

## HESSIAN RECOVERY FOR FINITE ELEMENT METHODS

HAILONG GUO, ZHIMIN ZHANG, AND REN ZHAO

ABSTRACT. In this article, we propose and analyze an effective Hessian recovery strategy for the Lagrangian finite element method of arbitrary order. We prove that the proposed Hessian recovery method preserves polynomials of degree  $k + 1$  on general unstructured meshes and superconverges at a rate of  $O(h^k)$  on mildly structured meshes. In addition, the method is proved to be ultraconvergent (two orders higher) for the translation invariant finite element space of any order. Numerical examples are presented to support our theoretical results.

### 1. INTRODUCTION

Post-processing is an important technique in scientific computing, where it is necessary to draw some useful information that have physical meanings such as velocity, flux, stress, etc., from the primary results of the computation. These quantities of interest usually involve derivatives of the primary data. Some popular post-processing techniques include the celebrated Zienkiewicz-Zhu superconvergent patch recovery (SPR) [28], polynomial preserving recovery (PPR) [16, 27], and edge based recovery [21], which were proposed to obtain accurate gradients with reasonable cost. Similarly, post-processing for second order derivatives, which are related to physical quantities such as momentum and Hessian, are also desirable. The Hessian matrix is particularly significant in adaptive mesh design, since it can indicate the direction where the function changes the most and guide us to construct anisotropic meshes to cope with the anisotropic properties of the solution of the underlying partial differential equation [3, 5]. It also plays an important role in finite element approximation of second order nonvariational elliptic problems [13], numerical solution of some fully nonlinear equations such as the Monge-Ampère equation [14, 18], and designing a nonlocal finite element technique [8].

There have been some works in literature on this subject. In 1998, Lakhany-Whiteman used a simple averaging twice at edge centers of the regular uniform triangular mesh to produce a superconvergent Hessian [12]. Later, some other researchers such as Agouzal et al. [1], Oval [20], and Aguilera et al. [2] also studied Hessian recovery. Comparison studies of existing Hessian recovery techniques are found in Vallet et al. [23] and Picasso et al. [22]. However, there is no systematic theory that guarantees convergence of recovered Hessian in general circumstances. Moreover, there are certain technical difficulties in obtaining rigorous convergence

---

Received by the editor July 17, 2014 and, in revised form, December 14, 2015.

2010 *Mathematics Subject Classification*. Primary 65N50, 65N30; Secondary 65N15.

*Key words and phrases*. Hessian recovery, gradient recovery, ultraconvergence, superconvergence, finite element method, polynomial preserving.

The second author is the corresponding author. The research of the second author was supported in part by the National Natural Science Foundation of China under grants 11471031, 91430216 U1530401, and the U.S. National Science Foundation through grant DMS-1419040.

proof for meshes other than the regular pattern triangular mesh. In a very recent work, Kamenski-Huang argued that it is not necessary to have very accurate or even convergent Hessian in order to obtain a good mesh [11].

Our current work is not targeted in the direction of adaptive mesh refinement; instead, our emphasis is to obtain accurate Hessian matrices via recovery techniques. We propose an effective Hessian recovery method and establish a solid theoretical analysis for such recovery methods. Our approach is to apply PPR twice to the primarily computed data. This idea is natural. However, the mathematical theory behind it is nontrivial and quite involved, especially in the ultraconvergence analysis of the recovered Hessian. A direct calculation of the gradient from the linear finite element space has linear convergent rate and the Hessian has no convergence at all. Our Hessian recovery method can achieve second order convergence under some uniform meshes, which is a very surprising result! In particular, the proposed method is the only one of all Hessian recovery methods that is ultraconvergent on the Chevron pattern uniform mesh.

The rest of the paper is organized as follows. We begin in Section 2 with an introduction of some notation and definition of polynomial preserving recovery. Then, in Section 3, we define the Hessian recovery operator and use two examples to show how it relates to the finite difference operator. Also, in this section, we analyze the consistence of the Hessian recovery operator by proving its polynomial preserving property. In Section 4, we prove superconvergence of our Hessian recovery operator on mildly unstructured mesh and ultraconvergence on translation invariant mesh. Section 5 is devoted to numerical comparison of the proposed Hessian recovery method with some popular Hessian recovery methods in the literature and illustration of our theoretical results. Finally, some conclusions are drawn in Section 6.

Throughout this article, the letter  $C$  or  $c$ , with or without subscript, denotes a generic constant which is independent of  $h$  and may not be the same at each occurrence. For convenience, we denote  $x \leq Cy$  by  $x \lesssim y$ .

## 2. PRELIMINARIES

In this section, we first introduce some notation and then briefly describe the polynomial preserving recovery (PPR) operator [16, 27], which is a basis of our Hessian recovery method.

**2.1. Notation.** Let  $\Omega$  be a bounded polygonal domain with Lipschitz boundary  $\partial\Omega$  in  $\mathbb{R}^2$ . Throughout this article, the standard notation for Sobolev spaces and their associate norms are adopted as in [4, 6]. A 2-index  $\alpha$  is a pair of nonnegative integers  $\alpha_i$ ,  $i = 1, 2$ . The length of  $\alpha$  is given by

$$|\alpha| = \sum_{i=1}^2 \alpha_i.$$

We adopt the same notation for derivatives as in the textbook by Evans [7]. We usually write  $u_x$  (or  $u_y$ ) for weak derivative  $\frac{\partial u}{\partial x}$  (or  $\frac{\partial u}{\partial y}$ ). Similarly,  $\frac{\partial^2 u}{\partial x \partial y} = u_{xy}$ ,  $\frac{\partial^3 u}{\partial x^2 \partial y} = u_{xxy}$ , etc. Given a 2-index  $\alpha$ , define

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}.$$

Given any nonnegative integer  $k$ ,  $D^k u$  is the tensor of all partial derivatives of order  $k$ . The Hessian operator  $H$  is denoted by

$$(2.1) \quad H = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} \end{pmatrix}.$$

For a subdomain  $\mathcal{A}$  of  $\Omega$ , let  $\mathbb{P}_m(\mathcal{A})$  be the space of polynomials of degree less than or equal to  $m$  over  $\mathcal{A}$  and let  $n_m$  be the dimension of  $\mathbb{P}_m(\mathcal{A})$  with  $n_m = \frac{1}{2}(m+1)(m+2)$ .  $W_p^k(\mathcal{A})$  denotes the Sobolev space with norm  $\|\cdot\|_{k,p,\mathcal{A}}$  and seminorm  $|\cdot|_{k,p,\mathcal{A}}$ . When  $p = 2$ , we denote simply  $H^k(\mathcal{A}) = W_2^k(\mathcal{A})$  and the subscript  $p$  is omitted.

For any  $0 < h < \frac{1}{2}$ , let  $\mathcal{T}_h$  be a shape regular triangulation of  $\bar{\Omega}$  with mesh size at most  $h$ , i.e.,

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K,$$

where  $K$  is a triangle. For any  $k \in \mathbb{N}$ , define the continuous finite element space  $S_h$  of order  $k$  as

$$S_h = \{v \in C(\bar{\Omega}) : v|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h\} \subset H^1(\Omega).$$

Let  $\mathcal{N}_h$  denote the set of mesh nodes. The standard Lagrange basis of  $S_h$  is denoted by  $\{\phi_z : z \in \mathcal{N}_h\}$  with  $\phi_z(z') = \delta_{zz'}$  for all  $z, z' \in \mathcal{N}_h$ . For any  $v \in H^1(\Omega) \cap C(\Omega)$ , let  $v_I$  be the interpolation of  $v$  in  $S_h$ , i.e.,  $v_I = \sum_{z \in \mathcal{N}_h} v(z)\phi_z$ .

For any vertex  $z$  and  $n \in \mathbb{Z}^+$ , let  $\mathcal{L}(z, n)$  denote the union of mesh elements in the first  $n$  layers around  $z$ , i.e.,

$$(2.2) \quad \mathcal{L}(z, n) := \bigcup \{\tau : \tau \in \mathcal{T}_h, \tau \cap \mathcal{L}(z, n-1) \neq \emptyset\},$$

where  $\mathcal{L}(z, 0) := \{z\}$ .

For  $\mathcal{A} \subset \Omega$ , let  $S_h(\mathcal{A})$  denote the restrictions of functions in  $S_h$  to  $\mathcal{A}$  and let  $S_h^{\text{comp}}(\mathcal{A})$  denote the set of those functions in  $S_h(\mathcal{A})$  with compact support in the interior of  $\mathcal{A}$  [24]. Let  $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega$  be separated by  $d \geq c_0 h$  and  $\ell$  be a direction, i.e., a unit vector in  $\mathbb{R}^2$ . Let  $\tau$  be a parameter, which will typically be a multiple of  $h$ . Let  $T_\tau^\ell$  denote translation by  $\tau$  in the direction  $\ell$ , i.e.,

$$(2.3) \quad T_\tau^\ell v(x) = v(x + \tau \ell),$$

and for an integer  $\nu$ ,

$$(2.4) \quad T_{\nu\tau}^\ell v(x) = v(x + \nu\tau \ell).$$

Following the definition of [24], the finite element space  $S_h$  is called translation invariant by  $\tau$  in the direction  $\ell$  if

$$(2.5) \quad T_{\nu\tau}^\ell v \in S_h^{\text{comp}}(\Omega), \quad \forall v \in S_h^{\text{comp}}(\Omega_1),$$

for some integer  $\nu$  with  $|\nu| < M$ . Equivalently,  $\mathcal{T}_h$  is called a translation invariant mesh. To clarify the matter, we consider five popular triangular mesh patterns: Regular, Chevron, Criss-cross, Union-Jack, and Equilateral patterns, as shown in Figure 1.

We see that:

- 1) Regular pattern is translation invariant by  $h$  in directions  $(1, 0)$  and  $(0, 1)$ , by  $2\sqrt{2}h$  in directions  $(\pm \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , and by  $\sqrt{5}h$  in directions  $(\frac{2\sqrt{5}}{5}, \pm \frac{\sqrt{5}}{5})$  and  $(\pm \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$ , etc.

