Numer. Math. (2018) 138:473–500
https://doi.org/10.1007/s00211-017-0912-8

Numerische Mathematik

Maximum-norms error estimates for high-order finite volume schemes over quadrilateral meshes

Wenming He[1,2] · Zhimin Zhang[3,4] · Qingsong Zou[5]

Received: 14 December 2016 / Revised: 18 June 2017 / Published online: 28 August 2017

Abstract In this paper, we perform L^∞ and $W^{1,\infty}$ error estimates for a class of bi-k finite volume schemes on a quadrilateral mesh for elliptic equations, where $k \geq 2$ is arbitrary. We show that the errors of the finite volume solution in both the L^∞ and $W^{1,\infty}$ norms converge to zero with optimal orders, provided the solution $u \in W^{k+2,\infty}$. Our analysis is based mainly on an estimate of the difference between the finite volume and the corresponding finite element bilinear forms, as well as some techniques derived

W. He: Research supported in part by NSFC Grant 11671304, and the Zhejiang Provincial Natural Science Foundation, China LY15A010015. Z. Zhang: Research supported in part by NSFC Grants 11471031 and 91430216, NASF Grant U1530401, and NSF Grant DMS-1419040. Q. Zou: Research supported in part by the following Grants: the special project High performance computing of National Key Research and Development Program 2016YFB0200604, NSFC 11571384, Guangdong Provincial NSF 2014A030313179, the Fundamental Research Funds for the Central Universities 16lgjc80.

✉ Qingsong Zou
mcszqs@mail.sysu.edu.cn

Wenming He
he_wenming@aliyun.com; hewenming@wzu.edu.cn

Zhimin Zhang
zmzhang@csrc.ac.cn; zzhang@math.wayne.edu

1 School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang 524048, Guangdong, China

2 Department of Mathematics, Wenzhou University, Wenzhou 320035, Zhejiang, China

3 Beijing Computational Science Research Center, Beijing 100193, China

4 Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

5 School of Data and Computer Science and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen University, Guangzhou 510275, China

for L^∞ and $W^{1,\infty}$ estimates of the finite element method. Our theoretical findings are supported by several numerical examples.

Mathematics Subject Classification Primary 65N30; Secondary 45N08

1 Introduction

As an important numerical tool for solving PDEs, the finite volume method (FVM) enjoys great popularity among scientific and engineering computations, see, e.g., [15, 23,25,33,36]. However, the FVM theory (cf., [2,3,5–9,11,16,19,26,27,30,31,35,40, 42,43]) is far less developed, especially for high-order schemes. This paper is one of a series that attempts to set up a mathematical foundation for high-order FVMs. Note that very recently, we studied the stability and H^1-norm errors of high-order FVMs in [44], and L^2 norm errors of high-order FVMs in [28].

In this article, we will investigate the maximum-norm errors of high-order FVMs. Precisely, we will study the L^∞ and $W^{1,\infty}$ norm error of a class of FVMs of any order for the elliptic model problem:

$$\mathcal{L}u \equiv -\nabla \cdot (\alpha \nabla u) = f \text{ in } \Omega, \quad \text{and} \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\Omega \subset \Re^2$ is a bounded domain, $\alpha = (a_{ij})_{2\times 2} \in (W^{k+1+\sigma,\infty}(\Omega))^{2\times 2}$ for some $\sigma > 0$ is a uniformly positive definite matrix in the sense that there exists $\delta > 0$ such that for all $x = (x_1, x_2)^T \in \Omega$, $\xi = (\xi_1, \xi_2)^T \in \Re^2$, one has $\xi^T \alpha(x)\xi \geq \delta|\xi|^2$. Here we choose homogenous Dirichlet boundary condition only for the sake of simplicity. Since the FVM is a local method which can be used for arbitrary boundary conditions, our maximum-norm error estimates in this paper can be extended to problems with nonhomogenous Dirichlet boundary conditions and other boundary conditions.

Since the 1970s, the maximum-norm error of the finite element method (FEM) has been intensively studied, see e.g. [14,34,37–39,41]. The superconvergence properties of the FEM based on these maximum-norm error estimates have been studied in [20–22,45] However, to the best of our knowledge, only one paper (see [12]) on the maximum-norm error of the finite volume method has been published. In their paper [12], Chou and Li showed that both the L^∞ and $W^{1,\infty}$-norm errors of the linear FVM converge with optimal orders, by using the fact that the linear FVM solution is superclose to the linear FEM solution. But for high-order (i.e. order $k \geq 2$) schemes, the FVM solution is usually significantly different from the FEM solution, so the approach in [12] is hard to extend to high-order FVM methods.

In this article, we estimate the maximum-norm error of the high-order FVMs in a different way. First, based on H^1 and L^2 error estimates [28,44] of high-order FVMs, we use the partition of unity technique in [38,39] to estimate local H^1-norm errors of high-order FVMs. This local error estimate is then applied to prove that the $W^{1,\infty}$-norm errors of high-order FVMs have optimal convergence rates. Since we do not have optimal-order error estimates for the negative-norm error of the FVM, we cannot establish L^2-norm local error estimates for the FVM, and thus we cannot apply the above $W^{1,\infty}$-norm techniques to estimate the L^∞-norm errors of high-order FVMs. In fact, the L^∞-norm error in our paper is established with the help of the *discrete Green function*. Unlike the definition based on the FEM bilinear form in [12], our *discrete Green function* is defined by using the FVM bilinear forms. By using a novel discussion of a property of the exact Green's function (see Proposition 4.1), we obtain a $W^{1,1}$-norm estimate of the approximation error of our discrete Green function. The optimal-order L^∞-norm error estimate of high-order FVMs is then established by using the above approximation error estimate of the discrete Green function.

We need to point out that our above optimal-convergence-order results are proved under the regularity assumption $u \in W^{k+2,\infty}$ which is one order higher than that required in the maximum-norm error estimate for the FEM solution. The main reason for this stronger regularity requirement is that the analysis in this paper is established based on the optimal convergence order of the L^2-norm estimate in [28]. Since there is a counterexample in [28] to show that when $u \in H^{k+1}$ but $u \notin H^{k+2}$, the L^2-norm error of the FVM solution does not converge with the optimal order $k+1$, we think the regularity assumption $u \in W^{k+2,\infty}$ in our paper is an almost sharp requirement for the maximum-norm error estimates for FVMs.

The rest of the paper is organised as below. In Sect. 2, the difference between the high-order FVM and FEM bilinear forms is estimated. In Sect. 3, by establishing some local estimate results, the $W^{1,\infty}$-norm error of high-order FVMs is shown to decay with optimal convergence order. Section 4 is dedicated to the L^∞ error estimate of the high-order FV schemes. For this purpose, some discrete Green's functions corresponding to finite volume schemes are defined and discussed. We present several numerical examples to support our theory in Sect. 5. Some concluding remarks are presented in the sixth and last section.

We close the section with a presentation of some notation which will be used in the rest of the paper. We denote by $W^{m,p}(S)$, $S \subset \Omega, m \geq 1, 1 \leq p \leq \infty$ the standard Sobolev spaces equipped with the semi-norm $|v|_{m,p,S} = \left(\int_S \sum_{|\alpha|=m} |D^\alpha v|^p dx\right)^{\frac{1}{p}}$ and the norm $\|v\|_{m,p,S} = \left(\sum_{l=0}^m |v|_{l,p,S}\right)^{\frac{1}{p}}$. For simplicity, we write $H^m(S) = W^{m,2}(S)$ and $|\cdot|_{m,S} = |\cdot|_{m,2,S}$. Moreover, we use the abbreviation $W^{m,p} = W^{m,p}(\Omega)$, $H^m = H^m(\Omega)$ and $|\cdot|_{m,p} = |\cdot|_{m,p,\Omega}, |\cdot|_m = |\cdot|_{m,\Omega}, \|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega}, \|\cdot\|_m = \|\cdot\|_{m,\Omega}$. In the rest of this paper, "$A \lesssim B$" means that A can be bounded by B multiplied by a constant which is independent of the parameters which A and B may depend on. "$A \sim B$" means "$A \lesssim B''$ and "$B \lesssim A$".

2 Any order FVMs over quadrilateral meshes

This section is dedicated to a presentation of a class FV schemes of any order that we introduced in [28,44] and an estimate of the difference between high-order FVMs and FEMs.

We start our presentation of FV schemes of any order with a description of the quadrilateral mesh. Let $\mathcal{T}_h = \{Q\}$ be a conforming and shape-regular quadrilateral partition of Ω. We assume $\mathcal{T}_h$ to be an h^2- *parallelogram mesh*, in the sense that for any $Q \in \mathcal{T}_h$, the distance between the midpoints of two diagonals of Q is $\mathcal{O}(h^2)$, see e.g. [1,10,13,42,44]. We denote by $\mathcal{N}_h$ and $\mathcal{E}_h$ the sets of all vertices and all edges of $\mathcal{T}_h$, respectively. Moreover, let $\mathcal{N}_h^\circ = \mathcal{N}_h \setminus \partial\Omega$, $\mathcal{E}_h^\circ = \mathcal{E}_h \setminus \partial\Omega$, $\mathcal{N}_h^b = \mathcal{N}_h \cap \partial\Omega$ and $\mathcal{E}_h^b = \mathcal{E}_h \cap \partial\Omega$ be the sets of interior vertices, internal edges, boundary vertices and boundary edges, respectively. For each $Q \in \mathcal{T}_h$, let F_Q be the bilinear transformation from the reference $\hat{Q} = [-1, 1]^2$ to Q. Let $\{g_i | i = 1, \ldots, k\}$ be the k Gauss points, i.e., zeros of L_k (the Legendre polynomial of degree k) on the interval $[-1, 1]$, and let $\{l_j | j = 0, \ldots, r\}$ be the $k+1$ Lobatto points of degree k in the interval $[-1, 1]$, that is, $l_0 = -1$, $l_k = 1$ and $\{l_m | m = 1, \ldots k-1\}$ are the $k-1$ zeros of L_k'. We define the set of Gauss points and Lobatto points in Q by $\mathcal{G}_Q = \left\{G_{i,j}^Q = F_Q(g_i, g_j) \middle| i, j \in \{1, \ldots k\}\right\}$ and $\mathcal{L}_Q = \left\{L_{i,j}^Q = F_Q(l_i, l_j) \middle| i, j \in \{0, 1, \ldots, k\}\right\}$, respectively. Moreover let $\mathcal{G} = \cup_{Q \in \mathcal{T}_h} \mathcal{G}_Q$ and $\mathcal{L} = \cup_{Q \in \mathcal{T}_h} \mathcal{L}_Q$ be the sets of all Gauss and Lobatto points in $\mathcal{T}_h$, respectively.

