numerical examples

In this section, we will simulate the following two-dimensional Helmholtz problem:

$$-\Delta u - k^2 u = f := \frac{\sin(kr)}{r} \qquad \text{in } \Omega, \tag{5.1}$$
$$\frac{\partial u}{\partial n} + \mathbf{i} k u = g \qquad \text{on } \Gamma. \tag{5.2}$$

Here Ω is the unit regular hexagon with center $(1,\sqrt{3}/2)$ and g is so chosen that the exact solution is

$$u=\frac{\cos(kr)}{r}-\frac{\cos k+\mathbf{i}\sin k}{k\big(J_0(k)+\mathbf{i}J_1(k)\big)}J_0(kr) \tag{5.3}$$

in polar coordinates, where $J_\nu(z)$ are Bessel functions of the first kind.

We shall use the uniform triangulation consisting of congruent and equilateral triangles of size h in the following numerical tests. Let $u_h=\{u_0,u_b\}$ be the numerical solution to (5.1)-(5.2).

We refer the reader to [11, 12, 14, 33] for this problem computed by other numerical methods on both triangular meshes and rectangular meshes, such as the continuous interior penalty finite element method and the interior penalty discontinuous Galerkin method.

5.1 Linear WG-FEM

We consider the first case that $u_h\in V_h^1$ with $p=1$. From (2.4), Lemma 3.1 and Lemma 3.3, we can easily get

$$\Big(\sum_{K\in\mathcal{T}_h}\|\nabla_{w,0,K}Q_hu-\nabla u\|^2_{L^2(K)}\Big)^{1/2}\lesssim h+kh.$$

By combining the inequality above and Corollary 4.1, the error of the WG finite element solution in the broken H^1-seminorm is bounded by

$$\Big(\sum_{K\in\mathcal{T}_h}\|\nabla_{w,0,K}u_h-\nabla u\|^2_{L^2(K)}\Big)^{1/2}\leq C_1kh+C_2k(kh)^2 \tag{5.4}$$

for some constants C_1 and C_2 independent of k and h if $k^{7/2}h^2\leq C_0$.

The second term on the right-hand side of (5.4) is the so-called pollution error. We now verify the error bounds by numerical results.

We first show the relative errors and convergence rates of linear WG-FEM solutions and linear interpolations for $k=10,50,200$ in Tables 1, 2 and 3. When $k=10$, the relative errors of WG-FEM solutions are almost equal to those of interpolations. However, when $k=50$ and 200, the relative errors of WG-FEM solutions are larger than those of interpolations although all the relative errors are almost equal when h is sufficiently small, which implies the existence of the pollution errors. We remark that the numerical tests for the pollution phenomenon of the finite element method have been done largely in the literature and the reader is referred to [11] and the references therein for more details.

We verify more precisely the pollution term in (5.4). To do so, we introduce the definition of the critical mesh size with respect to a given relative tolerance [11,32].

Table 1: The relative errors of linear WG-FEM solution u_h and linear interpolation u_I with $k=10$ where $|u-u_h|_{1,w}=\left(\sum_{K\in\mathcal{T}_h}\|\nabla_{w,0,K}u_h-\nabla u\|^2_{L^2(K)}\right)^{1/2}$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_I\|_1/\|u\|_1$	order
$\frac{1}{4}$	1.0645		0.5712	
$\frac{1}{8}$	0.3501	1.6044	0.3007	0.9257
$\frac{1}{16}$	0.1583	1.1447	0.1523	0.9815
$\frac{1}{32}$	0.0771	1.0374	0.0764	0.9954
$\frac{1}{64}$	0.0383	1.0094	0.0382	0.9988
$\frac{1}{128}$	0.0191	1.0024	0.0191	0.9997
$\frac{1}{256}$	0.0096	1.0006	0.0096	0.9999
$\frac{1}{512}$	0.0048	1.0001	0.0048	1.0000
$\frac{1}{1024}$	0.0024	1.0000	0.0024	1.0000

Table 2: The relative errors of linear WG-FEM solution u_h and linear interpolation u_I with $k=50$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_I\|_1/\|u\|_1$	order
$\frac{1}{4}$	0.9999		1.0098	
$\frac{1}{8}$	0.9942	0.0083	1.0274	-0.0249
$\frac{1}{16}$	1.1363	-0.1927	0.6994	0.5549
$\frac{1}{32}$	1.1803	-0.0548	0.3788	0.8846
$\frac{1}{64}$	0.3821	1.6272	0.1933	0.9710
$\frac{1}{128}$	0.1276	1.5825	0.0971	0.9927
$\frac{1}{256}$	0.0528	1.2719	0.0486	0.9982
$\frac{1}{512}$	0.0249	1.0875	0.0243	0.9995
$\frac{1}{1024}$	0.0122	1.0236	0.0122	0.9999