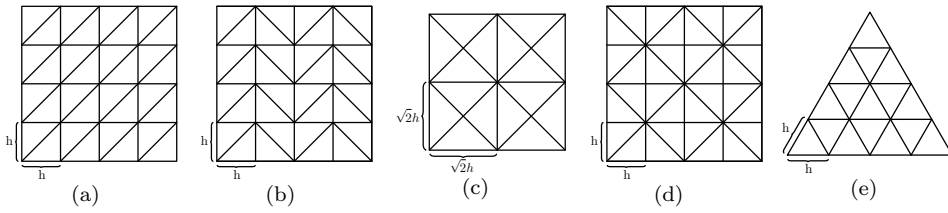


FIGURE 1. Five types of uniform meshes: (a) Regular pattern; (b) Chevron pattern; (c) Criss-cross pattern; (d) Union-Jack pattern; (e) Equilateral pattern

2) Chevron pattern is translation invariant by  $h$  in the direction  $(0, 1)$ , by  $2h$  in the direction  $(1, 0)$ , and by  $2\sqrt{2}h$  in directions  $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , and by  $\sqrt{5}h$  in directions  $(\pm\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$ , etc.

3) Criss-cross pattern is translation invariant by  $\sqrt{2}h$  in directions  $(1, 0)$  and  $(0, 1)$ , and by  $2h$  in directions  $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , etc.

4) Union-Jack pattern is translation invariant by  $2h$  in directions  $(1, 0)$  and  $(0, 1)$ , and by  $2\sqrt{2}h$  in directions  $(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , etc.

5) Equilateral pattern is translation invariant by  $h$  in directions  $(1, 0)$  and  $(\pm\frac{1}{2}, \frac{\sqrt{3}}{2})$ , and by  $\sqrt{3}h$  in directions  $(0, 1)$  and  $(\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$ , etc.

**2.2. Polynomial preserving recovery.** Let us introduce  $G_h : S_h \rightarrow S_h \times S_h$  the PPR gradient recovery operator. For any function  $u_h \in S_h$ ,  $G_h u_h$  is a function in  $S_h \times S_h$  which is uniquely determined by its values at nodes. If the values  $\{(G_h u_h)(z) : z \in \mathcal{N}_h\}$  are well defined, then define  $G_h u_h$  on the whole domain by

$$G_h u_h := \sum_{z \in \mathcal{N}_h} (G_h u_h)(z) \phi_z.$$

When  $z$  is a vertex, let  $\mathcal{K}_z$  be a patch of elements around  $z$ . Select all nodes in  $\mathcal{N}_h \cap \mathcal{K}_z$  as sampling points and fit a polynomial  $p_z \in \mathbb{P}_{k+1}(\mathcal{K}_z)$  in the least squares sense at those sampling points, i.e.,

$$(2.6) \quad p_z = \arg \min_{p \in \mathbb{P}_{k+1}(\mathcal{K}_z)} \sum_{\tilde{z} \in \mathcal{N}_h \cap \mathcal{K}_z} (u_h - p)^2(\tilde{z}).$$

Then the recovered gradient at  $z$  is defined as

$$(G_h u_h)(z) = \nabla p_z(z).$$

For the linear element, all nodes in  $\mathcal{N}_h$  are vertices and hence  $G_h u_h$  is well defined. However,  $\mathcal{N}_h$  may contain edge nodes or interior nodes for higher order elements. If  $z$  is an edge node which lies on an edge between two vertices  $z_1$  and  $z_2$ , we define

$$(G_h u_h)(z) = \beta \nabla p_{z_1}(z) + (1 - \beta) \nabla p_{z_2}(z),$$

where  $\beta$  is determined by the ratio of distances of  $z$  to  $z_1$  and  $z_2$ . If  $z$  is an interior node which lies in a triangle formed by three vertices  $z_1, z_2$ , and  $z_3$ , we define

$$(G_h u_h)(z) = \sum_{j=1}^3 \beta_j \nabla p_{z_j}(z),$$

where  $\beta_j$  is the barycentric coordinate of  $z$ .

To complete the definition of PPR, we need to define  $\mathcal{K}_z$ . If  $z$  is an interior vertex,  $\mathcal{K}_z$  is defined as the smallest  $\mathcal{L}(z, n)$  that guarantees the uniqueness of  $p_z$  in (2.6) [16, 27]. In the case that  $z$  is a boundary vertex, let  $n_0$  be the smallest positive integer such that  $\mathcal{L}(z, n_0)$  has at least one interior mesh vertex. Then, we define

$$\mathcal{K}_z = \mathcal{L}(z, n_0) \cup \{\mathcal{K}_{\tilde{z}} : \tilde{z} \in \mathcal{L}(z, n_0) \text{ and } \tilde{z} \text{ an interior vertex}\}.$$

*Remark 2.1.* In order to avoid numerical instability, a discrete least squares fitting process is carried out on a reference patch  $\omega_z$ .

### 3. HESSIAN RECOVERY METHOD

Given  $u \in S_h$ , let  $G_h u \in S_h \times S_h$  be the recovered gradient using PPR as defined in the previous section. We rewrite  $G_h u$  as

$$(3.1) \quad G_h u = \begin{pmatrix} G_h^x u \\ G_h^y u \end{pmatrix}.$$

In order to recover the Hessian matrix of  $u$ , we apply gradient recovery operator  $G_h$  to  $G_h^x u$  and  $G_h^y u$  one more time, respectively, and define the Hessian recovery operator  $H_h$  as follows:

$$(3.2) \quad H_h u = (G_h(G_h^x u), G_h(G_h^y u)) = \begin{pmatrix} G_h^x(G_h^x u) & G_h^x(G_h^y u) \\ G_h^y(G_h^x u) & G_h^y(G_h^y u) \end{pmatrix}.$$

Just as PPR, we obtain  $H_h : S_h \rightarrow S_h^2 \times S_h^2$  on the whole domain  $\Omega$  by interpolation after determining values of  $H_h u$  at all nodes in  $\mathcal{N}_h$ .

*Remark 3.1.* For the Hessian recovery operator  $H_h$  defined in (3.2), we shall prove that  $H_h$  is symmetric when the mesh is translation invariant and all sampling points are symmetric with respect to the recovered point. In the general case,  $H_h$  may not be symmetric. But we can overcome the asymmetry by symmetrizing the recovered Hessian matrix, i.e.,

$$H_h \leftarrow \frac{(H_h + H_h^T)}{2}.$$

This symmetrization process is easily implemented in practical and it certainly does not compromise the quality of approximation. Since the recovered Hessian converges to the actual Hessian which is symmetric, the skew-symmetric part of  $H_h$  should be relatively small. For simplicity in theoretical analysis, we still keep the definition (3.2).

*Remark 3.2.* The two gradient recovery operators in definition (3.2) of  $H_h$  can be different. Actually, we can define the Hessian recovery operator  $H_h$  as

$$H_h u = (\tilde{G}_h(G_h^x u), \tilde{G}_h(G_h^y u)).$$

By choosing  $G_h$  and  $\tilde{G}_h$  as PPR or SPR operators, we obtain four different Hessian recovery operators, i.e., PPR-PPR, PPR-SPR, SPR-PPR, and SPR-SPR. However, numerical tests have shown that PPR-PPR is the best one. Indeed, PPR-PPR is the only one that ultraconverges at all of the five different uniform meshes.

**3.1. Illustration.** To give the readers some intuition, we shall discuss two examples in detail. For the sake of simplicity, only linear elements on uniform meshes will be considered. In practice, the method can be applied to arbitrary meshes and higher order elements.

**Example 1.** Consider the regular pattern uniform mesh as in Figure 2. We want to recover the Hessian matrix at  $z_0$ . As deduced in [27], the recovered gradient at  $z_0$  is given by

$$(G_h u)(z_0) = \frac{1}{6h} \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u_2 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u_3 + \begin{pmatrix} -2 \\ -1 \end{pmatrix} u_4 + \begin{pmatrix} -1 \\ -2 \end{pmatrix} u_5 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_6 \right).$$

Here  $u_i = u(z_i)$  ( $i = 0, 1, \dots, 18$ ) represents the function value of  $u$  at node  $z_i$ . Thus, according to the definition (3.2) of the Hessian recovery operator  $H_h$ , we have

$$(3.3) \quad \begin{pmatrix} H_h^{xx} u \\ H_h^{xy} u \end{pmatrix} (z_0) = \frac{1}{6h} (2(G_h u)(z_1) + (G_h u)(z_2) - (G_h u)(z_3) - 2(G_h u)(z_4) - (G_h u)(z_5) + (G_h u)(z_6))$$

and

$$(3.4) \quad \begin{pmatrix} H_h^{yx} u \\ H_h^{yy} u \end{pmatrix} (z_0) = \frac{1}{6h} ((G_h u)(z_1) + 2(G_h u)(z_2) + (G_h u)(z_3) - (G_h u)(z_4) - 2(G_h u)(z_5) - (G_h u)(z_6)),$$

where

$$(G_h u)(z_1) = \frac{1}{6h} \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_7 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u_8 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u_2 + \begin{pmatrix} -2 \\ -1 \end{pmatrix} u_0 + \begin{pmatrix} -1 \\ -2 \end{pmatrix} u_{18} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_6 \right)$$

and  $(G_h u)(z_2), \dots, (G_h u)(z_6)$  follow a similar pattern. Direct calculation reveals that

$$\begin{aligned} (H_h^{xx} u)(z_0) &= \frac{1}{36h^2} (-12u_0 + 2u_1 - 4u_2 - 4u_3 + 2u_4 - 4u_5 - 4u_6 + 4u_7 + 4u_8 + u_9 \\ &\quad - 2u_{10} + u_{11} + 4u_{12} + 4u_{13} + 4u_{14} + u_{15} - 2u_{16} + u_{17} + 4u_{18}), \\ (H_h^{xy} u)(z_0) &= \frac{1}{36h^2} (6u_0 - u_1 + 5u_2 - u_3 - u_4 + 5u_5 - u_6 - 2u_7 + u_8 + u_9 \\ &\quad + u_{10} - 2u_{11} - 5u_{12} - 2u_{13} + u_{14} + u_{15} + u_{16} - 2u_{17} - 5u_{18}), \\ (H_h^{yx} u)(z_0) &= \frac{1}{36h^2} (6u_0 - u_1 + 5u_2 - u_3 - u_4 + 5u_5 - u_6 - 2u_7 + u_8 + u_9 \\ &\quad + u_{10} - 2u_{11} - 5u_{12} - 2u_{13} + u_{14} + u_{15} + u_{16} - 2u_{17} - 5u_{18}), \\ (H_h^{yy} u)(z_0) &= \frac{1}{36h^2} (-12u_0 - 4u_1 - 4u_2 + 2u_3 - 4u_4 - 4u_5 + 2u_6 + u_7 - 2u_8 + u_9 \\ &\quad + 4u_{10} + 4u_{11} + 4u_{12} + u_{13} - 2u_{14} + u_{15} + 4u_{16} + 4u_{17} + 4u_{18}). \end{aligned}$$

It is observed that  $(H_h^{xy} u)(z_0) = (H_h^{yx} u)(z_0)$ , which means the recovered Hessian matrix is symmetric, a property of the exact Hessian we would like to maintain.