We design our finite volume scheme by constructing the so-called *dual mesh* with Gauss points. We decompose each $Q \in \mathcal{T}_h$ into $(k+1)^2$ sub-quadrilaterals Q_P, $P \in \mathcal{L}_Q$ by connecting with a line segment each Gauss point on one edge to the one at the same position of its opposite edge. For any given Lobatto point $P \in \mathcal{L}$, we construct a control volume V_P as the union of all quadrilaterals Q_P containing the node P. The collection of all control volumes $\mathcal{T}_h^* = \{V_P | P \in \mathcal{L}\}$ constitutes the dual mesh of Ω.

Letting the standard FEM space

$$\mathcal{U}_h = \{v \in C(\Omega) \big| \hat{v}_Q = v \circ F_Q \in \mathbb{Q}_k(\hat{Q}),\ \forall Q \in \mathcal{T}_h,\ \text{and } v|_{\partial\Omega} = 0\},$$

where $\mathbb{Q}_k(\hat{Q})$ consists of all bi-k polynomials, the FV solution of (1) reads as: find $u^h \in \mathcal{U}_h$ such that the conservation law

$$-\int_{\partial V_P} (\alpha \nabla u^h) \cdot \mathbf{n} ds = \int_{V_P} f dxdy \tag{2}$$

holds on each control volume V_P, $P \in \mathcal{L}^\circ$, where $\mathbf{n}$ is the unit outward normal on the boundary ∂V_P.

The scheme (2) can be rewritten under the framework of Petrov–Galerkin method. Let the test space

$$\mathcal{V}_h = \text{Span}\{\psi_{V_P} \big| P \in \mathcal{L}^\circ\},$$

where $\mathcal{L}^\circ = \mathcal{L} \setminus \partial\Omega$ is the set of all interior Lobatto points and ψ_A is the characteristic function of some set $A \subset \Omega$ defined by $\psi_A(x) = 1$ if $x \in A$ and $\psi_A(x) = 0$ if $x \in \Omega \setminus A$. Note that $\dim \mathcal{V}_h = \#\mathcal{L}^0 = \dim \mathcal{U}_h$. Apparently each $w_h \in \mathcal{V}_h$ it can be written as

$$w_h = \sum_{P \in \mathcal{L}^\circ} w_P \psi_{V_P}$$

where the coefficient w_P is a constant. Multiplying (2) with w_h and then summing up for all $P \in \mathcal{L}^\circ$, we obtain

$$a_h(u^h, w_h) = (f, w_h), \quad \forall w_h \in \mathcal{V}_h, \tag{3}$$

where the FV bilinear form is defined for all $v \in H_0^1(\Omega)$, $w_h \in \mathcal{V}_h$ as

$$a_h(v, w_h) = -\sum_{p \in \mathcal{L}^\circ} w_P \int_{\partial V_P} (\alpha \nabla v) \cdot \mathbf{n} ds, \tag{4}$$

and $(\cdot, \cdot)$ is the standard L^2 inner product. Denoting by $\mathcal{E}_h'$ the set of interior edges of the dual partition $\mathcal{T}_h^*$, the bilinear form $a_h(\cdot, \cdot)$ can be rewritten as

$$a_h(v, w_h) = \sum_{E \in \mathcal{E}_h'} [w_h]_E \int_E \alpha \frac{\partial v}{\partial \mathbf{n}} ds, \quad \forall v \in H_0^1(\Omega), \quad w_h \in \mathcal{V}_h, \tag{5}$$

where $[w_h]_E = w_h\big|_{V_2} - w_h\big|_{V_1}$ denotes the jump of the w_h across the common edge $E = V_1 \cap V_2$ of two volumes $V_1, V_2 \in \mathcal{T}_h^*$ and $\mathbf{n}$ denotes the normal vector on E pointing from V_1 to V_2. Note that the similar bilinear finite volume scheme over quadrilateral meshes has been proposed and studied in [29,30].

We equip (c.f., [43]) a semi-norm in the test space $\mathcal{V}_h$ for all $w_h \in \mathcal{V}_h$ by

$$\left|w_h\right|_{1,p}' = \left(\sum_{E \in \mathcal{E}_h'} h_E^{2-p} \left| [w_h]_E \right|^p \right)^{\frac{1}{p}}, \quad p \geq 1,$$

where h_E is the diameter of an edge E, and a semi-norm in *broken* $W^{2,p}$ *space*

$$W_h^{2,p}(\Omega) = \left\{ v \in W^{1,p}(\Omega) : v|_Q \in W^{2,p}, \ \forall Q \in \mathcal{T}_h \right\}, \quad p \geq 1,$$

for all $v \in W_h^{2,p}(\Omega)$ by

$$\left|v\right|_{1,p}^h = \left(\sum_{Q \in \mathcal{T}_h} |v|_{1,p,Q}^p + h_Q^p |v|_{2,p,Q}^p \right)^{\frac{1}{p}}, \quad p \geq 1,$$

where h_Q is the diameter of Q.

Lemma 1 *The finite volume bilinear form $a_h(\cdot, \cdot)$ is variationally exact: If $u \in H_0^1(\Omega)$ is the solution of* (1)*, then for any $w_h \in \mathcal{V}_h$ we have*

$$a_h(u, w_h) = (f, w_h), \tag{6}$$

and $a_h(\cdot,\cdot)$ *is continuous in the sense that for all* $v \in H_0^1(\Omega)\cap W_h^{2,p}(\Omega)$ *and* $w_h \in \mathcal{V}_h$*,*

$$|a_h(v, w_h)| \lesssim |v|_{1,p}^h |w_h|'_{1,q}, \tag{7}$$

where $p, q \geq 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof The equality (6) holds apparently by multiplying (1) with an arbitrary function $w_h \in \mathcal{V}_h$ and applying Green's formula in each control volume. Next we will focus on the proof of (7).

Utilizing the Cauchy–Schwartz inequality, for all $v \in H_0^1(\Omega)$ and $w_h \in \mathcal{V}_h$, there holds

$$\begin{aligned} a_h(v, w_h) &\lesssim \sum_{E\in\mathcal{E}_h'} |[w_h]_E| \int_E \left|\frac{\partial v}{\partial \mathbf{n}}\right| ds \\ &= \sum_{E\in\mathcal{E}_h'} \int_E \left(h_E^{\frac{1}{p}} \left|\frac{\partial v}{\partial \mathbf{n}}\right|\right) \left(h_E^{\frac{1}{q}-1} |[w_h]_E|\right) ds \\ &\leq \sum_{E\in\mathcal{E}_h'} \left(\int_E h_E \left|\frac{\partial v}{\partial \mathbf{n}}\right|^p ds\right)^{\frac{1}{p}} \left(\int_E h_E^{1-q} |[w_h]_E|^q ds\right)^{\frac{1}{q}} \\ &\leq |w_h|'_{1,q} \left(\sum_{E\in\mathcal{E}_h'} h_E \int_E \left|\frac{\partial v}{\partial \mathbf{n}}\right|^p ds\right)^{\frac{1}{p}}. \end{aligned}$$

By the trace inequality, Young's inequality and shape regularity of $\mathcal{T}_h$, one obtains

$$\left(\int_{E\cap Q} h_E \left|\frac{\partial v}{\partial \mathbf{n}}\right|^p ds\right)^{\frac{1}{p}} \lesssim |v|_{1,p,Q} + h_Q |v|_{2,p,Q},$$

where $Q \in \mathcal{T}_h$ and $Q \cap E \neq \emptyset$. Since for any given $E \in \mathcal{E}_h'$, there are at most two elements $Q \in \mathcal{T}_h$ such that $Q \cap E \neq \emptyset$, we can deduce

$$\begin{aligned} a_h(v, w_h) &\lesssim |w_h|'_{1,q} \left(\sum_{E\in\mathcal{E}_h'} \sum_{Q\in\mathcal{T}_h, Q\cap E\neq\emptyset} |v|_{1,p,Q}^p + h_Q^p |v|_{2,p,Q}^p\right)^{\frac{1}{p}} \\ &\lesssim |w_h|'_{1,q} \left(\sum_{Q\in\mathcal{T}_h} |v|_{1,p,Q}^p + h_Q^p |v|_{2,p,Q}^p\right)^{\frac{1}{p}} \\ &= |v|_{1,p}^h |w_h|'_{1,q}, \end{aligned}$$

which implies the inequality (7). □

Since $\dim \mathcal{U}_h = \dim \mathcal{V}_h$, there exists [28,44] a linear bijection Π which maps each $v_h \in \mathcal{U}_h$ to $v_h^* = \Pi v_h \in \mathcal{V}_h$ such that the following boundedness of Π

$$|\Pi v_h|'_{1,p} \lesssim |v_h|_{1,p}, v_h \in \mathcal{U}_h, \quad p \geq 1, \tag{8}$$

and the coercivity property (See [44])

$$a_h\left(v_h, v_h^*\right) \gtrsim |v_h|^2, \quad v_h \in \mathcal{U}_h, \tag{9}$$

hold.

Moreover, it is shown in [44] that if $u \in H^{k+1}(\Omega)$,

$$|u - u^h|_1 \lesssim h^k \|u\|_{k+1}, \tag{10}$$

and is shown in [28] that if $u \in H^{k+1}(\Omega)$, $f \in H^k(\Omega)$, then

$$\|u - u^h\|_0 \lesssim h^{k+1}(\|u\|_{k+1} + \|f\|_k). \tag{11}$$

Introducing the elliptic projection $P_h : H_0^1(\Omega) \to \mathcal{U}_h$

$$a_h(P_h v, w) = a_h(v, w), \quad \forall v \in \mathcal{U}_h, \quad w \in \mathcal{V}_h, \tag{12}$$

and denoting $v^h = P_h v$ for simplicity, we derive from (10) that if $v \in H^{k+1}(\Omega)$,

$$|v - v^h|_1 \lesssim h^k \|v\|_{k+1}, \tag{13}$$

and from (11) that if $v \in H^{k+2}(\Omega)$, then

$$\|v - v^h\|_0 \lesssim h^{k+1} \|v\|_{k+2}. \tag{14}$$

On the other hand, it is well-known that for quadrilateral meshes, the FV bilinear form (or equivalently, the FV stiffness matrix) is significant different from its FE counterparts. However, we have that (c.f. [28,44]), for all $v_h \in \mathcal{U}_h$, $w_h \in \mathcal{U}_h^1 := \{v \in C(\Omega) \big| \hat{v}_Q = v \circ F_Q \in \mathbb{Q}_1(\hat{Q}), \forall Q \in \mathcal{T}_h$, and $v|_{\partial\Omega} = 0\}$,

$$\left|a(v_h, w_h) - a_h\left(v_h, w_h^*\right)\right| \lesssim h|v_h|_1|w_h|_{2,h}, \tag{15}$$

where for $w_h \in \mathcal{U}_h$, the *broken*-norm

$$|w_h|_{2,h} = \left(\sum_{\tau \in \mathcal{T}_h} |w_h|_{2,\tau}^2\right)^{\frac{1}{2}},$$

and for all $v, w \in H_0^1(\Omega)$,

$$a(v, w) = \int_\Omega \alpha \nabla v \nabla w dx.$$

Moreover, we have the following estimate.