Table 3: The relative errors of linear WG-FEM solution u_h and linear interpolation u_I with $k=200$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_I\|_1/\|u\|_1$	order
$\frac{1}{4}$	1.0001		1.0008	
$\frac{1}{8}$	0.9998	0.0004	0.9982	0.0038
$\frac{1}{16}$	1.0000	-0.0002	1.0093	-0.0159
$\frac{1}{32}$	0.9986	0.0021	1.0260	-0.0237
$\frac{1}{64}$	1.1552	-0.2102	0.7006	0.5503
$\frac{1}{128}$	1.3238	-0.1966	0.3798	0.8834
$\frac{1}{256}$	1.1095	0.2548	0.1938	0.9708
$\frac{1}{512}$	0.3419	1.6985	0.0974	0.9927
$\frac{1}{1024}$	0.0964	1.8261	0.0488	0.9982

Definition 5.1. Given a relative tolerance ε and a wave number k, the critical mesh size $h(k,\varepsilon)$ with respect to the relative tolerance ε is defined by the maximum mesh size such that the relative error of the WG finite element solution in the H^1-seminorm is less than or equal to ε

Clearly, if the pollution terms (5.4) are of order k^3h^2, then $h(k,\varepsilon)$ should be proportional to $k^{-3/2}$ for k sufficiently large. This is verified by Table 4.

Table 4: $h(k,0.5)$ and $h(k,0.1)$ for linear WG-FEM solutions.

k	$h(k,0.5)$	order	k	$h(k,0.1)$	order
6	0.2500		6	0.0625	
50	0.0182	-1.2362	30	0.0120	-1.0229
94	0.0073	-1.4457	54	0.0059	-1.2097
138	0.0041	-1.4710	78	0.0037	-1.2942
182	0.0028	-1.4800	102	0.0026	-1.3528
226	0.0020	-1.4880	126	0.0019	-1.3765
270	0.0015	-1.4809	150	0.0015	-1.3949
314	0.0012	-1.4862	174	0.0012	-1.4160
358	0.0010	-1.4893	198	0.0010	-1.4220

In this example, we set $\rho=20$ because of the assumption that ρ is positive for simplicity of proof. When ρ is non-negative, it is easy to see that the sesquilinear form $a(\cdot,\cdot)$ is at least a semi-positive definite operator, which is essential for the elliptic problems. However, because of the highly indefinite nature of the Helmholtz problem for large k, the WG finite element method may perform better with negative ρ. For comparison, we show the relative errors of the WG method solution with $\rho=-4.6$ and the FEM solution in Tables 5-7. By comparing with Tables 1-3, it is easy to see that the WG-FEM with $\rho=-4.6$ performs much better than the one with $\rho=20$ and the classical FEM.

5.2 Quadratic WG-FEM

We consider the first case that $u_h \in V_h^1$ with $p=2$ by setting $\rho=10$. It is easy to get the following inequality similar to (5.4)

$$\left(\sum_{K\in\mathcal{T}_h} \|\nabla_{w,1,K}u_h-\nabla u\|_{L^2(K)}^2\right)^{1/2} \le C_1(kh)^2+C_2k(kh)^4, \tag{5.5}$$

where C_1 and C_2 are constants independent of k and h under the mesh condition $k(kh)^3$.

Let u_I be the quadratic interpolation of u and let

$$|u-u_h|_{1,w}=\Big(\sum_{K\in\mathcal{T}_h} \|\nabla_{w,1,K}u_h-\nabla u\|_{L^2(K)}^2\Big)^{1/2}.$$

Table 5: The relative errors of linear WG-FEM solution u_h and linear FEM solution u_h^{FEM} with $k=10$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_h^{\mathrm{FEM}}\|_1/\|u\|_1$	order
$\frac{1}{4}$	0.6615		0.9189	
$\frac{1}{8}$	0.3083	1.1011	0.4348	1.0797
$\frac{1}{16}$	0.1532	1.0090	0.1776	1.2915
$\frac{1}{32}$	0.0765	1.0019	0.0800	1.1507
$\frac{1}{64}$	0.0382	1.0005	0.0387	1.0478
$\frac{1}{128}$	0.0191	1.0001	0.0192	1.0128
$\frac{1}{256}$	0.0096	1.0000	0.0096	1.0032
$\frac{1}{512}$	0.0048	1.0000	0.0048	1.0008
$\frac{1}{1024}$	0.0024	1.0000	0.0024	1.0002