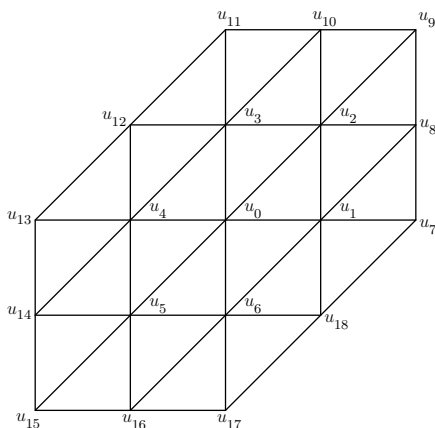


FIGURE 2. Regular pattern

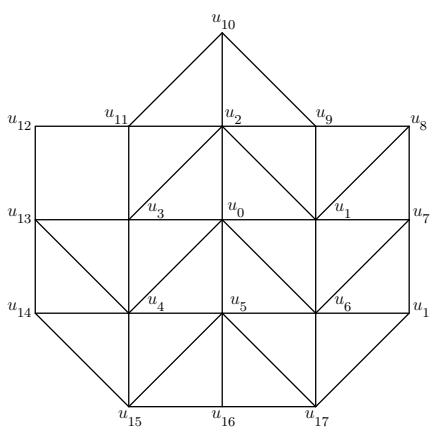


FIGURE 3. Chevron pattern

Using Taylor expansion, we can show that

$$\begin{aligned}
 (H_h^{xx}u)(z_0) &= u_{xx}(z_0) + \frac{h^2}{3}(u_{xxxx}(z_0) + u_{xxxy}(z_0) + u_{xxyy}(z_0)) + O(h^4), \\
 (H_h^{xy}u)(z_0) &= u_{xy}(z_0) + \frac{h^2}{3}(u_{xxxy}(z_0) + u_{xxyy}(z_0) + u_{xyyy}(z_0)) + O(h^4), \\
 (H_h^{yx}u)(z_0) &= u_{yx}(z_0) + \frac{h^2}{3}(u_{xxxy}(z_0) + u_{xxyy}(z_0) + u_{xyyy}(z_0)) + O(h^4), \\
 (H_h^{yy}u)(z_0) &= u_{yy}(z_0) + \frac{h^2}{3}(u_{xxyy}(z_0) + u_{xyyy}(z_0) + u_{yyyy}(z_0)) + O(h^4),
 \end{aligned}$$

which imply that  $H_h u$  provides a second order approximation of  $Hu$  at  $z_0$ .

**Example 2.** Consider the Chevron pattern uniform mesh as shown in Figure 3. Repeating the procedure as in Example 1, we derive the recovered Hessian matrix at  $z_0$  as

$$\begin{aligned}
 (H_h^{xx}u)(z_0) &= \frac{1}{144h^2}(-72u_0 + 36u_{13} + 36u_7), \\
 (H_h^{xy}u)(z_0) &= \frac{1}{144h^2}(-12u_1 + 12u_3 + 24u_4 - 24u_6 + 6u_7 \\
 &\quad + 36u_9 - 36u_{11} - 6u_{13} + 6u_{14} - 6u_{18}), \\
 (H_h^{yx}u)(z_0) &= \frac{1}{144h^2}(12u_1 - 12u_3 + 36u_4 - 36u_6 - 6u_7 \\
 &\quad + 6u_8 + 24u_9 - 24u_{11} - 6u_{12} + 6u_{13}), \\
 (H_h^{yy}u)(z_0) &= \frac{1}{144h^2}(-48u_0 - 10u_1 - 22u_2 - 10u_3 - 10u_4 + 18u_5 \\
 &\quad - 10u_6 - 2u_7 + u_8 + 10u_9 + 36u_{10} + 10u_{11} + u_{12} \\
 &\quad - 2u_{13} + u_{14} + 10u_{15} + 16u_{16} + 10u_{17} + u_{18}).
 \end{aligned}$$

In addition, we have the following Taylor expansion:

$$\begin{aligned} (H_h^{xx}u)(z_0) &= u_{xx}(z_0) + \frac{h^2}{3}u_{xxxx}(z_0) + \frac{2h^4}{45}u_{xxxxxx}(z_0) + O(h^5), \\ (H_h^{xy}u)(z_0) &= u_{xy}(z_0) + \frac{h^2}{12}(3u_{xxyy}(z_0) + 2u_{xyyy}(z_0)) - \frac{h^3}{24}u_{xxyyy}(z_0) + O(h^4), \\ (H_h^{yx}u)(z_0) &= u_{yx}(z_0) + \frac{h^2}{12}(3u_{xxyy}(z_0) + 2u_{xyyy}(z_0)) + \frac{h^3}{24}u_{xxyyy}(z_0) + O(h^4), \\ (H_h^{yy}u)(z_0) &= u_{yy}(z_0) + \frac{h^2}{6}(u_{xyyy}(z_0) + 2u_{yyyy}(z_0)) - \frac{5h^3}{72}u_{xyyyy}(z_0) + O(h^4). \end{aligned}$$

We conclude that  $H_h u$  is a second order approximation to the Hessian matrix. It is worth mentioning that, though  $H_h^{xy} \neq H_h^{yx}$  for the Chevron pattern uniform mesh, they are both second order finite difference schemes at  $z_0$ .

*Remark 3.3.* PPR-PPR is the only one among the four Hessian recovery methods mentioned in Remark 3.2 that provides second order approximation for the Chevron pattern uniform mesh.

Both Examples 1 and 2 indicate that for the linear element the PPR-PPR approach is equivalent to a finite difference scheme of second order accuracy at vertex  $z_0$ .

*Remark 3.4.* In a general sense, the recovery operator can be viewed as a finite difference operator on unstructured meshes. The practical usage of recovery operator is not only to obtain a better approximation and provide an asymptotically exact a posteriori error estimator, but also to design some new numerical solvers for PDEs. It provides a systematic way to construct finite difference schemes on general unstructured meshes. Actually, the Hessian recovery operator defined in (3.2) can be used to construct finite difference schemes for second order differential operators on unstructured meshes. We have made some progress in this direction and will report the results in a separate paper.

**3.2. Polynomial preserving property.** As we observed in previous subsection,  $H_h$  can be viewed as a finite difference scheme on unstructured meshes. For finite difference schemes, one of the most important properties is consistency. In this subsection, we shall prove the polynomial preserving property of the Hessian recovery operator  $H_h$  which leads to consistency.

For an arbitrary unstructured mesh, we can prove the following polynomial preserving property.

**Theorem 3.5.** *The Hessian recovery operator  $H_h$  preserves polynomials of degree  $k + 1$  for an arbitrary mesh.*

*Proof.* Suppose  $u$  is a polynomial of  $k + 1$  on  $\mathcal{K}_z$ , i.e.,  $u \in \mathbb{P}(\mathcal{K}_z)$ . According to Theorem 2.1 in [27],  $G_h$  preserves polynomials of degree  $k + 1$ . Then it follows that  $G_h u = \nabla u$  which is a polynomial of degree  $k$ . Therefore, we have

$$(3.5) \quad H_h u = (G_h(G_h^x u), G_h(G_h^y u)) = (G_h \frac{\partial u}{\partial x}, G_h \frac{\partial u}{\partial x}) = (\nabla \frac{\partial u}{\partial x}, \nabla \frac{\partial u}{\partial x}) = Hu.$$

It means that  $H_h$  preserves polynomials of degree  $k + 1$  which completes our proof. □



If the mesh  $\mathcal{T}_h$  is translation invariant, we have the following improved results.

**Theorem 3.6.** *If  $z$  is a node of a translation invariant mesh and a mesh symmetric center of the involved nodes, then  $H_h$  preserves polynomials of degree  $k + 2$  for odd  $k$ , and of degree  $k + 3$  for even  $k$ . In addition,  $H_h$  is symmetric.*

*Proof.* Since  $G_h$  is exact for polynomial of degree  $k + 1$ , it follows that

$$(3.6) \quad G_h^x u = D_x u + h^{k+1} \mathbf{a}^x \cdot D^{k+2} u + h^{k+2} \mathbf{b}^x \cdot D^{k+3} u + h^{k+3} \mathbf{c}^x \cdot D^{k+4} u + \dots,$$

$$(3.7) \quad G_h^y u = D_y u + h^{k+1} \mathbf{a}^y \cdot D^{k+2} u + h^{k+2} \mathbf{b}^y \cdot D^{k+3} u + h^{k+3} \mathbf{c}^y \cdot D^{k+4} u + \dots.$$