Proposition 1 *For all $v \in W^{k+1,\infty}(\Omega)$, we have*

$$\left|a(v, w_h) - a_h\left(v, w_h^*\right)\right| \lesssim h^{k+1} \|v\|_{k+1,\infty} |w_h|_{2,1,h}, \quad w_h \in \mathcal{U}_h^1, \tag{16}$$

where for $w_h \in \mathcal{U}_h$, the broken-norm

$$|w_h|_{2,1,h} = \sum_{\tau \in \mathcal{T}_h} |w_h|_{2,1,\tau},$$

where

$$|w_h|_{2,1,\tau} = \sum_{|\mathbf{m}|=2} \int_\tau |D^{\mathbf{m}} w_h| dx, \quad |\mathbf{m}| = m_1 + m_2, \quad \mathbf{m} = (m_1, m_2).$$

Proof As shown in [28], the FV bilinear form $a_h(\cdot, \Pi \cdot)$ can be regarded as a Gauss quadrature of the FE bilinear form $a(\cdot, \cdot)$. That is,

$$a(v, w_h) - a_h\left(v, w_h^*\right) = \sum_{Q \in \mathcal{T}_h} \tilde{a}_Q(v, w_h) - \tilde{a}_{h,Q}\left(v, w_h^*\right),$$

with the elementwise difference

$$\begin{aligned}\tilde{a}_{h,Q}\left(v, w_h^*\right) - \tilde{a}_Q(v, w_h) = &\int_{-1}^1 E_1(\Theta_1, \hat{y}) d\hat{y} + \sum_{j=1}^k A_j E_2(\Theta_1, g_j) \\ &+ \int_{-1}^1 E_2(\Theta_2, \hat{x}) d\hat{x} + \sum_{j=1}^k A_j E_1(\Theta_2, g_j),\end{aligned}$$

where $A_j, j = 1, \ldots, k$ are the weights of the k-point-Gauss-quadrature for computing the integral $\int_{-1}^1 z(x)dx$, and for all $Q \in \mathcal{T}_h$, the mapping F_Q is the bilinear transformation from the reference $\hat{Q} = [-1, 1]^2$ to Q, $\hat{v}_h = v_h \circ F_Q$, $v_h \in \mathcal{U}_h$, and the matrix $b = \left((-1)^{i+j} b_{i,j}\right)_{2\times 2}$ with $(b_{i,j})_{2\times 2} = J_Q^{-1}(DF_Q)(DF_Q)^T$, and for all $(\hat{x}, \hat{y}) \in \hat{Q}$,

$$\begin{aligned}\hat{V}_1(\hat{x}, \hat{y}) &= \hat{V}_1(-1, \hat{y}) + \int_{-1}^{\hat{x}} \left(b_{12}\frac{\partial \hat{v}}{\partial \hat{x}} + b_{11}\frac{\partial \hat{v}}{\partial \hat{y}}\right)(\hat{x}', \hat{y}) d\hat{x}', \\ \hat{V}_2(\hat{x}, \hat{y}) &= \hat{V}_2(\hat{x}, -1) + \int_{-1}^{\hat{y}} \left(b_{22}\frac{\partial \hat{v}}{\partial \hat{x}} + b_{21}\frac{\partial \hat{v}}{\partial \hat{y}}\right)(\hat{x}, \hat{y}') d\hat{y}',\end{aligned}$$

and $\Theta_i = \hat{\Theta}_{i,Q} = \frac{\partial \hat{w}_h}{\partial \hat{x}\partial \hat{y}} \hat{V}_i$, $i = 1, 2$, and for any function F defined on $\hat{Q}$, we define the error of Gauss quadrature along the $x-$ and $y-$ directions by

$$E_1(F, \hat{y}) = \int_{-1}^{1} F(\hat{x}, \hat{y}) d\hat{x} - \sum_{i=1}^{s+1} A_i F(g_i, \hat{y}),$$
$$E_2(F, \hat{y}) = \int_{-1}^{1} F(\hat{x}, \hat{y}) d\hat{y} - \sum_{j=1}^{s+1} A_j F(\hat{x}, g_j),$$

respectively.

Next we estimate $E_1(\Theta_1, \hat{y})$, $E_2(\Theta_2, \hat{x})$, $E_2(\Theta_1, \hat{x})$ and $E_1(\Theta_2, \hat{y})$ for $(\hat{x}, \hat{y}) \in \hat{Q}$. Without loss of generality, we only estimate $E_1(\Theta_1, \hat{y})$, since the estimates of $E_2(\Theta_2, \hat{x})$, $E_2(\Theta_1, g_i)$ and $E_1(\Theta_2, g_j)$ can be obtained in a similar way.

Since for any given $\hat{y} \in [-1, 1]$, $E_1(\Theta_1, \hat{y})$ is the difference between an exact integral and a numerical quadrature of order k, then we have

$$|E_1(\Theta_1, \hat{y})| \lesssim \left\| \frac{\partial^k}{\partial \hat{x}^k} \Theta_1(\cdot, \hat{y}) \right\|_{L^\infty}.$$

Noticing that $\frac{\partial \hat{w}_h}{\partial \hat{x}\partial \hat{y}}$ is a constant in $\hat{Q}$, we have

$$\frac{\partial^k}{\partial \hat{x}^k} \Theta_1 = \frac{\partial^2 \hat{w}_h}{\partial \hat{x}\partial \hat{y}} \frac{\partial^k \hat{V}_1}{\partial \hat{x}^k}. \tag{17}$$

Note that by ([28], Lemma 4.1)

$$|D^{\mathbf{i}} b_{lm}| \lesssim h^{|\mathbf{i}|+1}, \quad 1 \le l, \quad m \le 2,$$

where $\mathbf{i} = (i_1, i_2)$, $i_1, i_2 \ge 0$, $|\mathbf{i}| = i_1 + i_2$ and J_Q is the determinant of DF_Q, the Jacobi matrix of F_Q. Then by Leibnitz formula, one obtains

$$\begin{aligned}
\left| \frac{\partial^k}{\partial \hat{x}^k} \hat{V}_1 \right| &= \left| \frac{\partial^{k-1}}{\partial \hat{x}^{k-1}} \left(\frac{\partial \hat{v}}{\partial \hat{x}} b_{1,2} + \frac{\partial \hat{v}}{\partial \hat{x}} b_{1,1} \right) \right| \\
&\lesssim \sum_{l=0}^{k-1} \left| \left(\frac{\partial^{l+1} \hat{v}}{\partial \hat{x}^{l+1}} \frac{\partial^{k-1-l} b_{12}}{\partial \hat{x}^{k-1-l}} + \frac{\partial^{l+1} \hat{v}}{\partial \hat{y} \partial \hat{x}^{l}} \frac{\partial^{k-1-l} b_{11}}{\partial \hat{x}^{k-1-l}} \right) \right| \\
&\lesssim \sum_{l=0}^{k-1} h^{k-l} |\hat{v}|_{l+1,\infty,\hat{Q}},
\end{aligned} \tag{18}$$

and by the scaling arguments,

$$|\hat{v}|_{l+1,\infty,\hat{Q}} \lesssim h^{l+1} \|v\|_{l+1,\infty,Q}.$$

Combining (17) with (18), we deduce that

$$\left|\frac{\partial^k}{\partial \hat{x}^k}\hat{V}_1\right| \lesssim \sum_{l=0}^{k-1} h^{k-l}h^{l+1}\|v\|_{l+1,\infty,Q} \lesssim h^{k+1}\|v\|_{k,\infty,Q}.$$

Moreover, we have

$$\|\hat{w}_h\|_{2,\infty,\hat{Q}} \lesssim h^2\|w_h\|_{2,\infty,Q} \lesssim \|w_h\|_{2,1,Q}.$$

Therefore,

$$|E_1(\Theta_1, \hat{y})| \lesssim h^{k+1}\|u\|_{k,\infty,Q}\|w_h\|_{2,1,Q}.$$

By the same arguments, we obtain

$$|E_1(\Theta_2, \hat{y})| \lesssim h^{k+1}\|u\|_{k+1,\infty,Q}\|w_h\|_{2,1,Q},$$
$$|E_2(\Theta_1, \hat{x})| \lesssim h^{k+1}\|u\|_{k,\infty,Q}\|w_h\|_{2,1,Q},$$

and

$$|E_2(\Theta_1, \hat{y})| \lesssim h^{k+1}\|u\|_{k+1,\infty,Q}\|w_h\|_{2,1,Q}.$$

Summing up the above estimates over all elements $Q \in \mathcal{T}_h$, we obtain the (16). □

3 A $W^{1,\infty}$ error estimate

We begin with some notation. Letting $R_m = 2^m h$ for some integer $m \geq 0$, and for given $x_0 \in \Omega$, letting $m_0 = m_0(x_0) \lesssim |\ln h|$ satisfy

$$R_{m_0-1} < \max_{x\in\Omega} |x - x_0| \leq R_{m_0}, \tag{19}$$

we define $\Omega_{-1} = \Omega_{-2} = \emptyset$ and for $0 \leq m \leq m_0$,

$$\Omega_m = \bigcup_{e\subset B(x_0,R_m),e\in\mathcal{T}_h} e. \tag{20}$$

We suppose that the functions $\{\phi_m \in C^\infty(\Omega)|0 \leq m \leq m_0\}$ satisfy the following three properties:

(1) $0 \leq \phi_m \leq 1$, $\sum_{m=0}^{m_0} \phi_m = 1$ in the whole Ω;
(2) $\phi_m = 0$ in $\Omega_{m-2} \cup (\Omega \setminus \Omega_{m+1})$;
(3) $|\phi_m|_{l,\infty} \lesssim R_m^{-l}$, $\forall l \geq 1$.