Table 6: The relative errors of linear WG-FEM solution u_h and linear FEM solution u_h^{FEM} with $k=50$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_h^{\mathrm{FEM}}\|_1/\|u\|_1$	order
$\frac{1}{4}$	1.0033	0	1.0000	0
$\frac{1}{8}$	1.5714	-0.6474	1.0045	-0.0064
$\frac{1}{16}$	1.4720	0.0943	1.2034	-0.2606
$\frac{1}{32}$	0.4010	1.8761	1.4144	-0.2331
$\frac{1}{64}$	0.1953	1.0379	0.7520	0.9114
$\frac{1}{128}$	0.0974	1.0043	0.2212	1.7655
$\frac{1}{256}$	0.0486	1.0010	0.0700	1.6600
$\frac{1}{512}$	0.0243	1.0002	0.0274	1.3531
$\frac{1}{1024}$	0.0122	1.0001	0.0126	1.1249

Table 7: The relative errors of linear WG-FEM solution u_h and linear FEM solution u_h^{FEM} with $k=200$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_h^{\mathrm{FEM}}\|_1/\|u\|_1$	order
$\frac{1}{4}$	0.9999	0	1.0000	0
$\frac{1}{8}$	1.0004	-0.0008	1.0002	-0.0003
$\frac{1}{16}$	1.0009	-0.0007	1.0000	0.0002
$\frac{1}{32}$	1.3017	-0.3792	1.0011	-0.0016
$\frac{1}{64}$	1.4432	-0.1488	1.1832	-0.2411
$\frac{1}{128}$	0.4843	1.5752	1.2744	-0.1071
$\frac{1}{256}$	0.1970	1.2982	1.4710	-0.2069
$\frac{1}{512}$	0.0977	1.0119	0.7813	0.9128
$\frac{1}{1024}$	0.0488	1.0013	0.2094	1.8998

Table 8: The relative errors of quadratic WG-FEM solution u_h and quadratic interpolation u_I with $k=50$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_I\|_1/\|u\|_1$	order
$\frac{1}{4}$	1.4195		1.5032	
$\frac{1}{8}$	1.5994	-0.1722	1.0228	-0.5555
$\frac{1}{16}$	3.8820e-01	2.0427	3.3834e-01	1.5960
$\frac{1}{32}$	7.8465e-02	2.3067	9.0585e-02	1.9011
$\frac{1}{64}$	1.9479e-02	2.0101	2.3034e-02	1.9755
$\frac{1}{128}$	4.8766e-03	1.9980	5.7831e-03	1.9939
$\frac{1}{256}$	1.2198e-03	1.9992	1.4473e-03	1.9985
$\frac{1}{512}$	3.0501e-04	1.9998	3.6192e-04	1.9996

Table 9: The relative errors of quadratic WG-FEM solution u_h and quadratic interpolation u_I with $k=200$.

h	$\|u-u_h\|_{1,w}/\|u\|_1$	order	$\|u-u_I\|_1/\|u\|_1$	order
$\frac{1}{4}$	1.4118		1.4071	
$\frac{1}{8}$	1.4114	4.1659e-04	1.4108	-3.8128e-03
$\frac{1}{16}$	1.4145	-3.2113e-03	1.4976	-8.6115e-02
$\frac{1}{32}$	1.5863	-1.6533e-01	1.0354	5.3236e-01
$\frac{1}{64}$	9.0907e-01	8.0316e-01	3.4036e-01	1.6051
$\frac{1}{128}$	9.6938e-02	3.2293	9.1115e-02	1.9013
$\frac{1}{256}$	1.9912e-02	2.2834	2.3171e-02	1.9754
$\frac{1}{512}$	4.9106e-03	2.0197	5.8174e-03	1.9939

Table 8 and Table 9 show the relative errors and convergence rates of quadratic WG-FEM solutions and quadratic interpolations for $k=50$ and $k=200$. It is shown that the relative errors of quadratic WG-FEM solutions fit those of the corresponding interpolations very well even if $k=200$ (cf. Table 9), which means that the WG-FEM is a very efficient numerical methods for solving the Helmholtz problem with high wave number. However, we emphasize that the pollution errors are reduced greatly, but not eliminated. To illustrate this, we show the relative errors of both WG-FEM solutions and interpolations with fixed $kh=1$ and $kh=2$ shown in Fig. 1.