Notice that coefficients  $\mathbf{a}^x, \mathbf{a}^y, \mathbf{b}^x, \mathbf{b}^y, \dots$  depend only on the coordinates of nodes, since we recover gradient at nodes only. Thus for translation invariant meshes,  $\mathbf{a}^x, \mathbf{a}^y, \mathbf{b}^x, \mathbf{b}^y, \dots$  are constants. In addition, due to symmetry, it makes no difference if we perform  $G_h^x$  or  $G_h^y$  first. Hence,

$$(3.8) \quad \begin{aligned} (H_h^{xy} u)(z) &= (G_h^y(G_h^x u))(z) \\ &= G_h^y[D_x u(z) + h^{k+1} \mathbf{a}^x \cdot D^{k+2} u(z) + h^{k+2} \mathbf{b}^x \cdot D^{k+3} u(z) + \dots] \\ &= (G_h^y(D_x u))(z) + h^{k+1}(\mathbf{a}^x \cdot G_h^y(D^{k+2} u))(z) + h^{k+2}(\mathbf{b}^x \cdot G_h^y(D^{k+3} u))(z) + \dots \\ &= (D_y D_x u)(z) + h^{k+1}(\mathbf{a}^y \cdot D^{k+2} D_x u)(z) + h^{k+2}(\mathbf{b}^y \cdot D^{k+3} D_x u)(z) \\ &\quad + h^{k+1}(\mathbf{a}^x \cdot D_y(D^{k+2} u))(z) + h^{k+2}(\mathbf{b}^x \cdot D_y(D^{k+3} u))(z) + O(h^{k+3}) \\ &= (D_y D_x u)(z) + h^{k+1}[\mathbf{a}^y \cdot D^{k+2} D_x u + \mathbf{a}^x \cdot D_y(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\mathbf{b}^y \cdot D^{k+3} D_x u + \mathbf{b}^x \cdot D_y(D^{k+3} u)](z) + O(h^{k+3}). \end{aligned}$$

Notice that (3.8) is valid only at nodal points. Similarly,

$$(3.9) \quad \begin{aligned} (H_h^{yx} u)(z) &= (D_x D_y u)(z) + h^{k+1}[\mathbf{a}^x \cdot D^{k+2} D_y u + \mathbf{a}^y \cdot D_x(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\mathbf{b}^x \cdot D^{k+3} D_y u + \mathbf{b}^y \cdot D_x(D^{k+3} u)](z) + O(h^{k+3}), \end{aligned}$$

$$(3.10) \quad \begin{aligned} (H_h^{xx} u)(z) &= (D_x D_x u)(z) + h^{k+1}[\mathbf{a}^x \cdot D^{k+2} D_x u + \mathbf{a}^x \cdot D_x(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\mathbf{b}^x \cdot D^{k+3} D_x u + \mathbf{b}^x \cdot D_x(D^{k+3} u)](z) + O(h^{k+3}), \end{aligned}$$

$$(3.11) \quad \begin{aligned} (H_h^{yy} u)(z) &= (D_y D_y u)(z) + h^{k+1}[\mathbf{a}^y \cdot D^{k+2} D_y u + \mathbf{a}^y \cdot D_y(D^{k+2} u)](z) \\ &\quad + h^{k+2}[\mathbf{b}^y \cdot D^{k+3} D_y u + \mathbf{b}^y \cdot D_y(D^{k+3} u)](z) + O(h^{k+3}). \end{aligned}$$

(3.8)–(3.11) imply that the Hessian recovery operator  $H_h$  is exact for polynomials of degree  $k + 2$  for translation invariant meshes. Also, we observe  $H_h^{xy} = H_h^{yx}$  from (3.8) and (3.9).

Next we consider even order ( $k = 2r$ ) elements on translation invariant meshes, in which case

$$(3.12) \quad \mathbf{a}^x(z) = \mathbf{0}, \quad \mathbf{c}^x(z) = \mathbf{0}, \quad \mathbf{a}^y(z) = \mathbf{0}, \quad \mathbf{c}^y(z) = \mathbf{0},$$

$$(3.13) \quad D\mathbf{a}^x(z) = \mathbf{0}, \quad D\mathbf{c}^x(z) = \mathbf{0}, \quad D\mathbf{a}^y(z) = \mathbf{0}, \quad D\mathbf{c}^y(z) = \mathbf{0},$$

and  $\mathbf{b}^x, \mathbf{b}^y, \dots$  are constants in (3.7). Here the symbol  $D$  is understood as taking all partial derivatives to each entry of the vector. Consequently,

$$(3.14) \quad (G_h^y u)(z) = (D_y u)(z) + h^{k+2}(\mathbf{b}^y \cdot D^{k+3} u)(z) + O(h^{k+4}).$$

Also, (3.14) is valid only at nodal points. Plugging (3.6) into (3.14) yields

$$\begin{aligned} (H_h^{xy}u)(z) &= (G_h^y G_h^x u)(z) \\ &= (D_y G_h^x u)(z) + h^{k+2}(\mathbf{b}^y \cdot D^{k+3} G_h^x u)(z) + O(h^{k+4}) \\ &= D_y(D_x u + h^{k+1} \mathbf{a}^x \cdot D^{k+2} u + h^{k+2} \mathbf{b}^x \cdot D^{k+3} u + h^{k+3} \mathbf{c}^x \cdot D^{k+4} u \\ &\quad + \dots)(z) + h^{k+2}(\mathbf{b}^y \cdot D^{k+3} D_x u)(z) + O(h^{k+4}) \\ &= (D_y D_x u)(z) + h^{k+2}(\mathbf{b}^x \cdot D_y D^{k+3} u + \mathbf{b}^y \cdot D^{k+3} D_x u)(z) + O(h^{k+4}). \end{aligned}$$

In the last identity we have used (3.12) and (3.13).

The argument for the other three entries of the recovered Hessian matrix are similar. We conclude that the Hessian recovery operator  $H_h$  is exact for polynomials of degree up to  $k + 3$  when  $k$  is even and the mesh is translation invariant and symmetric with respect to  $x$  and  $y$ . □

*Remark 3.7.* It is worth mentioning that, except for the Chevron pattern, (3.8)–(3.11) are valid for the other four patterns of uniform meshes, since the recovered gradient  $G_h u$  produces the same stencil at each node.

*Remark 3.8.* According to [23], the best Hessian recovery method in the literature preserves polynomials of degree 2 for the linear element. Our method preserves polynomials of degree 2 on general unstructured meshes and preserves polynomials of degree 3 on translation invariant meshes for the linear element.

**Theorem 3.9.** *Let  $u \in W_\infty^{k+2}(\mathcal{K}_z)$ ; then*

$$\|Hu - H_h u\|_{0,\infty,\mathcal{K}_z} \lesssim h^k |u|_{k+2,\infty,\mathcal{K}_z}.$$

*If  $z$  is a node of translation invariant mesh and a mesh symmetric center of the involved nodes and  $u \in W_\infty^{k+3}(\mathcal{K}_z)$ , then*

$$|(Hu - H_h u)(z)| \lesssim h^{k+1} |u|_{k+3,\infty,\mathcal{K}_z}.$$

*Moreover, if  $u \in W_\infty^{k+4}(\mathcal{K}_z)$  and  $k$  is an even number, then*

$$|(Hu - H_h u)(z)| \lesssim h^{k+2} |u|_{k+4,\infty,\mathcal{K}_z}.$$

*Proof.* It follows directly from Theorems 3.5, 3.6, and the Bramble-Hilbert lemma. □

#### 4. SUPERCONVERGENCE ANALYSIS

In this section, we first use the supercloseness between the gradient of the finite element solution  $u_h$  and the gradient of the interpolation  $u_I$  [3, 5, 9, 10, 25, 26], and properties of the PPR operator [15, 27] to establish the superconvergence property of our Hessian recovery operator on mildly structured mesh. Then we utilize the tool of superconvergence by difference quotients from [24] to prove the proposed Hessian recovery method is ultraconvergent for the translation invariant finite element space of any order.

In this section, we consider the following variational problem: find  $u \in H^1(\Omega)$  such that

$$(4.1) \quad B(u, v) = \int_\Omega (\mathcal{D}\nabla u + \mathbf{b}u) \cdot \nabla v + cuv dx = f(v), \quad \forall v \in H^1(\Omega).$$

Here  $\mathcal{D}$  is a  $2 \times 2$  symmetric positive definite matrix,  $\mathbf{b}$  is a vector,  $c$  is a number and  $f(\cdot)$  is a linear functional on  $H^1(\Omega)$ . All coefficient functions are assumed to be smooth.

In order to insure (4.1) has a unique solution, we assume the bilinear form  $B(\cdot, \cdot)$  satisfies the continuity condition

$$(4.2) \quad |B(u, v)| \leq \nu \|u\|_{1,\Omega} \|v\|_{1,\Omega},$$

for all  $u, v \in H^1(\Omega)$ . We also assume the inf-sup conditions [3, 4, 6]

$$(4.3) \quad \inf_{u \in H^1(\Omega)} \sup_{v \in H^1(\Omega)} \frac{B(u, v)}{\|u\|_{1,\Omega} \|v\|_{1,\Omega}} = \sup_{u \in H^1(\Omega)} \inf_{v \in H^1(\Omega)} \frac{B(u, v)}{\|u\|_{1,\Omega} \|v\|_{1,\Omega}} \geq \mu > 0.$$

The finite element approximation of (4.1) is to find  $u_h \in S_h$  satisfying

$$(4.4) \quad B(u_h, v_h) = f(v_h), \quad \forall v_h \in S_h.$$

To insure a unique solution for (4.4), we assume the inf-sup conditions

$$(4.5) \quad \inf_{u \in S_h} \sup_{v \in S_h} \frac{B(u, v)}{\|u\|_{1,\Omega} \|v\|_{1,\Omega}} = \sup_{u \in S_h} \inf_{v \in S_h} \frac{B(u, v)}{\|u\|_{1,\Omega} \|v\|_{1,\Omega}} \geq \mu > 0.$$

From (4.1) and (4.4), it is easy to see that

$$(4.6) \quad B(u - u_h, v) = 0$$

for any  $v \in S_h$ . In particular, (4.6) holds for any  $v \in S_h^{\text{comp}}(\Omega)$ .

**4.1. Linear element.** The linear finite element space  $S_h$  on quasi-uniform mesh  $\mathcal{T}_h$  is considered in this subsection.

**Definition 4.1.** The triangulation  $\mathcal{T}_h$  is said to satisfy condition  $(\sigma, \alpha)$  if there exist a partition  $\mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$  of  $\mathcal{T}_h$  and positive constants  $\alpha$  and  $\sigma$  such that every two adjacent triangles in  $\mathcal{T}_{h,1}$  form an  $O(h^{1+\alpha})$  parallelogram and

$$\sum_{T \in \mathcal{T}_{h,2}} |T| = O(h^\sigma).$$

An  $O(h^{1+\alpha})$  parallelogram is a quadrilateral shifted from a parallelogram by  $O(h^{1+\alpha})$ .

For general  $\alpha$  and  $\sigma$ , Xu and Zhang [26] proved the following theorem.