Letting $I_h = I_h^k$ be the $k-$th degree interpolating operator over $\mathcal{T}_h$, for any given $v \in H^1(\Omega)$, we define

$$v_m = (v - I_h v)\phi_m \quad \text{and} \quad v_m^h = P_h v_m.$$

Lemma 2 *There hold the estimates*

$$\left|v_m^h\right|_1 \lesssim h^k R_m \|v\|_{k+1,\infty}, \quad v \in W^{k+1,\infty}(\Omega), \tag{21}$$

$$\left\|v_m^h\right\|_0 \lesssim h^{k+1} R_m |v|_{k+2,\infty}, \quad v \in W^{k+2,\infty}(\Omega). \tag{22}$$

Proof Noting $v_m^h = \left(v_m^h - v_m\right) + v_m$, we next estimate $v_m^h - v_m$ and v_m seperately. For any $l \geq 1$,

$$\begin{aligned}|v_m|_l &= \left(\sum_{e\in\Omega_{m+1}} |v_m|_{l,e}^2\right)^{\frac{1}{2}} \leq \left(\sum_{e\in\Omega_{m+1}} \sum_{j=0}^{l} |v - I_h v|_{j,e}^2 \cdot R_m^{-2(l-j)}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{e\in\Omega_{m+1}} \sum_{j=0}^{l} |v|_{l,e}^2 h^{2(l-j)} \cdot R_m^{-2(l-j)}\right)^{\frac{1}{2}} \\ &\lesssim |v|_{l,\Omega_{m+1}} \lesssim R_m |v|_{l,\infty}.\end{aligned} \tag{23}$$

Consequently, by (13) and (14),

$$\left|v_m - v_m^h\right|_1 \lesssim h^k \|v_m\|_{k+1} \lesssim h^k R_m \|v\|_{k+1,\infty}, \tag{24}$$

$$\left\|v_m - v_m^h\right\|_0 \lesssim h^{k+1} \|v_m\|_{k+2} \lesssim h^{k+1} R_m \|v\|_{k+2,\infty}. \tag{25}$$

On the other hand,

$$|v_m|_1 \lesssim |v - I_h v|_{1,\Omega_{m+1}} + R_m^{-1} \|v - I_h v\|_{0,\Omega_{m+1}} \lesssim h^k R_m |v|_{k+1,\infty}, \tag{26}$$

$$\|v_m\|_0 \lesssim \|v - I_h v\|_{0,\Omega_{m+1}} \lesssim h^{k+1} R_m |v|_{k+1,\infty}. \tag{27}$$

The statements (21) and (22) follow from (24), (26) and (25), (27) respectively. □

The following *strengthened inverse inequality* will play an important role in our $W^{1,\infty}$ error estimate.

Lemma 3 *Let* $D_1 \subset D_2 \subset \Omega$, $D_2 = D_1 \cup \{e \in \mathcal{T}_h | \text{dist}(e, D_1) \leq h\}$. *Let* $v \in H^1(\Omega)$ *satisfy: (1)* $v = 0$ *in* D_2*; (2)* $v^h = P_h v \in \mathcal{U}_h$ *satisfy* $|v^h|_{1,D_2\setminus D_1} \leq \epsilon |v^h|_{1,D_1}$ *and* $|v^h|_{0,D_2\setminus D_1} \leq \epsilon |v^h|_{0,D_1}$ *for sufficiently small* $\epsilon > 0$. *Then*

$$|v^h|_{1,D_1} \lesssim \sqrt{\epsilon} h^{-1} \|v^h\|_{0,D_1}. \tag{28}$$

Proof By the definition of D_2, there exists a $\varphi \in C^\infty(\Omega)$ satisfying: (i) $0 \le \varphi(x) \le 1, \forall x \in \Omega$; (ii) $\varphi = 1$ in D_1 and $\varphi = 0$ in $\Omega \setminus D_2$; (iii) $|\varphi|_{1,\infty} \lesssim h^{-1}$. Then

$$a_h(v^h, (I_h(v^h\varphi))^*) = a_h(v, (I_h(v^h\varphi))^*) = 0,$$

and thus,

$$\begin{aligned} |v^h|^2_{1,D_1} &\le |I_h(v^h\varphi)|^2_{1,D_2} \\ &\lesssim a_h(I_h(v^h\varphi), (I_h(v^h\varphi))^*) \\ &= a_h(I_h(v^h\varphi) - v^h, (I_h(v^h\varphi))^*) \\ &\lesssim |I_h(v^h\varphi) - v^h|_{1,D_2} |I_h(v^h\varphi)|_{1,D_2} \\ &\lesssim |v^h\varphi - v^h|_{1,D_2} |v^h\varphi|_{1,D_2}. \end{aligned}$$

For sufficiently small $\epsilon > 0$,

$$\begin{aligned} |v^h\varphi - v^h|_{1,D_2} &\lesssim |v^h|_{1,D_2\setminus D_1} + h^{-1}|v^h|_{0,D_2\setminus D_1} \\ &\lesssim \epsilon(|v^h|_{1,D_1} + h^{-1}|v^h|_{0,D_1}), \end{aligned}$$

and

$$\begin{aligned} |v^h\varphi|_{1,D_2} &\lesssim |v^h|_{1,D_1} + |v^h|_{1,D_2\setminus D_1} + h^{-1}|v^h|_{0,D_2\setminus D_1} \\ &\lesssim |v^h|_{1,D_1} + \epsilon|v^h|_{1,D_1} + h^{-1}\epsilon|v^h|_{0,D_1} \\ &\lesssim |v^h|_{1,D_1} + h^{-1}\epsilon|v^h|_{0,D_1}. \end{aligned}$$

Then,

$$\begin{aligned} |v^h|^2_{1,D_1} &\lesssim \epsilon\left(|v^h|_{1,D_1} + h^{-1}|v^h|_{0,D_1}\right)\left(|v^h|_{1,D_1} + h^{-1}\epsilon|v^h|_{0,D_1}\right) \\ &\lesssim \epsilon\left(|v^h|^2_{1,D_1} + h^{-2}|v^h|^2_{0,D_1} + h^{-2}\epsilon^2|v^h|^2_{0,D_1}\right) \\ &\lesssim \epsilon\left(|v^h|^2_{1,D_1} + h^{-2}|v^h|^2_{0,D_1}\right). \end{aligned}$$

Consequently, for sufficiently small ϵ, (28) is valid. □

We are now in a perfect position to show the following $W^{1,\infty}$ estimate.

Theorem 1 *Let $u \in W^{k+2,\infty}(\Omega)$ be the solution of (1) and $u_h \in \mathcal{U}_h$ be the FV solution of (2). If $u \in W^{k+2,\infty}(\Omega)$, then*

$$\|u - u^h\|_{1,\infty} \lesssim h^k |\ln h|^2 \|u\|_{k+2,\infty}. \tag{29}$$

Proof A decomposition

$$u = I_h u + (u - I_h u), \tag{30}$$

and the fact that $(I_h u)^h = I_h u$ yield

$$u - u^h = u - I_h u - (u - I_h u)^h. \tag{31}$$

Noticing

$$|u - I_h u|_{1,\infty} \lesssim h^k |u|_{k+1,\infty},$$

we estimate (29) by bounding

$$(u - I_h u)^h = \sum_{m=0}^{m_0} u_m^h, \tag{32}$$

where $u_m = (u - I_h u)\phi_m$ and u_m^h is the finite volume projection of u_m.

We next bound $|\nabla u_m^h(x_0)|$ for all m.

If

$$\left\| u_m^h \right\|_{1,\Omega_{m-3}} \leq R_m^{-1} h \left\| u_m^h \right\|_{1,\Omega_{m-2}}, \tag{33}$$

then by (21),

$$\left\| u_m^h \right\|_{1,\Omega_{m-3}} \lesssim h^{k+1} \|u\|_{k+1,\infty}. \tag{34}$$

If

$$\left\| u_m^h \right\|_{0,\Omega_{m-3}} \leq R_m^{-1} h \left\| u_m^h \right\|_{0,\Omega_{m-2}}, \tag{35}$$

then by (22),

$$\left\| u_m^h \right\|_{0,\Omega_{m-3}} \lesssim h^{k+2} \|u\|_{k+2,\infty},$$

and by the inverse inequality, (34) is also valid.

If neither (33) nor (35) is valid, then

$$\left\| u_m^h \right\|_{1,\Omega_{m-3}} \geq R_m^{-1} h \left\| u_m^h \right\|_{1,\Omega_{m-2}},$$
$$\left\| u_m^h \right\|_{0,\Omega_{m-3}} \geq R_m^{-1} h \left\| u_m^h \right\|_{0,\Omega_{m-2}}$$

hold simultaneously. Let

$$D_j^m = \bigcup_{e \in \mathcal{T}_h, \operatorname{dist}(e, \Omega_{m-3}) \leq jh} e.$$

Then $\Omega_{m-2} = D^m_{j_0}$ with $j_0 = 2^{m-3}$. Letting $\mu = (h/R_m)^{\frac{1}{j_0}} = 2^{-\frac{8m}{2^m}}$, there exists $0 \leq j \leq j_0 - 1$ such that

$$\left|u^h_m\right|_{1,D^m_j} \geq \mu^2 \left|u^h_m\right|_{1,D^m_{j+1}}$$

and

$$\left\|u^h_m\right\|_{0,D^m_j} \geq \mu^2 \left\|u^h_m\right\|_{0,D^m_{j+1}}$$

hold simultaneously.

In other words,

$$\left|u^h_m\right|_{1,D^m_{j+1}\setminus D^m_j} \leq \frac{\sqrt{1-\mu^4}}{\mu^2} \left|u^h_m\right|_{1,D^m_j},$$

and

$$\left\|u^h_m\right\|_{0,D^m_{j+1}\setminus D^m_j} \leq \frac{\sqrt{1-\mu^4}}{\mu^2} \left\|u^h_m\right\|_{0,D^m_j}.$$

When $m \gtrsim \ln|\ln h|$, $\left(\frac{m}{2^m}\right)^2$ is small and $\frac{\sqrt{1-\mu^4}}{\mu^2} \sim \left(\frac{m}{2^m}\right)^2$, then by (28) and (22),

$$\left\|u^h_m\right\|_{1,D^m_j} \lesssim \frac{m}{2^m} h^{-1} \left\|u^h_m\right\|_{0,D^m_j} \leq |\ln h| h^{k+1} \|u\|_{k+2,\infty}. \tag{36}$$

When $m \lesssim \ln|\ln h|$, $R_m \sim 2^m h \lesssim |\ln h|$, we have

$$\left\|u^h_m\right\|_{1,\Omega_{m-2}} \lesssim h^k R_m \|u\|_{k+1,\infty} \lesssim h^k |\ln h| \|u\|_{k+1,\infty}. \tag{37}$$

Recalling (34), (36) and (37) and using the inverse inequality, we obtain

$$\left|\overline{\nabla} u^h_m(x_0)\right| \lesssim h^k |\ln h| \|u\|_{k+2,\infty}$$

and thus

$$\left|\overline{\nabla}(u - I_h u)^h)(x_0)\right| \leq \sum_{m=1}^{m_0} \left|\overline{\nabla} u^h_m(x_0)\right| \lesssim h^k |\ln h|^2 \|u\|_{k+2,\infty}.$$

Then (29) holds by noticing that x_0 is an arbitrary point in Ω. □

4 An L^∞ estimate

Since we do not have the H^{-1} norm error estimate for the finite volume solution u_h, we can not use the same method in the previous section to obtain our L^∞ error estimate. In this section, we will the use the Green's function and the *discrete* Green's function to bound the L^∞-norm error of the finite volume solution.

For any given x_0 in Ω, let $G_{x_0} \in W_0^{1,1}(\Omega)$ be the standard Green's function defined by

$$a(v, G_{x_0}) = v(x_0),\ \forall v \in W_0^{1,\infty}(\Omega)$$

and let $G_{x_0}^h \in \mathcal{U}_h$ be the FV *discrete Green*'s function defined by

$$a_h\left(\upsilon, \Pi G_{x_0}^h\right) = \upsilon(x_0), \quad \upsilon \in S_0^h(\Omega).$$

Next we present a property of G_{x_0} of which the proof has been given in [22,45] for the special case that $\mathcal{L}$ is the Laplace operator.

