Solutions to
EXAM 3    MAT 5420

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Directions: There are nine problems, worth a total of 100 points. Please show all work clearly. Justify all of your answers. Time limit: 55 minutes.

1. (5 points) Find an element of order 3 in $\mathbb{Z}_7$.

There are no such elements. By Lagrange’s Theorem, $\mathbb{Z}_7$ cannot contain an element of order 3.

In $\mathbb{Z}_7^\times$, the elements of order 3 are $[2]_7$ and $[4]_7$.

2. (10 points) List all subgroups of $\mathbb{Z}_{18}$, showing each subgroup exactly once.

You do not need to list all of the elements in each subgroup as long as you describe it accurately.

The divisors of 18 are 1, 2, 3, 6, 9, and 12. There is a subgroup of each of these orders, and they are generated by $[0], [9], [6], [3], [2],$ and $[1]$ respectively.

That is, the subgroups of $\mathbb{Z}_{18}$ are $\langle [0] \rangle, \langle [9] \rangle, \langle [6] \rangle, \langle [3] \rangle, \langle [2] \rangle$, and $\langle [1] \rangle$.

3. (10 points) How many subgroups of order 5 are contained in $\mathbb{Z}_5 \times \mathbb{Z}_5$? Be sure to explain your reasoning.

All nonidentity elements of $\mathbb{Z}_5 \times \mathbb{Z}_5$ have order 5. There are 24 such elements. Each of these elements generates a subgroup of order 5, and every such subgroup has 4 generators. So there are exactly $6 = 24/4$ subgroups of order 5 in $\mathbb{Z}_5 \times \mathbb{Z}_5$.

4. (15 points) Let $G$ be an abelian group. Prove that if $H$ and $K$ are subgroups of $G$, then the set $L = \{g \in G \mid g = hk \text{ for some } h \in H \text{ and } k \in K\}$ is a subgroup of $G$.

Proof: Since $e \in H$ and $e \in K$ we have $e = ee \in L$. Therefore $L \neq \emptyset$.

If $x,y \in L$, then $x = hk$ where $h \in H$ and $k \in K$, and $y = h'k'$ where $h' \in H$ and $k' \in K$. Then $xy^{-1} = hk(h'k')^{-1} = hk(k'^{-1}h'^{-1})$.

Since $G$ is abelian, $hkk'^{-1}h'^{-1} = hh'^{-1}kk'^{-1}$. Since $H$ and $K$
are subgroups of $G$, $hh'^{-1} \in H$ and $kk'^{-1} \in K$. Therefore $hh'^{-1} \cdot kk'^{-1} \in L$. This shows that $xy^{-1} \in L$ whenever $x$ and $y \in L$.

We showed above that $L$ is nonempty, therefore $L$ is a subgroup of $G$.

5. (10 points) Give an example of a group $G$ and subgroups $H$ and $K$ such that $\{hk \mid h \in H, k \in K\}$ is not a subgroup of $G$.

One example: In $S_3$, if $H = \langle (1,2) \rangle$ and $K = \langle (1,3) \rangle$, then $HK = \{(1),(1,2),(1,3),(1,3,2)\}$ is not a subgroup of $G$.

Note that by the previous problem, $G$ must be nonabelian.

6. (15 points) Let $A$ be the subgroup of $GL_2(\mathbb{R})$ containing all matrices of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$. Prove that $\mathbb{Z} \cong A$. You are not required to prove that $A$ is a group.

Let $f : \mathbb{Z} \to A$ be defined by $f(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

We claim that $f$ is an isomorphism.

For $a, b \in \mathbb{Z}$, $f(a) \cdot f(b) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = f(a+b)$.

Thus, $f(a+b) = f(a)f(b)$, the appropriate condition for a function from an additive group to a multiplicative group.

To show that $f$ is one-to-one, it suffices to show that if $f(n) = I$, the identity element of $A$, then $n = 0$, the identity element of $\mathbb{Z}$. If $f(n) = I$, then $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $n = 0$. Thus $f$ is one-to-one.

Now let $x \in A$, then $x = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z}$. That is, $x = f(n)$. This shows that $f$ is onto.

7. (10 points) Let $G = A_4$, the group of all even permutations of $\{1, 2, 3, 4\}$, and let $H = \langle (1,2,3) \rangle$. Let $\sim$ be the equivalence relation defined on $G$ by $x \sim y \iff xy^{-1} \in H$. Find the equivalence class of $(2, 3, 4)$ under $\sim$. 
The equivalence class of \((2, 3, 4)\) under \(\sim\) consists of all elements \(a \in G\) such that \(a(2, 3, 4)^{-1} \in H\). \(a \in [(2, 3, 4)]\) if and only if \(a = h(2, 3, 4)\) for some \(h \in H\).

So \([(2, 3, 4)] = \{(2, 3, 4), (1, 2, 3)(2, 3, 4), (1, 3, 2)(2, 3, 4)\} = \{(2, 3, 4), (1, 2)(3, 4), (1, 3, 4)\}.

8. (15 points) Let \(G = \mathbb{Z}_3 \times \mathbb{Z}_4\), and let \(g = (1, 2) \in G\).

(a) Find the order of \(g\).

By computation, \(g + g = 2g = (2, 0)\), \(3g = 2g + g = (0, 2)\), \(4g = (1, 0)\), \(5g = (2, 2)\), and \(6g = (0, 0)\). Therefore \(g\) has order 6.

(b) List the elements of \(\langle g \rangle\).

As computed above, \(\langle g \rangle = \{(0, 0), (1, 2), (2, 0), (0, 2), (1, 0), (2, 2)\}\).

9. (10 points) Describe two infinite groups which are not isomorphic and explain why they are not isomorphic. You should be able to find at least one such pair among the following groups: \(\mathbb{R}\) under addition, \(\mathbb{R}^\times\) under multiplication, \(\mathbb{R}^+\), the positive real numbers, under multiplication, \(GL_2(\mathbb{R})\), \(\mathbb{Z}\).

The one pair of isomorphic groups on the list is (\(\mathbb{R}\) under addition, \(\mathbb{R}^\times\) under multiplication). One isomorphism between these is the exponentiation map, \(x \rightarrow e^x\), where \(e\) is the base of natural logarithms.

Every element of \(\mathbb{R}\) under addition has infinite order. In \(\mathbb{R}^\times\), the element \(-1\) has order 2, so \(\mathbb{R}\) and \(\mathbb{R}^\times\) are not isomorphic.

\(GL_2(\mathbb{R})\) is nonabelian and is not isomorphic to any of the other groups listed.

\(\mathbb{Z}\) is cyclic and is not isomorphic to any of the other groups listed.