**Theorem 4.2.** *Let  $u$  be the solution of (4.1), let  $u_h \in S_h$  be the finite element solution of (4.4), and let  $u_I \in S_h$  be the linear interpolation of  $u$ . If the triangulation  $\mathcal{T}_h$  satisfies condition  $(\sigma, \alpha)$  and  $u \in H^3(\Omega) \cap W_\infty^2(\Omega)$ , then*

$$|u_h - u_I|_{1,\Omega} \lesssim h^{1+\rho} (|u|_{3,\Omega} + |u|_{2,\infty,\Omega}),$$

where  $\rho = \min(\alpha, \sigma/2, 1/2)$ .

Using the above result, we are able to obtain a convergence rate for our Hessian recovery operator.

**Theorem 4.3.** *Suppose that the solution of (4.1) belongs to  $H^3(\Omega) \cap W_\infty^2(\Omega)$  and  $\mathcal{T}_h$  satisfies condition  $(\sigma, \alpha)$ , then we have*

$$\|Hu - H_h u_h\|_{0,\Omega} \leq h^\rho \|u\|_{3,\infty,\Omega}.$$

*Proof.* We decompose  $Hu - H_h u_h$  as  $(Hu - H_h u) + H_h(u_I - u_h)$ , since  $H_h u = H_h u_I$ . Using the triangle inequality and the definition of  $H_h$ , we obtain

$$\begin{aligned} \|Hu - H_h u_h\|_{0,\Omega} &\leq \|Hu - H_h u\|_{0,\Omega} + \|H_h(u_I - u_h)\|_{0,\Omega} \\ &= \|Hu - H_h u\|_{0,\Omega} + \|G_h(G_h(u_I - u_h))\|_{0,\Omega}. \end{aligned}$$

The first term in the above expression is bounded by  $h|u|_{3,\infty,\Omega}$  according to Theorem 3.9. Since  $G_h$  is a bounded linear operator [16], it follows that

$$\|H_h(u_I - u_h)\|_{0,\Omega} \lesssim \|\nabla(G_h(u_I - u_h))\|_{0,\Omega}$$

Notice that  $G_h(u_I - u_h)$  is a function in  $S_h$  and hence the inverse estimate [4, 6] can be applied. Thus,

$$\|H_h(u_I - u_h)\|_{0,\Omega} \lesssim h^{-1} \|G_h(u_I - u_h)\|_{0,\Omega} \lesssim h^{-1} \|u_I - u_h\|_{1,\Omega}$$

and hence Theorem 4.2 implies that

$$\|H_h(u_I - u_h)\|_{0,\Omega} \lesssim h^\rho \|u\|_{3,\infty,\Omega}.$$

Combining the above two estimates completes our proof. □

**4.2. Quadratic element.** We proceed to quadratic finite element space  $S_h$ . According to [10], a triangulation  $\mathcal{T}_h$  is strongly regular if any two adjacent triangles in  $\mathcal{T}_h$  form an  $O(h^2)$  approximate parallelogram. Huang and Xu proved the following superconvergence results in [10].

**Theorem 4.4.** *If the triangulation  $\mathcal{T}_h$  is uniform or strongly regular, then*

$$|u_h - u_I|_{1,\Omega} \lesssim h^3 |u|_{4,\Omega}.$$

Based on the above theorem, we obtain the following superconvergent result.

**Theorem 4.5.** *Suppose that the solution of (4.1) belongs to  $H^4(\Omega)$  and  $\mathcal{T}_h$  is uniform or strongly regular. Then we have*

$$\|Hu - H_h u_h\|_{0,\Omega} \leq h^2 \|u\|_{4,\Omega}.$$

*Proof.* The proof is similar to the proof of Theorem 4.3 by using Theorem 4.4 and the inverse estimate. □

*Remark 4.6.* Theorem 4.5 can be generalized to mildly structured meshes as in [10].

**4.3. Translation invariant element of any order.** In this subsection, we establish the ultraconvergence theory of Hessian recovery operator  $H_h$  for the translation invariant finite element space.

First, we observe that the Hessian recovery operator results in a difference quotient. It is due to the fact that  $G_h$  is a difference quotient [27] and the composition of two difference quotients is still a difference quotient. Let us take the linear element on uniform triangular mesh of the regular pattern as an example; see Figure 2. The recovered second order derivative at a nodal point  $z$  is

$$\begin{aligned} (H_h^{xx} u_h)(z) &= \frac{1}{36h^2} (-12u_0 + 2u_1 - 4u_2 - 4u_3 + 2u_4 - 4u_5 - 4u_6 + 4u_7 + 4u_8 + u_9 \\ &\quad - 2u_{10} + u_{11} + 4u_{12} + 4u_{13} + 4u_{14} + u_{15} - 2u_{16} + u_{17} + 4u_{18}). \end{aligned}$$

Let  $\phi_j$  be the nodal shape functions. Since  $\phi_z(z') = \delta_{zz'}$ , it follows that

$$\begin{aligned} & (H_h^{xx}u_h)\phi_0(x, y) \\ &= \frac{1}{36h^2}[-12u_0\phi_0(x, y) + 2u_1\phi_1(x + h, y) - 4u_2\phi_2(x + h, y + h) \\ &\quad - 4u_3\phi_3(x, y + h) + 2u_4\phi_4(x - h, y) - 4u_5\phi_5(x - h, y - h) \\ &\quad - 4u_6\phi_6(x, y - h) + 4u_7\phi_7(x + 2h, y) + 4u_8\phi_8(x + 2h, y + h) \\ &\quad + u_9\phi_9(x + 2h, y + 2h) - 2u_{10}\phi_{10}(x + h, y + 2h) + u_{11}\phi_{11}(x, y + 2h) \\ &\quad + 4u_{12}\phi_{12}(x - h, y + h) + 4u_{13}\phi_{13}(x - 2h, y) + 4u_{14}\phi_{14}(x - 2h, y - h) \\ &\quad + u_{15}\phi_{15}(x - 2h, y - 2h) - 2u_{16}\phi_{16}(x - h, y - 2h) + u_{17}\phi_{17}(x, y - 2h) \\ &\quad + 4u_{18}\phi_{18}(x + h, y - h)]. \end{aligned}$$

The translations are in the directions of  $\ell_1 = (1, 0)$ ,  $\ell_2 = (0, 1)$ ,  $\ell_3 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $\ell_4 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ ,  $\ell_5 = (\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$ , and  $\ell_6 = (\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5})$ . Therefore, we can express the recovered second order derivative as

$$(4.7) \quad (H_h^{xx}u_h)(z) = \sum_{|\nu| \leq M} \sum_{i=1}^6 C_{\nu, h}^i u_h(z + \nu h \ell_i),$$

for some integer  $M$ .

Based on such an observation, we can prove the following lemma.

**Lemma 4.7.** *Let all the coefficients in the bilinear operator  $B(\cdot, \cdot)$  be constant; let  $\Omega_1 \subset\subset \Omega$  be separated by  $d = O(1)$ ; let the finite element space  $S_h$ , which includes piecewise polynomials of degree  $k$ , be translation invariant in the directions required by the Hessian recovery operator  $H_h$  on  $\Omega_1$ ; and let  $u \in W_\infty^{k+3}(\Omega)$ . Assume that Theorem 5.2.2 from [24] is applicable. Then on any interior region  $\Omega_0 \subset\subset \Omega_1$ , we get*

$$(4.8) \quad \|H_h(u - u_h)\|_{0, \infty, \Omega_0} \lesssim \left(\ln \frac{1}{h}\right)^{\bar{r}} h^{k+1} \|u\|_{k+3, \infty, \Omega} + \|u - u_h\|_{-s, q, \Omega}$$

for some  $s \geq 0$  and  $q \geq 1$ . Here  $\bar{r} = 1$  for the linear element and  $\bar{r} = 0$  for the higher order element.

*Proof.* Since all coefficients in the bilinear form  $B(\cdot, \cdot)$  are constant, it follows that

$$B(T_{\nu\tau}^\ell(u - u_h), v) = B(u - u_h, T_{-\nu\tau}^\ell v) = B(u - u_h, (T_{\nu\tau}^\ell)^* v) = 0.$$

Notice that  $H_h^{xx}$  is a difference operator constructed from translation of type (4.7). Then we have

$$(4.9) \quad B(H_h^{xx}(u - u_h), v) = B(u - u_h, (H_h^{xx})^* v) = 0, \quad v \in S_h^{\text{comp}}(\Omega_1).$$

Therefore, Theorem 5.5.2 of [24] (with  $F \equiv 0$ ) implies that

$$(4.10) \quad \begin{aligned} \|H_h^{xx}(u - u_h)\|_{0, \infty, \Omega_0} &\lesssim \left(\ln \frac{d}{h}\right)^{\bar{r}} \min_{v \in S_h} \|H_h^{xx}u - v\|_{0, \infty, \Omega_1} \\ &\quad + d^{-s-\frac{2}{q}} \|H_h^{xx}(u - u_h)\|_{-s, q, \Omega_1}. \end{aligned}$$

Note that  $H_h^{xx}u \in S_h$  and hence the first term on the right-hand side of (4.10) can be estimated by standard approximation theory under the assumption that the finite element space includes piecewise polynomial of degree  $k$ :

$$(4.11) \quad \min_{v \in S_h} \|H_h^{xx}u - v\|_{0, \infty, \Omega_1} \lesssim h^{k+1} |u|_{k+3, \infty, \Omega_1},$$

provided  $u \in W_\infty^{k+3}(\Omega)$ ; see [4, 6]. It remains to attack the second term on the right-hand side of (4.10). Note that

$$(4.12) \quad \|H_h^{xx}(u - u_h)\|_{-s,q,\Omega_1} = \sup_{\phi \in C_0^\infty(\Omega_1), \|\phi\|_{s,q',\Omega_1}=1} (H_h^{xx}(u - u_h), \phi).$$