Finally, We refer the reader to [38] for more numerical examples for various Helmholtz problems solved by WG-FEMs.

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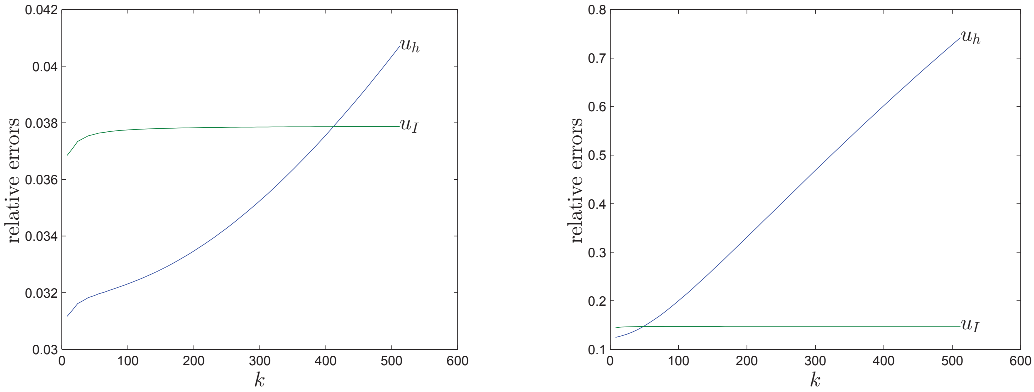

Figure 1: The relative errors of quadratic WG-FEM solutions and interpolations for $kh=1$ (left) and $kh=2$ (right).

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References

[1] M. Ainsworth, *Discrete dispersion relation for hp-version finite element approximation at high wave number*, SIAM J. Numer. Anal., 42 (2004), pp. 553–575.

[2] A. Aziz and R. Kellogg, *A scattering problem for the Helmholtz equation*, in Advances in Computer Methods for Partial Differential Equations-III, vol. 1, 1979, pp. 93–95.

[3] I. Babuška, F. Ihlenburg, E. Paik, and S. Sauter, *A generalized finite element method for solving the Helmholtz equation in two dimensions with minimal pollution*, Comput. Methods Appl. Mech. Engrg., 128 (1995), pp. 325–359.

[4] I. Babuška and S. Sauter, *Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers?*, SIAM Rev., 42 (2000), pp. 451–484.

[5] S. Brenner and L. Scott, *The mathematical theory of finite element methods*, Springer, New York, third ed., 2008.

[6] H. Chen, P. Lu, and X. Xu, *A hybridizable discontinuous Galerkin method for the Helmholtz equation with high wave number*, SIAM J. Numer. Anal.,, 51 (2013), pp. 2166–2188.

[7] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Pub. Co., New York, 1978.

[8] L. Demkowicz, J. Gopalakrishnan, I. Muga, and J. Zitelli, *Wavenumber explicit analysis of a DPG method for the multidimensional Helmholtz equation*, Comput. Methods Appl. Mech. Engrg., 214 (2012), pp. 126–138.

[9] A. Deraemaeker, I. Babuška, and P. Bouillard, *Dispersion and pollution of the FEM solution for the Helmholtz equation in one, two and three dimensions*, Internat. J. Numer. Methods Engrg., 46 (1999), pp. 471–499.

[10] J. DOUGLAS JR, J. SANTOS, AND D. SHEEN, *Approximation of scalar waves in the space-frequency domain*, Math. Models Methods Appl. Sci., 4 (1994), pp. 509–531.

[11] Y. DU AND H. WU, *Preasymptotic error analysis of higher order fem and cip-fem for Helmholtz equation with high wave number*, SIAM J. Numer. Anal., 53 (2015), pp. 782–804.

[12] Y. DU AND L. ZHU, *Preasymptotic error analysis of high order interior penalty discontinuous Galerkin methods for the Helmholtz equation with high wave number*, J. Sci. Comput., 67 (2016), pp. 130–152.

[13] B. ENGQUIST AND A. MAJDA, *Radiation boundary conditions for acoustic and elastic wave calculations*, Comm. Pure Appl. Math., 32 (1979), pp. 313–357.

[14] X. FENG AND H. WU, *Discontinuous Galerkin methods for the Helmholtz equation with large wave numbers*, SIAM J. Numer. Anal., 47 (2009), pp. 2872–2896.

[15] ——, *hp-discontinuous Galerkin methods for the Helmholtz equation with large wave number*, Math. Comp., 80 (2011), pp. 1997–2024.