Proposition 2 *For any $x_0 \in \Omega$ and $r > 0$, there holds*

$$\frac{\|G_{x_0}\|_{0,\infty,\Omega\backslash B(x_0,r)}}{|\ln r| + 1} + \|G_{x_0}\|_{1,\Omega\backslash B(x_0,r)} + r\|G_{x_0}\|_{2,\Omega\backslash B(x_0,r)} + r^2\|G_{x_0}\|_{3,\Omega\backslash B_{x_0}^r} \lesssim 1, \tag{38}$$

where

$$B_{x_0}^r = \{x \in \Omega \,|\, \min\{|x - x_0|, \rho(x, M)\} \le r\},$$

and M is the set of vertices on $\partial\Omega$, $\rho(x, M) = \min_{y\in M} \|x - y\|$.

Proof Let $\tilde{G}_{x_0} \in W_p^1(\Re^2)$, $1 \le p < 2$ be the Green's function in the whole $\Re^2$ defined by

$$\int_{\Re^2} \alpha \nabla \tilde{G}_{x_0} \nabla \psi dx = \psi(x_0) \quad \forall \psi \in W_{\bar{p}}^1(\Re^2) \cap H_0^1(\Re^2),$$

where $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. By [24,32], there hold

$$|\tilde{G}_{x_0}(x)| \lesssim \Big|\ln|x - x_0|\Big| + 1, \quad \forall x \ne x_0, \tag{39}$$

and

$$\left|D^i \tilde{G}_{x_0}(x)\right| \lesssim \frac{1}{|x - x_0|^i}, \quad \forall i \ge 1, \quad \forall x \ne x_0. \tag{40}$$

Let $E_{x_0} = G_{x_0} - \tilde{G}_{x_0}$, then $E_{x_0} = -\tilde{G}_{x_0} \le 0$ on $\partial\Omega$, and $\mathcal{L}E_{x_0} = 0$ in Ω. Therefore $E_{x_0} \le 0$ in the whole Ω and

$$|G_{x_0}(x)| = G_{x_0}(x) \le \tilde{G}_{x_0}(x).$$

Then by (39), we have that for all $r > 0$,

$$\|G_{x_0}\|_{0,\infty,\Omega\setminus B(x_0,r)} \lesssim |\ln r| + 1. \tag{41}$$

For a given $r > 0$, let $\psi \in C^\infty(\Omega)$ be the cutoff function such that $\psi = 1$ in $B(x_0, r)$, $\psi = 0$ in $\Omega \setminus B(x_0, 2r)$, and $|\nabla^i \psi| \lesssim r^{-i}$ for any positive integer i. We assume that $\hat{x} \in \Omega$ is a fixed point satisfying $|x_0 - \hat{x}| = r$. We define

$$G_2(x) = \psi(x)(\tilde{G}_{x_0}(\hat{x}) - \tilde{G}_{x_0}(x)), \quad \forall x \in \Omega.$$

and let

$$g(x) = -\psi(x)(\tilde{G}_{x_0}(\hat{x}) - \tilde{G}_{x_0}(x)), \quad x \in \partial\Omega.$$

We now split G_{x_0} into

$$G_{x_0} = G_2 + G_3 + G_4,$$

such that G_3 satisfies

$$\begin{cases} \mathcal{L}G_3(x) = \mathcal{L}(G_{x_0} - G_2)(x), & x \in \Omega, \\ G_3(x) = 0, & x \in \partial\Omega, \end{cases}$$

and G_4 satisfies

$$\begin{cases} \mathcal{L}G_4(x) = 0, & x \in \Omega, \\ G_4(x) = g(x), & x \in \partial\Omega. \end{cases}$$

We next estimate $\|G_3\|_{2,\Omega\setminus B(x_0,r)}$. Obviously, $\mathcal{L}G_3 = 0$ in $B(x_0, r) \cup (\Omega \setminus B(x_0, 2r))$ and $G_3 = 0$ on $\partial\Omega$. Moreover by (40),

$$|\tilde{G}_{x_0}(\hat{x}) - \tilde{G}_{x_0}(x)| \le |x - \hat{x}|\|\tilde{G}_{x_0}\|_{1,\infty,B(x_0,2r)\setminus B(x_0,r)} \lesssim 1, x \in B(x_0, 2r) \setminus B(x_0, r).$$

Using (40) and the properties of ψ, we have

$$\|G_2\|_{2,\infty,B(x_0,2r)\setminus B(x_0,r)} + \|\mathcal{L}G_2\|_{0,\infty,B(x_0,2r)\setminus B(x_0,r)} \lesssim \frac{1}{r^2}. \tag{42}$$

Then by the shifting theorem in [17],

$$\begin{aligned} \|G_3\|_2 &\lesssim \|\mathcal{L}G_3\|_0 = \|\mathcal{L}G_3\|_{0,B(x_0,2r)\setminus B(x_0,r)} \\ &= \|\mathcal{L}G_2\|_{0,B(x_0,2r)\setminus B(x_0,r)} \lesssim r\|\mathcal{L}G_2\|_{0,\infty,B(x_0,2r)\setminus B(x_0,r)} \lesssim \frac{1}{r}. \end{aligned} \tag{43}$$

Next we estimate $\|G_4\|_{2,\Omega\setminus B(x_0,r))}$ which depends heavily on g. Noticing that g might be *weak* singular when x_0 is very close to $\partial\Omega$, we let $\hat{z} \in \partial\Omega$ satisfy $|x_0 - \hat{z}| = \rho(x_0, \partial\Omega)$.

In the case that $|x_0 - \hat{z}| \leq \frac{r}{2}$, we have

$$B(x_0, 2r) \setminus B(x_0, r) \subset B(\hat{z}, 3r) \setminus B\left(\hat{z}, \frac{r}{2}\right).$$

Moreover, by the definition of ψ and g,

$$g(x) = 0, \quad \text{if} \quad |x - \hat{z}| \geq 3r.$$

Then, by ([4], Theorem 5.4),

$$\begin{aligned}\|G_4\|_{2,\Omega\setminus B(x_0,r)} &= \|G_4\|_{2,B(x_0,2r)\setminus B(x_0,r)} \leq \|G_4\|_{2,B(\hat{z},3r)\setminus B(\hat{z},\frac{r}{2})} \\ &\lesssim r^{-2}\|G_4\|_{\overline{\kappa}_0^2(\Omega)} \lesssim r^{-2}\|g\|_{\overline{\kappa}^{\frac{3}{2}}_{-\frac{1}{2}}(\partial\Omega)} \lesssim r^{-1},\end{aligned}$$

where the weighted norms are defined for two given positive integers a, b by

$$\|u\|_{\overline{\kappa}_a^b(\Omega)} = \left(\sum_{|\mathbf{i}|\leq b} \int_\Omega (|D^{\mathbf{i}}u(x)||x - \hat{z}|^{|\mathbf{i}|-a})^2 dx\right)^{\frac{1}{2}},$$

and

$$\|u\|_{\overline{\kappa}_a^b(\partial\Omega)} = \left(\sum_{|\mathbf{i}|\leq b} \int_{\partial\Omega} (|D^{\mathbf{i}}u(x)||x - \hat{z}|^{|\mathbf{i}|-a})^2 ds\right)^{\frac{1}{2}},$$

with $\mathbf{i} = (i_1, i_2)$ and $|\mathbf{i}| = i_1 + i_2$.

In the case that $|x_0 - \hat{z}| \geq \frac{r}{2}$, we have that $\|g\|_{\frac{3}{2},\infty,\partial\Omega} \lesssim r^{-\frac{3}{2}}$. Then,

$$\|G_4\|_{2,\Omega} \lesssim \|g\|_{\frac{3}{2},\partial\Omega} \lesssim r^{-1}.$$

That is, in both cases, we have

$$\|G_4\|_{2,\Omega\setminus B(x_0,r)} \lesssim r^{-1}. \tag{44}$$

Combining the estimates (42), (43), and (44), we have

$$\|G_{x_0}\|_{2,\Omega\setminus B(x_0,r)} \leq \|G_3\|_{2,\Omega} + \|G_2\|_{2,\Omega\setminus B(x_0,r)} + \|G_4\|_{2,\Omega\setminus B(x_0,r)} \lesssim r^{-1}. \tag{45}$$

Similarly to (45), we have

$$\|G_{x_0}\|_{1,\Omega\setminus B(x_0,r)} \lesssim 1. \tag{46}$$

In the following, we estimate $\|G_{x_0}\|_{3,\Omega\setminus B^r_{x_0}}$. Let χ be the cutoff function such that $\chi = 1$ in $B^{3r/2}_{x_0} \setminus B^r_{x_0}$, and $\chi = 0$ in $B^{r/2}_{x_0} \cup \left(\Omega \setminus B^{2r}_{x_0}\right)$, and $\|\chi\|_{W^{b,\infty}(\Omega)} \lesssim r^{-b}$ for any positive integer b. Let

$$G_r = \|G_{x_0}\|_{0,1,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} \left(\int_{B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} dx\right)^{-1}$$

and $\overline{G}_{x_0} = \chi(G_{x_0} - G_r)$. Noticing (46), there holds

$$\|G_{x_0} - G_r\|_{0,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} \lesssim r\|G_{x_0}\|_{1,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} \lesssim r. \tag{47}$$

By [4, Theorem 6.5], (38) and (47), we obtain

$$\begin{aligned}
\|\overline{G}_{x_0}\|_{\kappa^3_2(\Omega)} &\lesssim \|\mathcal{L}\overline{G}_{x_0}\|_{\kappa^1_2(\Omega)} + \|\overline{G}_{x_0}\|_{\kappa^0_{-1}(\Omega)} \\
&\lesssim \|(G_{x_0} - G_r)\mathcal{L}\chi\|_{\kappa^1_2(\Omega)} + \|\nabla G_{x_0}\cdot\nabla\chi\|_{\kappa^1_2(\Omega)} \\
&\quad + r^{-1}\|G_{x_0} - G_r\|_{0,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} \\
&\lesssim r^2 r^{-3}\|G_{x_0} - G_r\|_{0,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} + r^2 r^{-2}\|G_{x_0}\|_{1,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} \\
&\quad + r^2 r^{-1}\|G_{x_0}\|_{2,B^{2r}_{x_0}\setminus B^{r/2}_{x_0}} + 1 \\
&\lesssim r^2(r^{-3}r + r^{-2} + r^{-1}r^{-1}) + 1 \lesssim 1.
\end{aligned} \tag{48}$$

Therefore,

$$\|\nabla^3 G_{x_0}\|_{L^2\left(B^{3r/2}_{x_0}\setminus B^r_{x_0}\right)} \lesssim r^{-2}\|\overline{G}_{x_0}\|_{\kappa^3_2(\Omega)} \lesssim r^{-2}. \tag{49}$$

Let T_0 be the smallest positive integer such that $\Omega \subset B^0_{(3/2)^{T_0}r}$. Let $r_0 = r, r_1 = \frac{3r}{2}, \ldots, r_{T_0} = (3/2)^{T_0}r$. Furthermore, from (49) it follows

$$\|\nabla^3 G_{x_0}\|_{L^2\left(\Omega\setminus B^r_{x_0}\right)} \lesssim \sum_{j=0}^{T_0-1} \|\nabla^3 G_{x_0}\|_{L^2\left(B^{r_{j+1}}_{x_0}\setminus B^{r_j}_{x_0}\right)} \lesssim \sum_{j=0}^{T_0-1} \left(\frac{4}{9}\right)^j r^{-2} \lesssim r^{-2}.$$

This implies the desired result

$$r^2\|G_{x_0}\|_{3,\Omega\setminus B^r_{x_0}} \lesssim 1. \tag{50}$$

The proposition is then proved by combining (41), (45), (46) and (50). □

Remark 1 From the above arguments, we can also derive that for any $x_0 \in \Omega$ and $r > 0$, there holds

$$\frac{\|G_{x_0}\|_{0,1,B(x_0,r)}}{r^2(|\ln r|+1)} + r^{-1}\|G_{x_0}\|_{1,1,\Omega\backslash B(x_0,r)} + \|G_{x_0}\|_{2,1,\Omega\backslash B(x_0,r)} + r\|G_{x_0}\|_{3,1,\Omega\backslash B^r_{x_0}} \lesssim 1. \tag{51}$$

For the FV discrete Green's function, we have the following estimate.