Here  $\frac{1}{q} + \frac{1}{q'} = 1$  and

$$(4.13) \quad \begin{aligned} (H_h^{xx}(u - u_h), \phi) &= (u - u_h, (H_h^{xx})^* \phi) \\ &\lesssim \|u - u_h\|_{0,\infty,\Omega_1} \|(H_h^{xx})^* \phi\|_{0,1,\Omega_1} \\ &\lesssim \|u - u_h\|_{0,\infty,\Omega_1}, \end{aligned}$$

where we use the fact that  $\|(H_h^{xx})^* \phi\|_{0,1,\Omega_1}$  is bounded uniformly with respect to  $h$  when  $s \geq 1$ . Again, we apply Theorem 5.5.1 from [24] to  $\|u - u_h\|_{0,\infty,\Omega_1}$  with  $\Omega_1 \subset\subset \Omega$  separated by  $d$ ; then

$$(4.14) \quad \begin{aligned} \|u - u_h\|_{0,\infty,\Omega_1} &\lesssim \left(\ln \frac{d}{h}\right)^{\bar{r}} \min_{v \in S_h} \|u - v\|_{0,\infty,\Omega} \\ &\quad + d^{-s-\frac{2}{q}} \|u - u_h\|_{-s,q,\Omega}. \end{aligned}$$

If the separation parameter  $d = O(1)$ , then we combine (4.10), (4.11) and (4.14) to obtain

$$(4.15) \quad \|H_h^{xx}(u - u_h)\|_{0,\infty,\Omega_0} \lesssim \left(\ln \frac{1}{h}\right)^{\bar{r}} h^{k+1} \|u\|_{k+3,\infty,\Omega} + \|u - u_h\|_{-s,q,\Omega}.$$

Following the same argument, we can establish the same result for  $H_h^{xy}, H_h^{yx}$ , and  $H_h^{yy}$ . Therefore, our proof is completed by replacing  $H_h^{xx}$  with  $H_h$  in (4.15).  $\square$

Now we are in a perfect position to prove our main result for the translation invariant finite element space of any order.

**Theorem 4.8.** *Let all the coefficients in the bilinear operator  $B(\cdot, \cdot)$  be constant; let  $\Omega_1 \subset\subset \Omega$  be separated by  $d = O(1)$ ; let the finite element space  $S_h$ , which includes piecewise polynomials of degree  $k$ , be translation invariant in the directions required by the Hessian recovery operator  $H_h$  on  $\Omega_1$ ; and let  $u \in W_\infty^{k+3}(\Omega)$ . Assume that Theorem 5.2.2 from [24] is applicable. Then on any interior region  $\Omega_0 \subset\subset \Omega_1$ , we get*

$$(4.16) \quad \|Hu - H_h u_h\|_{0,\infty,\Omega_0} \lesssim \left(\ln \frac{1}{h}\right)^{\bar{r}} h^{k+1} \|u\|_{k+3,\infty,\Omega} + \|u - u_h\|_{-s,q,\Omega}$$

for some  $s \geq 0$  and  $q \geq 1$ .

*Proof.* We decompose

$$(4.17) \quad Hu - H_h u_h = (Hu - (Hu)_I) + ((Hu)_I - H_h u) + H_h(u - u_h),$$

where  $(Hu)_I \in S_h^2 \times S_h^2$  is the standard Lagrange interpolation of  $Hu$  in the finite element space  $S_h$ . By the standard approximation theory, we obtain

$$(4.18) \quad \|Hu - (Hu)_I\|_{0,\infty,\Omega} \lesssim h^{k+1} |Hu|_{k+1,\infty,\Omega} \lesssim h^{k+1} |u|_{k+3,\infty,\Omega}.$$

For the second term, using Theorem 3.9, we have

$$(4.19) \quad \begin{aligned} \|(Hu)_I - H_h u\|_{0,\infty,\Omega_0} &= \left\| \sum_{z \in \mathcal{N}_h} ((Hu)(z) - (H_h u)(z)) \phi_z \right\|_{0,\infty,\Omega_0} \\ &\lesssim \max_{z \in \mathcal{N}_h \cap \Omega_0} |(Hu)(z) - (H_h u)(z)| \\ &\lesssim h^{k+1} |u|_{k+3,\infty,\Omega}. \end{aligned}$$

The last term in (4.17) is bounded by (4.8). The conclusion follows by combining (4.8), (4.18) and (4.19).  $\square$

*Remark 4.9.* Theorem 4.8 is an ultraconvergence result under the condition

$$\|u - u_h\|_{-s,q,\Omega} \lesssim h^{k+\sigma}, \quad \sigma > 0.$$

The reader is referred to [19] for negative norm estimates.

### 5. NUMERICAL TESTS

In this section, two numerical examples are provided to illustrate our Hessian recovery method. The first one is designed to demonstrate the polynomial preserving property of the proposed Hessian recovery method. The second one is devoted to a comparison of our method and some existing Hessian recovery methods in the literature on both uniform and unstructured meshes.

In order to evaluate the performance of Hessian recovery methods, we split mesh nodes  $\mathcal{N}_h$  into  $\mathcal{N}_{h,1}$  and  $\mathcal{N}_{h,2}$ , where  $\mathcal{N}_{h,2} = \{z \in \mathcal{N}_h : \text{dist}(z, \partial\Omega) \leq L\}$  denotes the set of nodes near the boundary and  $\mathcal{N}_{h,1} = \mathcal{N}_h \setminus \mathcal{N}_{h,2}$  denotes the remaining interior nodes. Now, we can define

$$\Omega_{h,1} = \bigcup \{ \tau \in \mathcal{T}_h : \tau \text{ has all of its vertices in } \mathcal{N}_{h,1} \},$$

and  $\Omega_{h,2} = \Omega \setminus \Omega_{h,1}$ . In the following examples we choose  $L = 0.1$ .

Let  $\tilde{G}_h$  be the weighted average recovery operator. Then we define

$$H_h^{ZZ} u_h = (\tilde{G}_h(\tilde{G}_h^x u_h), \tilde{G}_h(\tilde{G}_h^y u_h))$$

and

$$H_h^{LS} u_h = (\tilde{G}_h(G_h^x u_h), \tilde{G}_h(G_h^y u_h)).$$

For any nodal point  $z$ , fit a quadratic polynomial  $p_z$  at  $z$  as PPR. Then  $H_h^{QF}$  is defined as

$$H_h^{QF} u_h(z) = \begin{pmatrix} \frac{\partial^2 p_z}{\partial x^2}(0,0) & \frac{\partial^2 p_z}{\partial x \partial y}(0,0) \\ \frac{\partial^2 p_z}{\partial y \partial x}(0,0) & \frac{\partial^2 p_z}{\partial y^2}(0,0) \end{pmatrix}.$$

$H_h^{ZZ}$ ,  $H_h^{LS}$ , and  $H_h^{QF}$  are the first three Hessian recovery methods in [22]. To compare them, define

$$\begin{aligned} De &= \|H_h u_h - Hu\|_{0,\Omega_{1,h}}, & De^{ZZ} &= \|H_h^{ZZ} u_h - Hu\|_{0,\Omega_{1,h}}, \\ De^{LS} &= \|H_h^{LS} u_h - Hu\|_{0,\Omega_{1,h}}, & De^{QF} &= \|H_h^{QF} u_h - Hu\|_{0,\Omega_{1,h}}, \end{aligned}$$

where  $u_h$  is the finite element solution.

**Example 1.** Consider the following function:

$$(5.1) \quad u(x, y) = \sin(\pi x)\sin(\pi y), \quad (x, y) \in \Omega = (0, 1) \times (0, 1).$$

Let  $u_I$  be the standard Lagrangian interpolation of  $u$  in the finite element space. To validate Theorem 3.9, we apply the Hessian recovery operator  $H_h$  to  $u_I$  and consider the discrete maximum error of  $H_h u_I - Hu$  at all vertices in  $N_{1,h}$ . First, linear elements on uniform meshes are taken into account. Figures 4–7 display the numerical results. The numerical errors decrease at a rate of  $O(h^2)$  for four different pattern uniform meshes. It means the proposed Hessian recovery method preserves polynomials of degree 3 for linear elements on uniform meshes.

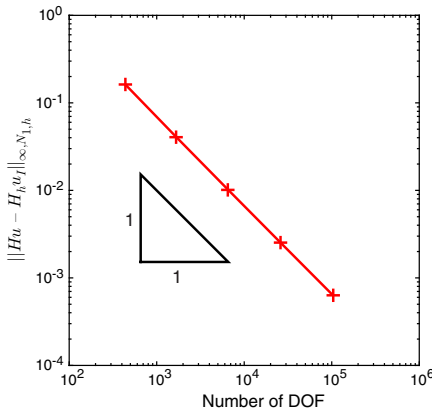


FIGURE 4. Numerical result of Example 1 for the linear element on Regular pattern uniform mesh

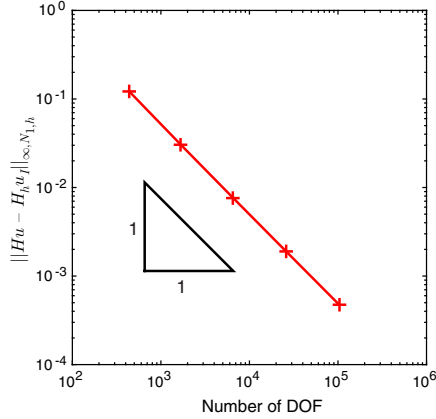


FIGURE 5. Numerical result of Example 1 for the linear element on Chevron pattern uniform mesh

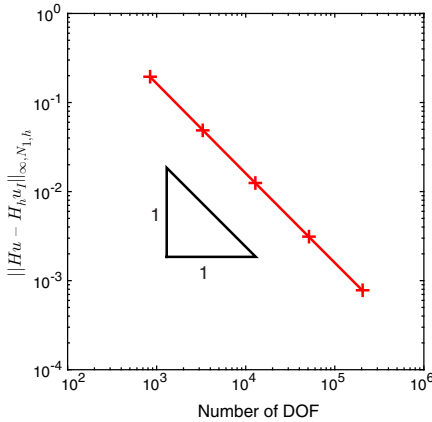


FIGURE 6. Numerical result of Example 1 for the linear element on Criss-cross pattern uniform mesh

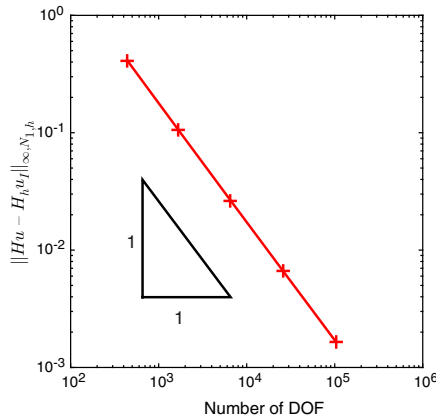


FIGURE 7. Numerical result of Example 1 for the linear element on Union-Jack pattern uniform mesh

Next, we consider unstructured meshes. We start from an initial mesh generated by EasyMesh [17] as shown in Figure 8 followed by four levels of refinement using bisection. Figure 9 shows that the recovered Hessian  $H_h u_I$  converges to the exact Hessian at rate  $O(h)$ . This coincides with the result in Theorem 3.6 that  $H_h$  only preserves polynomials of degree 2 on general unstructured meshes.