[16] I. HARARI, *Reducing spurious dispersion, anisotropy and reflection in finite element analysis of time-harmonic acoustics*, Comput. Meth. Appl. Mech. Engrg., 140 (1997), pp. 39–58.

[17] F. IHLENBURG, *Finite element analysis of acoustic scattering*, vol. 132 of Applied Mathematical Sciences, Springer-Verlag, New York, 1998.

[18] F. IHLENBURG AND I. BABUŠKA, *Finite element solution of the Helmholtz equation with high wave number. I. The h-version of the FEM*, Comput. Math. Appl., 30 (1995), pp. 9–37.

[19] ——, *Finite element solution of the Helmholtz equation with high wave number. II. The h-p version of the FEM*, SIAM J. Numer. Anal., 34 (1997), pp. 315–358.

[20] J. MELENK, A. PARSANIA, AND S. SAUTER, *General DG-methods for highly indefinite Helmholtz problems*, Journal of Scientific Computing, 57 (2013), pp. 536–581.

[21] J. M. MELENK AND S. SAUTER, *Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions*, Math. Comp., 79 (2010), pp. 1871–1914.

[22] ——, *Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation*, SIAM J. Numer. Anal., 49 (2011), pp. 1210–1243.

[23] P. MONK, *Finite element methods for Maxwell's equations*, Oxford University Press, New York, 2003.

[24] L. MU, J. WANG, AND X. YE, *A new weak Galerkin finite element method for the Helmholtz equation*, IMA Journal of Numerical Analysis, 35 (2014), pp. 1228–1255.

[25] L. MU, J. WANG, X. YE, AND S. ZHAO, *A numerical study on the weak Galerkin method for the helmholtz equation*, Communications in Computational Physics, 15 (2014), pp. 1461–1479.

[26] A. SCHATZ, *An observation concerning Ritz–Galerkin methods with indefinite bilinear forms*, Math. Comp., 28 (1974), pp. 959–962.

[27] J. SHEN AND L. WANG, *Analysis of a spectral-Galerkin approximation to the Helmholtz equation in exterior domains*, SIAM J. Numer. Anal., 45 (2007), pp. 1954–1978.

[28] L. THOMPSON, *A review of finite-element methods for time-harmonic acoustics*, J. Acoust. Soc. Am., 119 (2006), pp. 1315–1330.

[29] L. THOMPSON AND P. PINSKY, *Complex wavenumber Fourier analysis of the p-version finite element method*, Comput. Mech., 13 (1994), pp. 255–275.

[30] J. WANG AND C. WANG, *Weak Galerkin finite element methods for elliptic pdes (in chinese)*, Sci. Sin. Math, 45 (2015), pp. 1061–1092.

[31] J. WANG AND X. YE, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comp. Appl. Math., 214 (2013), pp. 103–115.

[32] H. WU, *Pre-asymptotic error analysis of CIP-FEM and FEM for Helmholtz equation with high wave*

number. Part I: Linear version, IMA J. Numer. Anal., 34 (2014), pp. 1266–1288.

[33] L. ZHU AND Y. DU, *Pre-asymptotic error analysis of hp-interior penalty discontinuous Galerkin methods for the Helmholtz equation with large wave number*, Comput. Math. Appl., 70 (2015), pp. 917–933.

[34] L. ZHU AND H. WU, *Pre-asymptotic error analysis of CIP-FEM and FEM for Helmholtz equation with high wave number. Part II: hp version*, SIAM J. Numer. Anal., 51 (2013), pp. 1828–1852.

[35] J. ZITELLI, I. MUGA, L. DEMKOWICZ, J. GOPALAKRISHNAN, D. PARDO, AND V. CALO, *A class of discontinuous Petrov-Galerkin methods. Part IV: The optimal test norm and time-harmonic wave propagation in 1D*, J. Comput. Phys., 230 (2011), pp. 2406 – 2432.

[36] R. ZHANG AND Q. ZHAI, *A weak Galerkin finite element scheme for the biharmonic equa- tions by using polynomials of reduced order*, J. Sci. Comput., 64(2015), pp. 559 – 585.

[37] Q. ZHAI, R. ZHANG AND L. MU, *A new weak Galerkin finite element scheme for the Brinkman model*, Commun. Comput. Phys., 19(2016), pp. 1409 – 1434.

[38] L. MU, J. WANG, X. YE AND S. ZHANG, *A weak Galerkin finite element method for the Maxwell equations*, (English summary) J. Sci. Comput., 65(2015), pp. 363 – 386.