Lemma 4 *There holds*

$$\left\|G^h_{x_0}\right\|_1 \lesssim 1 + |\ln h|^{\frac{1}{2}}. \tag{52}$$

Proof Let $\sigma = 1 + |\ln h|$. By the inverse estimates, we have

$$\begin{aligned}\left\|G^h_{x_0}\right\|_{0,\infty} &= \max_{e\in\mathcal{T}_h}\left\|G^h_{x_0}\right\|_{0,\infty,e} \lesssim \max_{e\in\mathcal{T}_h}\left\|G^h_{x_0}\right\|_{0,\sigma,e} h^{-\frac{2}{\sigma}}\\ &\lesssim \sigma^{\frac{1}{2}}\left\|G^h_{x_0}\right\|_1 \lesssim (1+|\ln h|)^{\frac{1}{2}}\left\|G^h_{x_0}\right\|_1.\end{aligned}$$

Then by the fact that

$$\left\|G^h_{x_0}\right\|_1^2 \lesssim a_h\left(G^h_{x_0}, \Pi G^h_{x_0}\right) \le z\left\|G^h_{x_0}\right\|_{0,\infty},$$

the inequality (52) follows. □

Next we estimate the difference between the exact Green's function G_{x_0} and $G^h_{x_0}$. As in the previous section, we will heavily use the interpolating operator I_h. However since $G_{x_0}(x_0) = +\infty$, the standard Lagrange interpolation of G_{x_0} is not well defined at x_0. Therefore, we modify a little bit the definition of interpolation of G_{x_0} as below (see also [21]). Let $\nu_{x_0} \in C^\infty(\Omega)$ be the cutoff function satisfying $\nu_{x_0}(y) = 0$ if $y \in B(x_0, h/2)$; and $\nu_{x_0}(y) = 1$ if $y \in \Omega \setminus B(x_0, h)$; and $\|\nu_{x_0}\|_{m,\infty} \lesssim h^{-m}$, $m = 1, 2$. We define the interpolation

$$G^I_{x_0} = I_h(\nu G_{x_0}).$$

For $r > 0$, set

$$B^r_{x_0} = B(x_0, r) \cap \Omega.$$

It is easy to derive from (51) that

$$\begin{aligned}\left\|G_{x_0} - G^I_{x_0}\right\|_{0,1} &\le \|G_{x_0} - \nu G_{x_0}\|_{0,1} + \|\nu G_{x_0} - I_h(\nu G_{x_0})\|_{0,1}\\ &\le \|G_{x_0}\|_{0,1,B^h_{x_0}} + h^2\|\nu G_{x_0}\|_{2,1}\\ &\lesssim \|G_{x_0}\|_{0,1,B^h_{x_0}} + h^2\|G_{x_0}\|_{2,1,\Omega\backslash B^{h/2}_{x_0}}\\ &\quad + h\|G_{x_0}\|_{1,1,B^h_{x_0}\backslash B^{h/2}_{x_0}} + \|G_{x_0}\|_{0,1,,B^h_{x_0}\backslash B^{h/2}_{x_0}}\\ &\lesssim h^2|\ln h|.\end{aligned} \tag{53}$$

Similarly, we have

$$\left|G_{x_0} - G^I_{x_0}\right|_{1,1} \lesssim h|\ln h| + h \lesssim h|\ln h|, \tag{54}$$

and

$$\left|G_{x_0} - G^I_{x_0}\right|_{2,1} \lesssim |\ln h|. \tag{55}$$

Next we will estimate the error $G_{x_0} - G^h_{x_0}$. To this end, we recall the definition of m_0 and R_m $(0 \le m \le m_0)$ in the previous section, and let the functions $\{\overline{\phi}_m \in C^\infty(\Omega) | 0 \le m \le m_0\}$ satisfy the following three properties:

(1) $0 \le \overline{\phi}_m \le 1$, $\sum_{m=0}^{m_0} \overline{\phi}_m = 1$ in the whole Ω;
(2) $\overline{\phi}_m = 0$ in $B^{R_{m-2}}_{x_0} \cup \left(\Omega \setminus B^{R_{m+1}}_{x_0}\right)$, if $m \ge 2$; and $\overline{\phi}_m = 0$ in $\Omega \setminus B^{R_{m+1}}_{x_0}$ if $m \le 1$.
(3) $|\overline{\phi}_m|_{l,\infty} \lesssim R_m^{-l}$, $\forall l \ge 1$.

Note that

$$G_{x_0} - G^h_{x_0} = G_{x_0} - G^I_{x_0} - \left(G_{x_0} - G^I_{x_0}\right)^h = \sum_{m=0}^{m_0} \left(\theta_m - \theta^h_m\right), \tag{56}$$

where

$$\theta_m = \left(G_{x_0} - G^I_{x_0}\right) \overline{\phi}_m, 0 \le m \le m_0.$$

In the following, we bound $\left\|G_{x_0} - G^h_{x_0}\right\|_{1,1}$ by estimating $\theta_m - \theta^h_m$. It is easy to deduce from (38) that for $m \ge 2$, for $l = 1, 2$ we have

$$\begin{aligned} |\theta_m|_l &\lesssim \sum_{j=0}^{l} |\overline{\phi}_m|_{1,\infty} \left|G_{x_0} - G^I_{x_0}\right|_{l-j, B^{R_{m+1}}_{x_0} \setminus B^{R_{m-2}}_{x_0}} \\ &\lesssim \sum_{j=0}^{l} \frac{1}{R_m^j} h^{j+2-l} |G_{x_0}|_{2, B^{R_{m+1}}_{x_0} \setminus B^{R_{m-2}}_{x_0}} \lesssim \frac{h^{2-l}}{R_m}, \end{aligned}$$

and

$$\begin{aligned} |\theta_m|_{3,1} &\lesssim \sum_{j=0}^{3} |\overline{\phi}_m|_{j,\infty} \left|G_{x_0} - G^I_{x_0}\right|_{3-j,1, B^{R_{m+1}}_{x_0} \setminus B^{R_{m-2}}_{x_0}} \\ &\lesssim \sum_{j=1}^{3} \frac{1}{R_m^j} h^{j-1} |G_{x_0}|_{2,1, B^{R_{m+1}}_{x_0} \setminus B^{R_{m-2}}_{x_0}} + |G_{x_0}|_{3,1, B^{R_{m+1}}_{x_0} \setminus B^{R_{m-2}}_{x_0}} \\ &\lesssim \frac{1}{R_m}. \end{aligned}$$

We are ready to estimate $\left\|\theta_m - \theta^h_m\right\|_0$ and $\left\|\theta_m - \theta^h_m\right\|_1$.

Lemma 5 *For $m \geq 2$, we have*

$$\left|\theta_m - \theta_m^h\right|_1 \lesssim \frac{h}{R_m}, \tag{57}$$

$$\left\|\theta_m - \theta_m^h\right\|_0 \lesssim \frac{h^2}{R_m}. \tag{58}$$

Proof Noticing that

$$\left|\theta_m - \theta_m^h\right|_1 \lesssim h|\theta_m|_2 \lesssim \frac{h}{R_m},$$

we obtain (57). To show (58), we construct the auxiliary problem

$$\mathcal{L}w = \theta_m - \theta_m^h \text{ in } \Omega, \quad \text{and} \quad w = 0 \text{ on } \partial\Omega.$$

Then

$$\|w\|_2 \lesssim \left\|\theta_m - \theta_m^h\right\|_0,$$

and

$$\begin{aligned}\left\|\theta_m - \theta_m^h\right\|_0^2 &\lesssim a\left(\theta_m - \theta_m^h, w - I_h^1 w\right) + a\left(\theta_m - \theta_m^h, I_h^1 w\right)\\ &\lesssim h\left|\theta_m - \theta_m^h\right|_1 \left\|\theta_m - \theta_m^h\right\|_0 + a\left(\theta_m - \theta_m^h, I_h^1 w\right) - a_h\left(\theta_m - \theta_m^h, \left(I_h^1 w\right)^*\right).\end{aligned}$$

Noticing that by (15),

$$\begin{aligned}&a\left(\theta_m - \theta_m^h, I_h^1 w\right) - a_h\left(\theta_m - \theta_m^h, \left(I_h^1 w\right)^*\right)\\ &= \left(L\theta_m, I_h^1 w - \left(I_h^1 w\right)^*\right) + a_h\left(\theta_m^h, \left(I_h^1 w\right)^*\right) - a\left(\theta_m^h, I_h^1 w\right)\\ &\lesssim h|L\theta_m|_{1,1}\left\|I_h^1 w - \left(I_h^1 w\right)^*\right\|_{0,\infty} + a_h\left(\theta_m^h, \left(I_h^1 w\right)^*\right) - a\left(\theta_m^h, I_h^1 w\right)\\ &\lesssim h^2|\theta_m|_{3,1}\left|I_h^1 w\right|_{1,\infty} + h\left|\theta_m^h\right|_1 \left|I_h^1 w\right|_2\\ &\lesssim h^2|\theta_m|_{3,1}\left\|I_h^1 w\right\|_2 + h\left|\theta_m - \theta_m^h\right|_1 \left|I_h^1 w\right|_2 + h|\theta_m|_1\left|I_h^1 w\right|_2\\ &\lesssim \left\|\theta_m - \theta_m^h\right\|_0 \frac{h^2}{R_m}.\end{aligned}$$

Then we obtain (58). □

Lemma 6 *There holds*

$$\left\|G_{x_0} - G_{x_0}^h\right\|_{1,1} \lesssim h|\ln h|^2. \tag{59}$$

Proof From the previous lemma, we observe that for $m \geq 2$, there holds

$$h \left\| I_h\theta_m - \theta_m^h \right\|_1 + \left\| I_h\theta_m - \theta_m^h \right\|_0 \lesssim R_m^{-1} h^2. \tag{60}$$

Notice the fact that,for all $j \geq m+1$,

$$\theta_m(x) = 0, \quad \forall x \in \Omega \setminus B_{x_0}^{R_j}. \tag{61}$$

Moreover, for all $j \geq m+1$,

$$\left\| I_h\theta_m - \theta_m^h \right\|_{1,\Omega\setminus B_{x_0}^{R_j}} \lesssim R_j^{-1} \left\| \theta_m - \theta_m^h \right\|_0 \lesssim (R_j R_m)^{-1} h^2, \tag{62}$$

where we have used the same arguments of proving (36).