Then we turn to the quadratic element. We test the discrete error of recovered Hessian  $H_h u_I$  and the exact Hessian  $Hu$  using uniform meshes of regular pattern and the same Delaunay meshes. Similarly, we define  $\|\cdot\|_{\infty, h}$  as a discrete maximum



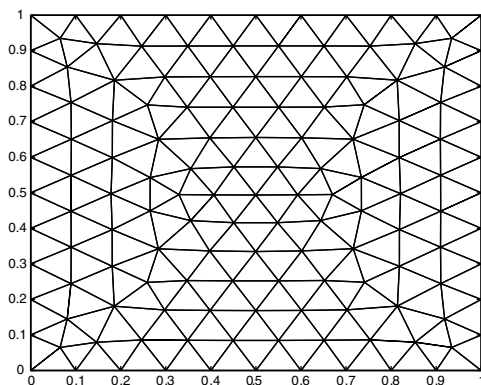


FIGURE 8. An example of unstructured mesh generated by EasyMesh based on Delaunay triangulation

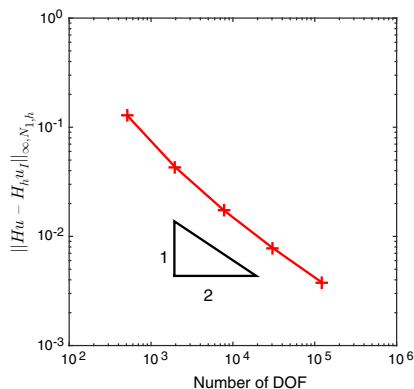


FIGURE 9. Numerical result of Example 1 for the linear element on Delaunay mesh with regular refinement

norm at all vertices and edge centers in an interior region  $\Omega_{1,h}$ . The result of uniform mesh of regular pattern is reported in Figure 10. As predicted by Theorem 3.9,  $H_h u_I$  converges to  $Hu$  at a rate of  $O(h^4)$  which implies  $H_h$  preserves polynomials of degree 5 for the quadratic element on uniform triangulation. For unstructured mesh, we observe that  $H_h u_I$  approximates  $Hu$  at a rate of  $O(h^2)$  from Figure 11.

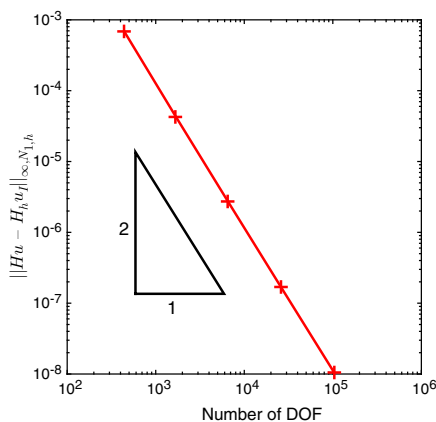


FIGURE 10. Numerical result of Example 1 for the quadratic element on regular pattern uniform mesh

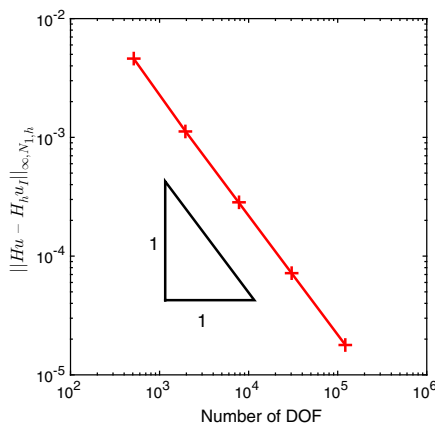


FIGURE 11. Numerical result of Example 1 for the quadratic element on Delaunay mesh with regular refinement

**Example 2.** We consider the following elliptic equation:

$$(5.2) \quad \begin{cases} -\Delta u = 2\pi^2 \sin \pi x \sin \pi y, & \text{in } \Omega = [0, 1] \times [0, 1], \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The exact solution is  $u(x, y) = \sin(\pi x) \sin(\pi y)$ . First, the linear element is considered. In Table 1, we report the numerical results for regular pattern meshes. All four methods ultraconverge at a rate of  $O(h^2)$  in the interior subdomain. The fact that  $H_h^{LS}$  and  $H_h^{ZZ}$  perform as good as  $H_h$  is not a surprise since it is well known that the polynomial preserving recovery is the same as the weighted average for uniform triangular mesh of the regular pattern.

The results of the Chevron pattern is shown in Table 2.  $H_h u_h$  approximates  $Hu$  at a rate of  $O(h^2)$  while  $H_h^{LS} u_h$ ,  $H_h^{ZZ} u_h$  and  $H_h^{QF} u_h$  approximate  $Hu$  at a rate of  $O(h)$ . It is observed that our method out-performs the other three Hessian recovery methods on the Chevron pattern uniform meshes. To the best of our knowledge, the proposed PPR-PPR Hessian recovery is the only method to achieve  $O(h^2)$  superconvergence for the linear element under the Chevron pattern triangular mesh.

TABLE 1. Numerical comparison of several Hessian recovery methods for the linear element on regular pattern uniform mesh

Dof	$De$	order	$De^{ZZ}e$	order	$De^{LS}$	order	$De^{QF}$	order
121	7.93e-001	–	9.73e-001	–	7.93e-001	–	4.01e-001	–
441	2.02e-001	1.06	2.02e-001	1.22	2.02e-001	1.06	1.03e-001	1.05
1681	5.10e-002	1.03	5.10e-002	1.03	5.10e-002	1.03	2.61e-002	1.03
6561	1.28e-002	1.02	1.28e-002	1.02	1.28e-002	1.02	6.53e-003	1.02
25921	3.20e-003	1.01	3.20e-003	1.01	3.20e-003	1.01	1.63e-003	1.01
103041	8.00e-004	1.00	8.00e-004	1.00	8.00e-004	1.00	4.08e-004	1.00

TABLE 2. Numerical comparison of several Hessian recovery methods for the linear element on Chevron pattern uniform mesh

Dof	$De$	order	$De^{ZZ}e$	order	$De^{LS}$	order	$De^{QF}$	order
121	6.51e-001	–	7.98e-001	–	7.82e-001	–	9.03e-001	–
441	1.34e-001	1.22	2.12e-001	1.03	2.34e-001	0.93	4.30e-001	0.57
1681	3.38e-002	1.03	7.96e-002	0.73	9.87e-002	0.64	2.11e-001	0.53
6561	8.46e-003	1.02	3.57e-002	0.59	4.68e-002	0.55	1.05e-001	0.51
25921	2.11e-003	1.01	1.73e-002	0.53	2.30e-002	0.52	5.23e-002	0.51
103041	5.29e-004	1.00	8.57e-003	0.51	1.15e-002	0.50	2.62e-002	0.50

Then the Criss-cross pattern mesh is considered and results are displayed in Table 3. An  $O(h^2)$  convergence rate is observed for our recovery method,  $H_h^{LS}$  and  $H_h^{ZZ}$  while no convergence rate is observed for  $H_h^{QF}$ . The results for the Union-Jack pattern mesh is very similar to the Criss-cross pattern mesh except that our recovery method superconverges at a rate of  $O(h^2)$  as shown in Table 4.

TABLE 3. Numerical comparison of several Hessian recovery methods for the linear element on Criss-cross pattern uniform mesh

Dof	$De$	order	$De^{ZZ}e$	order	$De^{LS}$	order	$De^{QF}$	order
221	5.49e-001	—	3.57e-001	—	4.40e-001	—	7.14e-001	—
841	1.28e-001	1.09	8.03e-002	1.12	1.04e-001	1.08	6.17e-001	0.11
3281	3.22e-002	1.01	2.01e-002	1.02	2.62e-002	1.01	5.95e-001	0.03
12961	8.06e-003	1.01	5.04e-003	1.01	6.55e-003	1.01	5.90e-001	0.01
51521	2.02e-003	1.00	1.26e-003	1.00	1.64e-003	1.00	5.89e-001	0.00
205441	5.04e-004	1.00	3.15e-004	1.00	4.09e-004	1.00	5.88e-001	0.00

TABLE 4. Numerical comparison of several Hessian recovery methods for the linear element on Union-Jack pattern uniform mesh

Dof	$De$	order	$De^{ZZ}e$	order	$De^{LS}$	order	$De^{QF}$	order
121	1.25e+000	—	8.40e-001	—	9.87e-001	—	1.05e+000	—
441	3.16e-001	1.06	1.77e-001	1.20	2.48e-001	1.07	6.95e-001	0.32
1681	7.96e-002	1.03	4.46e-002	1.03	6.24e-002	1.03	6.14e-001	0.09
6561	2.00e-002	1.02	1.12e-002	1.02	1.56e-002	1.02	5.95e-001	0.02
25921	5.00e-003	1.01	2.80e-003	1.01	3.91e-003	1.01	5.90e-001	0.01
103041	1.25e-003	1.00	6.99e-004	1.00	9.78e-004	1.00	5.89e-001	0.00

TABLE 5. Numerical comparison of several Hessian recovery methods for the linear element on Delaunay mesh with regular refinement

Dof	$De$	order	$De^{ZZ}e$	order	$De^{LS}$	order	$De^{QF}$	order
139	4.31e-001	—	4.38e-001	—	4.40e-001	—	3.26e-001	—
513	1.38e-001	0.87	2.20e-001	0.53	1.49e-001	0.83	1.79e-001	0.46
1969	5.39e-002	0.70	2.36e-001	-0.05	5.85e-002	0.69	8.88e-002	0.52
7713	2.38e-002	0.60	1.62e-001	0.28	2.55e-002	0.61	4.35e-002	0.52
30529	1.14e-002	0.54	1.13e-001	0.26	1.19e-002	0.56	2.15e-002	0.51
121473	5.59e-003	0.51	7.97e-002	0.25	5.73e-003	0.53	1.07e-002	0.51

Now, we turn to unstructured mesh generated by EasyMesh [17] as in the previous examples. Numerical data are listed in Table 5.  $H_h$ ,  $H_h^{LS}$  and  $H_h^{QF}$  converge at a rate of  $O(h)$  while  $H_h^{ZZ}$  only converges at a rate of  $O(h^{0.5})$ .

The results above indicate clearly that our Hessian recovery method converges at rate  $O(h)$  on general Delaunay meshes, which is predicted by Theorem 4.3. On uniform meshes, we can obtain  $O(h^2)$  ultraconvergence on an interior sub-domain as predicted by Theorem 4.8.

In the end, we consider the quadratic element. Note that our Hessian recovery method is well defined for arbitrary order elements. However, the extension of the other three methods to the quadratic element is not straightforward or even impossible and hence only our method is implemented here. We report the numerical results in Figure 12 for regular pattern uniform mesh. A rate of about  $O(h^{3.2})$

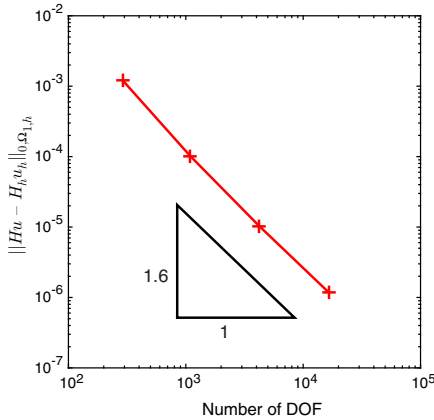


FIGURE 12. Numerical result of Example 2 for the quadratic element on regular pattern uniform mesh

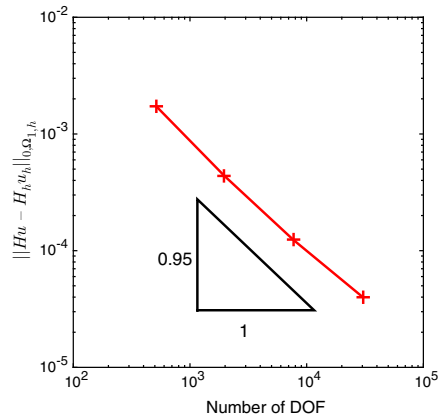


FIGURE 13. Numerical result of Example 2 for the quadratic element on Delaunay mesh with regular refinement

order convergence is observed, which is a bit better than the theoretical result predicted by Theorem 4.8. Figure 13 shows the result for Delaunay mesh generated by EasyMesh [17]. A rate of about  $O(h^{1.9})$  superconvergence is observed.

## 6. CONCLUDING REMARKS

In this work, we introduced a Hessian recovery method for arbitrary order Lagrange finite elements. Theoretically, we proved that the PPR-PPR Hessian recovery operator  $H_h$  preserves polynomials of degree  $k + 1$  on general unstructured meshes and preserves polynomials of degree  $k + 2$  on translation invariant meshes. This polynomial preserving property, combined with the supercloseness property of the finite element method, enabled us to prove convergence and superconvergence results for our Hessian recovery method on mildly structured meshes. Moreover, we proved the ultraconvergence result for the translation invariant finite element space of any order by using the argument of superconvergence by difference quotient from [24].

## REFERENCES

- [1] A. Agouzal and Yu. Vassilevski, *On a discrete Hessian recovery for  $P_1$  finite elements*, J. Numer. Math. **10** (2002), no. 1, 1–12, DOI 10.1515/JNMA.2002.1. MR1905846
- [2] N. E. Aguilera and P. Morin, *On convex functions and the finite element method*, SIAM J. Numer. Anal. **47** (2009), no. 4, 3139–3157, DOI 10.1137/080720917. MR2551161
- [3] R. E. Bank and J. Xu, *Asymptotically exact a posteriori error estimators. I. Grids with superconvergence*, SIAM J. Numer. Anal. **41** (2003), no. 6, 2294–2312 (electronic), DOI 10.1137/S003614290139874X. MR2034616
- [4] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954

- [5] W. Cao, *Superconvergence analysis of the linear finite element method and a gradient recovery postprocessing on anisotropic meshes*, Math. Comp. **84** (2015), no. 291, 89–117, DOI 10.1090/S0025-5718-2014-02846-9. MR3266954
- [6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4. MR0520174
- [7] L. C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR2597943
- [8] X. Gan and J.E. Akin, *Superconvergent second order derivative recovery technique and its application in a nonlocal damage mechanics model*, Finite Elements in Analysis and Design **35** (2014), 118–127.
- [9] C. Huang and Z. Zhang, *Polynomial preserving recovery for quadratic elements on anisotropic meshes*, Numer. Methods Partial Differential Equations **28** (2012), no. 3, 966–983, DOI 10.1002/num.20669. MR2902073
- [10] Y. Huang and J. Xu, *Superconvergence of quadratic finite elements on mildly structured grids*, Math. Comp. **77** (2008), no. 263, 1253–1268, DOI 10.1090/S0025-5718-08-02051-6. MR2398767
- [11] L. Kamenski and W. Huang, *How a nonconvergent recovered Hessian works in mesh adaptation*, SIAM J. Numer. Anal. **52** (2014), no. 4, 1692–1708, DOI 10.1137/120898796. MR3233938
- [12] A. M. Lakhany and J. R. Whiteman, *Superconvergent recovery operators: derivative recovery techniques*, Finite element methods (Jyväskylä, 1997), Lecture Notes in Pure and Appl. Math., vol. 196, Dekker, New York, 1998, pp. 195–215. MR1602865
- [13] O. Lakkis and T. Pryer, *A finite element method for second order nonvariational elliptic problems*, SIAM J. Sci. Comput. **33** (2011), no. 2, 786–801, DOI 10.1137/100787672. MR2801189
- [14] O. Lakkis and T. Pryer, *A finite element method for nonlinear elliptic problems*, SIAM J. Sci. Comput. **35** (2013), no. 4, A2025–A2045, DOI 10.1137/120887655. MR3085125
- [15] A. Naga and Z. Zhang, *A posteriori error estimates based on the polynomial preserving recovery*, SIAM J. Numer. Anal. **42** (2004), no. 4, 1780–1800 (electronic), DOI 10.1137/S0036142903413002. MR2114301
- [16] A. Naga and Z. Zhang, *The polynomial-preserving recovery for higher order finite element methods in 2D and 3D*, Discrete Contin. Dyn. Syst. Ser. B **5** (2005), no. 3, 769–798, DOI 10.3934/dcdsb.2005.5.769. MR2151732
- [17] B. Niceno, *EasyMesh Version 1.4: A Two-Dimensional Quality Mesh Generator*, <http://www.dinma.univ.trieste.it/nirftc/research/easymesh>.
- [18] M. Neilan, *Finite element methods for fully nonlinear second order PDEs based on a discrete Hessian with applications to the Monge-Ampère equation*, J. Comput. Appl. Math. **263** (2014), 351–369, DOI 10.1016/j.cam.2013.12.027. MR3162358
- [19] J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp. **28** (1974), 937–958. MR0373325
- [20] J. S. Owall, *Function, gradient, and Hessian recovery using quadratic edge-bump functions*, SIAM J. Numer. Anal. **45** (2007), no. 3, 1064–1080, DOI 10.1137/060648908. MR2318802
- [21] B. Poulriot, M. Fortin, A. Fortin, and É. Chamberland, *On a new edge-based gradient recovery technique*, Internat. J. Numer. Methods Engrg. **93** (2013), no. 1, 52–65, DOI 10.1002/nme.4374. MR3006282
- [22] M. Picasso, F. Alauzet, H. Borouchaki, and P.-L. George, *A numerical study of some Hessian recovery techniques on isotropic and anisotropic meshes*, SIAM J. Sci. Comput. **33** (2011), no. 3, 1058–1076, DOI 10.1137/100798715. MR2800564
- [23] M.-G. Vallet, C.-M. Manole, J. Dompierre, S. Dufour, and F. Guibault, *Numerical comparison of some Hessian recovery techniques*, Internat. J. Numer. Methods Engrg. **72** (2007), no. 8, 987–1007, DOI 10.1002/nme.2036. MR2360556
- [24] L. B. Wahlbin, *Superconvergence in Galerkin finite element methods*, Lecture Notes in Mathematics, vol. 1605, Springer-Verlag, Berlin, 1995. MR1439050
- [25] H. Wu and Z. Zhang, *Can we have superconvergent gradient recovery under adaptive meshes?*, SIAM J. Numer. Anal. **45** (2007), no. 4, 1701–1722, DOI 10.1137/060661430. MR2338406
- [26] J. Xu and Z. Zhang, *Analysis of recovery type a posteriori error estimators for mildly structured grids*, Math. Comp. **73** (2004), no. 247, 1139–1152 (electronic), DOI 10.1090/S0025-5718-03-01600-4. MR2047081

- [27] Z. Zhang and A. Naga, *A new finite element gradient recovery method: superconvergence property*, SIAM J. Sci. Comput. **26** (2005), no. 4, 1192–1213 (electronic), DOI 10.1137/S1064827503402837. MR2143481
- [28] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery and a posteriori error estimates. I. The recovery technique*, Internat. J. Numer. Methods Engrg. **33** (1992), no. 7, 1331–1364, DOI 10.1002/nme.1620330702. MR1161557

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

*Current address:* Department of Mathematics, University of California, Santa Barbara, California 93106

*E-mail address:* [hlguo@math.ucsb.edu](mailto:hlguo@math.ucsb.edu)

BEIJING COMPUTATIONAL SCIENCE RESEARCH CENTER, BEIJING 100193, CHINA – AND – DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

*E-mail address:* [zmzhang@csrc.ac.cn](mailto:zmzhang@csrc.ac.cn)

*E-mail address:* [zzhang@math.wayne.edu](mailto:zzhang@math.wayne.edu)

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

*E-mail address:* [rzhao@math.wayne.edu](mailto:rzhao@math.wayne.edu)