It follows that

$$\begin{aligned}
\left\| I_h\theta_m - \theta_m^h \right\|_{1,1} &\leq \left\| I_h\theta_m - \theta_m^h \right\|_{1,1,B_{x_0}^{R_{m+1}}} + \left\| I_h\theta_m - \theta_m^h \right\|_{1,1,\Omega\setminus B_{x_0}^{R_{m+1}}} \\
&\lesssim R_m \left\| I_h\theta_m - \theta_m^h \right\|_1 + \sum_{j=i+1}^{m_0} \left\| I_h\theta_m - \theta_m^h \right\|_{1,1,B_{x_0}^{R_{j+1}}\setminus B_{x_0}^{R_j}} \\
&\lesssim h + \sum_{j=m+1}^{m_0} R_m^{-1} h^2 \\
&\lesssim h + h^2 |\ln h| R_m^{-1},
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{m=2}^{m_0} \left\| \theta_m - \theta_m^h \right\|_{1,1} &\leq \sum_{m=2}^{m_0} \| I_h\theta_m - \theta_m \|_{1,1} + \sum_{m=2}^{m_0} \left\| I_h\theta_m - \theta_m^h \right\|_{1,1} \\
&\lesssim \sum_{m=2}^{m_0} \left(h + h^2 |\ln h| R_m^{-1} \right) \lesssim h|\ln h|.
\end{aligned} \tag{63}$$

In the following, we estimate the difference $\|(\theta_0 + \theta_1) - (\theta_0 + \theta_1)^h\|_{1,1}$. For simplicity, we denote $\eta = \theta_0 + \theta_1$. We first observe that

$$\begin{aligned}
\|\eta^h\|_1 &\leq \left\| G_{x_0}^h \right\|_1 + \sum_{m=2}^{m_0} \left\| \theta_m - \theta_m^h \right\|_1 + \sum_{m=2}^{m_0} \|\theta_m\|_1 \\
&\lesssim |\ln h| + |\ln h| + |\ln h| \lesssim |\ln h|.
\end{aligned}$$

and from (53) and (54), we have

$$\|\theta_0 + \theta_1\|_{1,1} \lesssim h|\ln h|.$$

Next turn to the estimation of $\|\eta^h\|_0$ with $\eta = \theta_0 + \theta_1$. Let $\chi \in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$L\chi = \eta^h. \tag{64}$$

Then

$$\begin{aligned}(\eta^h, \eta^h) &= a(\eta^h, \chi) = a\left(\eta^h, \chi - I_h^1\chi\right) + a\left(\eta^h, I_h^1\chi\right)\\ &= a\left(\eta^h, \chi - I_h^1\chi\right) + a_h\left(\eta^h, (I_h^1\chi)^*\right) + \left(a\left(\eta^h, I_h^1\chi\right) - a_h\left(\eta^h, \left(I_h^1\chi\right)^*\right)\right.\\ &= J_1 + J_2 + J_3.\end{aligned} \tag{65}$$

Noticing that

$$\begin{aligned}|J_1| &\lesssim \|\eta^h\|_1 \left\|\chi - I_h^1\chi\right\|_1 \lesssim \|\eta^h\|_1 h\|\chi\|_2 \lesssim h\|\eta^h\|_1\|\eta^h\|_0,\\ |J_2| &= \left|a_h\left(\eta, \left(I_h^1\chi\right)^*\right)\right| \lesssim \|\eta\|_{1,1}\left\|I_h^1\chi\right\|_{1,\infty} \lesssim h|\ln h|\|\chi\|_2 \lesssim h|\ln h|\|\eta^h\|_0,\end{aligned}$$

and

$$|J_3| \lesssim h|\ln h|\|\eta^h\|_0,$$

we derive from (65) that

$$\|\eta^h\|_0 \lesssim h|\ln h|. \tag{66}$$

Therefore, by the same reasoning in the previous section, we obtain that for all $m \geq 1$,

$$\|\eta^h\|_{1, B_{x_0}^{R_{m+1}} \setminus B_{x_0}^{R_m}} \lesssim R_m^{-1}\|\eta^h\|_0 \lesssim R_m^{-1} h|\ln h|,$$

which implies that

$$\|\eta^h\|_{1,1, B_{x_0}^{R_{m+1}} \setminus B_{x_0}^{R_m}} \lesssim h|\ln h|.$$

Consequently,

$$\|\eta^h\|_{1,1} = \|\eta^h\|_{1,1,B(x_0,h)} + \sum_{m=0}^{m_0-1} \|\eta^h\|_{1,1, B_{x_0}^{R_{m+1}} \setminus B_{x_0}^{R_m}} \lesssim h|\ln h|^2.$$

Noticing (53) and (54), we finally obtain

$$\|\eta - \eta^h\|_{1,1} \lesssim h|\ln h|^2. \tag{67}$$

The statement (59) follows directly from (63) and (67). □

We are now in a perfect position to estimate $\|u-u^h\|_{0,\infty}$.

Theorem 2 *Under the same assumption as in Theorem 1, we have*

$$\|u-u^h\|_{0,\infty} \lesssim h^{k+1}|\ln h|^2\|u\|_{k+2,\infty}. \tag{68}$$

Proof First, for $0 \le m \le 2$, there hold

$$\left\|u_m-u_m^h\right\|_0 \lesssim h^{k+1}\|u_m\|_{k+2} \lesssim h^{k+2}\|u\|_{k+2,\infty},$$

and consequently

$$\begin{aligned}\left\|u_m-u_m^h\right\|_{0,\infty} &\le \|u_m-I_hu_m\|_{0,\infty}+\left\|I_hu_m-u_m^h\right\|_{0,\infty}\\ &\lesssim h^{k+1}\|u\|_{k+2,\infty}+h^{-1}\left\|I_hu_m-u_m^h\right\|_0\\ &\lesssim h^{k+1}\|u\|_{k+2,\infty}+h^{-1}\left(\|I_hu_m-u_m\|_0+\left\|u_m-u_m^h\right\|_0\right)\\ &\lesssim h^{k+1}\|u\|_{k+2,\infty}+h^{-1}h^{k+2}\|u\|_{k+2,\infty}\\ &\lesssim h^{k+1}\|u\|_{k+2,\infty}.\end{aligned} \tag{69}$$

Secondly, setting

$$\bar{u}(x)=\sum_{m=3}^{m_0}u_m(x),$$

we next estimate $(\bar{u}-\bar{u}^h)(x_0)$ by splitting

$$\begin{aligned}(I_h\bar{u}-\bar{u}^h)(x_0) &= a_h\left(I_h\bar{u}-\bar{u}^h,\Pi G_{x_0}^h\right)\\ &= a_h\left(I_h\bar{u}-\bar{u},\Pi G_{x_0}^h\right)\\ &= a_h\left(I_h\bar{u}-\bar{u},\Pi G_{x_0}^I\right)+a_h\left(I_h\bar{u}-\bar{u},\Pi\left(G_{x_0}^h-G_{x_0}^I\right)\right)\\ &= K_1+K_2.\end{aligned} \tag{70}$$

It is easy to deduce from (59) that

$$\begin{aligned}|K_2| &\lesssim \|I_h\bar{u}-\bar{u}\|_{1,\infty}\left\|G_{x_0}^h-G_{x_0}^I\right\|_{1,1}\\ &\lesssim h|\ln h|^2h^k\|u\|_{k+1,\infty}\\ &\lesssim h^{k+1}|\ln h|^2\|u\|_{k+1,\infty}.\end{aligned} \tag{71}$$

We now turn to the estimation of K_1.

$$\begin{aligned} K_1 &= a_h\left(I_h\bar{u}-\bar{u},\Pi\left(G_{x_0}^I-I_h^1G_{x_0}^I\right)\right)+a_h\left(I_h\bar{u}-\bar{u},\left(I_h^1G_{x_0}^I\right)^*\right)\\ &= a_h\left(I_h\bar{u}-\bar{u},\Pi\left(G_{x_0}^I-I_h^1G_{x_0}^I\right)\right)+\Big[a_h\left(I_h\bar{u}-\bar{u},\left(I_h^1G_{x_0}^I\right)^*\right)\\ &\quad -a\left(I_h\bar{u}-\bar{u},I_h^1G_{x_0}^I\right)\Big]+a\left(I_h\bar{u}-\bar{u},I_h^1G_{x_0}^I-G_{x_0}^I\right)+a\left(I_h\bar{u}-\bar{u},G_{x_0}^I\right)\\ &= K_{1,1}+K_{1,2}+K_{1,3}+K_{1,4}. \end{aligned}$$

Noticing that

$$\begin{aligned} |K_{1,1}| &\lesssim \|I_h\bar{u}-\bar{u}\|_{1,\infty}\left\|G_{x_0}^I-I_h^1G_{x_0}^I\right\|_{1,1}\\ &\lesssim h^k\|u\|_{k+1,\infty}h|\ln h|\\ &\lesssim h^{k+1}|\ln h|\|u\|_{k+1,\infty},\\ |K_{1,2}| &\lesssim h^{k+1}\|I_h\bar{u}-\bar{u}\|_{k,\infty}\left\|G_{x_0}^I\right\|_{2,1} \quad \text{(By Lemma 2.2)}\\ &\lesssim h^{k+1}\|u\|_{k+1,\infty},\\ |K_{1,3}| &\lesssim h|I_h\bar{u}-\bar{u}|_{1,\infty}\\ &\lesssim h^{k+1}\|u\|_{k+1,\infty},\\ |K_{1,4}| &\lesssim \|I_h\bar{u}-\bar{u}\|_{0,\infty,\Omega\backslash B(x_0,R_2)}\|G_{x_0}\|_{2,1,\Omega\backslash B(x_0,R_2)}\\ &\lesssim h^{k+1}|\ln h|\|u\|_{k+1,\infty}, \end{aligned}$$

we have

$$|K_1| \lesssim h^{k+1}|\ln h|\|u\|_{k+1,\infty}.$$

This, together with (70) and (71), gives

$$|(I_h\bar{u}-\bar{u}^h)(x_0)| \lesssim h^{k+1}|\ln h|^2\|u\|_{k+1,\infty}.$$

Noticing that the interpolation error $|(\bar{u}-I_h\bar{u})(x_0)|$ has the optimal convergence rate and that x_0 is an arbitrary point in Ω, we obtain

$$\|\bar{u}-\bar{u}^h\|_{0,\infty} \lesssim h^{k+1}|\ln h|^2\|u\|_{k+1,\infty}. \tag{72}$$

The desired result (68) is a direct result of (72) and (69). □

5 Numerical experiments

We test a simple example that

$$-\Delta u = 2\pi^2\sin(\pi x)*\sin(\pi y) \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega,$$

which admits the exact solution

$$u = \sin(\pi x) * \sin(\pi y)$$

in the domain $\Omega = [0, 1]^2$.

We test our FVM scheme (2) for $k = 2, 3, 4$ on a uniform triangular mesh. We test our FVM scheme (2) for $k = 2, 3, 4$ on a uniform triangular mesh with $h = N^{-1}$. Here we divide [0, 1] into N equally spaced subintervals in both the x- and y-directions. The errors and convergence orders are depicted in Tables 1, 2 and 3, respectively. We observed that both the $W^{1,\infty}$ and L^∞ errors converge with optimal orders, which is consistent with Theorems 1 and 2.

Table 1 k = 2

N	$\|u - u_h\|_{0,\infty}$	Rate	$\lvert u - u_h\rvert_{1,\infty}$	Rate
2	3.02e−02		6.53e−01	
4	3.97e−03	2.93	1.97e−01	1.73
8	5.04e−04	2.98	4.52e−02	2.12
16	6.22e−05	3.02	1.07e−02	2.08
32	7.69e−06	3.02	2.60e−03	2.04
64	9.55e−07	3.01	6.41e−04	2.02

Table 2 k = 3

N	$\|u - u_h\|_{0,\infty}$	Rate	$\lvert u - u_h\rvert_{1,\infty}$	Rate
2	3.93e−03		1.65e−01	
4	3.36e−04	3.55	2.42e−02	2.78
8	2.38e−05	3.82	3.13e−03	2.95
16	1.53e−06	3.96	3.95e−04	2.99
32	9.65e−08	3.99	4.95e−05	3.00
64	6.04e−09	4.00	6.19e−06	3.00

Table 3 k = 4

N	$\|u - u_h\|_{0,\infty}$	Rate	$\lvert u - u_h\rvert_{1,\infty}$	Rate
2	2.65e−04		1.31e−02	
4	7.72e−06	5.10	8.92e−04	3.88
8	2.39e−07	5.02	5.08e−05	4.13
16	7.33e−09	5.03	2.99e−06	4.09
32	2.26e−10	5.02	1.80e−07	4.05
64	7.03e−12	5.01	1.11e−08	4.03

6 Concluding remarks

The analysis of high-order FV schemes is a challenging task. This paper is the third one in a series that attempts to set up a mathematical foundation for a family of high-order FV schemes. In two previous works [28,44], we analyzed the stability, H^1 error and L^2 error of FV schemes of any order over quadrilateral meshes. In this article, we report our results on L^∞ and $W^{1,\infty}$ error estimates. In a forthcoming paper [18], we analyze the local error of the high order FV schemes over quadrilateral meshes.

The L^∞ error estimates of high-order FVMs is significantly different from, and much more complicated than, that of the first-order FVM. The analysis in the current paper depends on a deep understanding of the relationship between high-order FVMs and FEMs and a thorough knowledge of the maximum-norm-estimate techniques for FEMs.

Acknowledgements The authors are grateful to Dr. Li Guo for her careful reading and corrections of the paper, and to Dr. Waixiang Cao for her help in numerical experiments.

References

1. Arnold, D., Boffi, D., Falk, R.: Approximation by quadrilateral finite elements. Math. Comput. **71**, 909–922 (2002)
2. Bank, R.E., Rose, D.J.: Some error estimates for the box scheme. SIAM Numer. Anal. **24**, 777–787 (1987)
3. Barth, T., Ohlberger, M.: Finite volume methods: foundation and analysis. In: Encyclopedia of Computational Mechanics, vol. 1, chapter 15. Wiley (2004)
4. Bacuta, C., Nistor, V., Zikatanov, L.T.: Improving the rate of convergence of high order finite elements on polyhedra I: a priori estimates. Numer. Funct. Anal. Optim. **26**, 613–639 (2005)
5. Cai, Z.: On the finite volume element method. Numer. Math. **58**, 713–735 (1991)
6. Cao, W., Zhang, Z., Zou, Q.: Superconvergence of any order finite volume schemes for 1D general elliptic equations. J. Sci. Comput. **56**, 1–25 (2013)
7. Chatzipantelidis, P.: A finite volume method based on the Crouzeix–Raviart element for elliptic PDEs in two dimensions. Numer. Math. **82**, 409–432 (1999)
8. Chen, L.: A new class of high order finite volume methods for second order elliptic equations. SIAM Numer. Anal. **47**, 4021–4043 (2010)
9. Chen, Z., Wu, J., Xu, Y.: Higher-order finite volume methods for elliptic boundary value problems. Adv. Comput. Math. **37**(2), 191–253 (2013)
10. Chou, S.H., He, S.: On the regularity and uniformness conditions on quadrilateral grids. Comput. Meth. Appl. Mech. Eng. **191**, 5149–5158 (2002)
11. Chou, S.H., Kwak, D.Y., Li, Q.: L^p error estimates and superconvergence for covolume or finite volume element methods. Numer. Methods Patial Differ. Equ. **19**, 463–486 (2003)
12. Chou, S.H., Li, Q.: Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: a unified approach mathematics of computation. Math. Comp. **69**, 103–120 (2000)
13. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North Holland, Amsterdam (1978)
14. Ciarlet, P.G., Raviart, P.A.: Maximum principle and uniform convergence for the finite element method. Comput. Methods Appl. Mech. Eng. **2**, 17–31 (1973)
15. Emonot, P.: Methods de volumes elements finis: applications aux equations de Navier–Stokes et resultats de convergence. Lyon (1992)
16. Eymard, R., Gallouet, T., Herbin, R.: Finite volume methods. In: Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, VII, pp. 713–1020. North-Holland, Amsterdam (2000)
17. Grisvard, P.: Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain. In: Hubbard, B. (ed.) Numerical Solution of Partial Differential Equations III, SYNSPADE 1975, pp. 207–274. Academic Press, New York (1976)

18. Guo, L., Li, H., Zou, Q.: Local error estimates of high order finite volume schemes. SIAM J Numer. Anal. (2017, submitted)
19. Hackbusch, W.: On first and second order box methods. Computing **41**, 277–296 (1989)
20. He, W., Guan, X., Cui, J.: The local superconvergence of the trilinear element for the three-dimensional Poisson problem. J. Math. Anal. Appl. **388**, 863–872 (2012)
21. He, W., Shen, J., Cui, J.: The local ultraconvergence for high degree Galerkin finite element methods. J. Math. Anal. Appl. **446**, 62–86 (2017)
22. He, W., Zhang, Z.: $2k$ superconvergence of Q_k finite element by anisotropic mesh approximation in weighted Sobolev spaces. Math. Comp. **86**, 1693–1718 (2017)
23. Hyman, J.M., Knapp, R., Scovel, J.C.: High order finite volume approximations of differential operators on nonuniform grids. Physica D **60**, 112–138 (1992)
24. John, F.: General properties of solutions of linear elliptic partial differential equations. In: Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 113–175. Oklahoma A&M College, Stillwater, Okla (1951)
25. Lazarov, R., Michev, I., Vassilevski, P.: Finite volume methods for convection–diffusion problems. SIAM J. Numer. Anal. **33**, 31–55 (1996)
26. LeVeque, R.J.: Finite Volume Methods for Hyperbolic Problems. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2002)
27. Li, R., Chen, Z., Wu, W.: The Generalized Difference Methods for Partial differential Equations. Marcel Dikker, New York (2000)
28. Lin, Y., Yang, M., Zou, Q.: L^2 error estimates for a class of any order finite volume schemes over quadrilateral meshes. SIAM J. Numer. Anal. **53**, 2009–2029 (2015)
29. Li, Y., Li, R.: Generalized difference methods on arbitrary quadrilateral networks. J. Comput. Math. **17**, 653–672 (1999)
30. Lv, J., Li, Y.: L^2 error estimate of the finite volume element methods on quadrilateral meshes. Adv. Comput. Math. **33**, 129–148 (2010)
31. Lv, J., Li, Y.: Optimal biquadratic finite volume element methods on quadrilateral meshes. SIAM J. Numer. Anal. **50**, 2397–2399 (2012)
32. Miranda, C.: Partial Differential Equations of Elliptic Type, 2nd edn. Springer, Berlin (1970)
33. Nicolaides, R.A., Porsching, T.A., Hall, C.A.: Covolume methods in computational fluid dynamics. In: Hafez, M., Oshima, K. (eds.) Computational Fluid Dynamics Review, pp. 279–299. Wiley, New York (1995)
34. Nitsche, J.A.: L^∞- convergence of finite element approximations. In: Proceedings Second Conference on Finite Elements. Rennes (1975)
35. Ollivier-Gooch, C., Altena, M.: A high-order-accurate unconstructed mesh finite-volume scheme for the advection–diffusion equation. J. Comput. Phys. **181**, 729–752 (2002)
36. Patanker, S.V.: Numerical Heat Transfer and Fluid Flow. Series on Computational Methods in Mechanics and Thermal Science. McGraw Hill, New York (1980)
37. Rannacher, R., Scott, R.: Some optimal error estimate for piecewise linear finite element approximations. Math. Comput. **38**, 437–445 (1982)
38. Schatz, A.H., Wahlbin, L.B.: Maximum norm estimates in the finite element method on plane polygonal domains, part I. Math. Comput. **32**, 33–110 (1978)
39. Schatz, A.H., Wahlbin, L.B.: Maximum norm estimates in the finite element method on plane polygonal domains, part II. Math. Comput. **33**, 465–492 (1979)
40. Schmidt, T.: Box schemes on quadrilateral meshes. Computing **51**, 271–292 (1993)
41. Scott, R.: Optimal L^∞ estimates for the finite element method on irregular meshes. Math. Comput. **30**, 681–697 (1976)
42. Süli, E.: The accuracy of cell vertex finite volume methods on quadrilateral meshes. Math. Comput. **59**(359–382), 3 (1992)
43. Xu, J., Zou, Q.: Analysis of linear and quadratic simplitical finite volume methods for elliptic equations. Numer. Math. **111**, 469–492 (2009)
44. Zhang, Z., Zou, Q.: Vertex-centered finite volume schemes of any order over quadrilateral meshes for elliptic boundary value problems. Numer. Math. **130**, 363–393 (2015)
45. Zhu, Q., Lin, Q.: Theory of Superconvergence of Finite Elements. Hunan Science and Technology Press, Hunan (1989). **(in Chinese)**